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JOHN ROLFE ISBELL

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Introduction. The main subject of this paper is the inductive dimension $\delta \operatorname{Ind} \mu X$ of uniform spaces μX. This is defined similarly to topological dimension Ind, but instead of separation one uses the notion of a set H, arbitrarily small uniform neighborhoods of which uniformly separate given sets A, B. For finite dimensional metric spaces M (i.e. the large dimension $\Delta d\, M$ is finite) $\delta \operatorname{Ind}$ coincides with the covering dimensions Δd and δd. For general spaces μX we have $\delta \operatorname{Ind} \mu X \geqq \delta d\, \mu X$. For all known examples (including the examples for $\Delta d \neq \delta d$ and, in compact spaces, Ind $\neq$ dim) $\delta \operatorname{Ind}$ coincides with δd.

The last section of the paper concerns the dimension theory of uniformisable spaces; it organizes alternative definitions and formulates problems, giving limited results on some of the problems. Covering dimension dim has been successfully generalized by Smirnov [17]; here we add to Smirnov's theory a generalization of Aleksandrov's theorem characterizing dim by separating n-tuples of pairs (A_i, B_i) of disjoint closed sets by closed sets C_i with empty intersection. The notion of min dim, mentioned in Part I [7], is formally defined: min dim X is the minimum of $\Delta d\, \mu X$ over all compatible uniformities μ. Equivalently, it is the minimum of dim Y over spaces Y containing X. The question when min dim $X = \dim X$, i.e. when X cannot be embedded in a space of lower dimension, is stressed. The Lindelöf property implies this, but the question is open for metrizable spaces and more generally for spaces admitting a complete uniformity.

It is shown that every completely metrizable space can be homeomorphically embedded as a closed set in a countable product of finite-dimensional polyhedra. Combined with results of [9] this means that every completely metrizable space is an inverse limit of polyhedra of the same or lower dimension. The question is still open whether a 1-dimensional completely metrizable space can be an inverse limit of discrete spaces.

An announcement of the results on $\delta \operatorname{Ind}$ appeared in [8].

1. Inductive dimension.

In a uniform space μX, a set U is said to *δ-separate* two sets A, B, if $X - U$ is the union of two sets A', B', respectively containing A and B, such that A' is far from B'. (That is, $X - A'$ is a uniform neighborhood of B'. Proximity notions are convenient here, and the prefix δ is meant to draw attention to the fact

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that the concept is a proximity invariant.) A set W is said to *free* A and B if W is far from $A \cup B$ and every uniform neighborhood of W which is disjoint from $A \cup B$ δ-separates A and B.

1.1. *A set W frees A and B if and only if the closure of W frees the closures of A and B.*

Since the far sets and the uniform neighborhoods are the same for the given sets as for their closures, this is obvious.

Inductive dimension $\delta \operatorname{Ind} \mu X$ of a uniform space μX is defined as follows. $\delta \operatorname{Ind} \mu X = -1$ means that X is empty. Recursively, $\delta \operatorname{Ind} \mu X \leqq n$ if every two far sets in μX are freed by some subspace μW such that $\delta \operatorname{Ind} \mu W \leqq n-1$. Then $\delta \operatorname{Ind} \mu X = n$ means that $\delta \operatorname{Ind} \mu X \leqq n$ but not $\delta \operatorname{Ind} \mu X \leqq n-1$; and $\delta \operatorname{Ind} \mu X = \infty$ means that for no n is $\delta \operatorname{Ind} \mu X \leqq n$.

The definition is framed to parallel the definition of topological dimension Ind as closely as seems reasonable, taking into account *Yu.* Smirnov's observation [17] that the reals cannot be δ-separated by a zero-dimensional set. It is interesting, but as far as I know not useful, to note the following equivalence.

A *chain* of sets from A to B is a sequence $C_1, \cdots, C_n$, such that $A \cap C_1 \neq 0$, $B \cap C_n \neq 0$, and $C_i \cap C_{i+1} \neq 0$ for $i = 1, \cdots, n-1$.

1.2. *Suppose that H is far from $A \cup B$. Then H frees A and B if and only if there are arbitrarily fine uniform coverings $\mathscr{U}$ such that every chain of elements of $\mathscr{U}$ from A to B includes an element which meets H.*

Proof. Suppose that H frees A and B, and let $\mathscr{V}$ be any uniform covering fine enough so that the $\mathscr{V}$-neighborhood U of H is disjoint from $A \cup B$. Then U δ-separates A and B into far sets A', B'. These have uniform neighborhoods A'', B'', which are still far from each other and disjoint from H. Let $\mathscr{U}$ consist of the collection of all elements of $\mathscr{V}$ which meet H, a uniform covering of A'' finer than $\mathscr{V}$, and a uniform covering of B'' finer then $\mathscr{V}$.

Conversely, if the required coverings exist, then for each uniform neighborhood U of H there is a uniform covering $\mathscr{U}$ such that the $\mathscr{U}$-neighborhood V of H is contained in U and δ-separates the set A' of all points of $X - V$ which can be joined to A by chains of elements of $\mathscr{U}$ avoiding H from the remainder $B' = X - V - A'$, which contains B. If U is disjoint from $A \cup B$, this implies that U δ-separates A and B.

1.3. *If μX is a dense subspace of μY, then $\delta \operatorname{Ind} \mu X \geqq \delta \operatorname{Ind} \mu Y$.*

Proof. Suppose that $\delta \operatorname{Ind} \mu X \leqq n$, and let A, B, be two far sets in μY. Let C and D be uniform neighborhoods of A, B, which are still far from each other. Let $E = C \cap X$, $F = D \cap X$. Let W be a subset of μX freeing E and F, with $\delta \operatorname{Ind} \mu W \leqq n - 1$. Then W frees A and B in μY. To check this it suffices to consider any closed uniform neighborhood V of W which is disjoint from $C \cup D$. Since $V \cap X$ is a uniform neighborhood of S in the space μX, $\mu X - V$ is a sum of far sets H, K, containing E and F respectively. Since $\mu Y - V$ is open, its intersection with X is dense in it; therefore the relative closures of H and K have union $Y - V$, and they are far sets containing the relative closures of E and F, which in turn contain A and B, respectively.

From 1.1 and 1.3 we see that the function $\delta \operatorname{Ind}$ would not be changed if we changed the definition to refer only to closed sets.

Note also that $\delta \operatorname{Ind} \mu X \geqq \delta \operatorname{Ind} \nu Y$ whenever μX can be mapped upon a dense subspace of νY by a δ-isomorphism; in particular, for the Samuel compactification $\beta\mu X$, $\delta \operatorname{Ind} \mu X \geqq \delta \operatorname{Ind} \beta\mu X$.

1.4. THEOREM. *For every uniform space μX, $\delta \operatorname{Ind} \mu X \geqq \delta d\, \mu X$.*

Proof. It suffices to prove this for compact spaces, in view of the last remark and the theorem $\delta d\, \beta\mu X = \delta d\, \mu X$ [6]. Here δd becomes dim (though $\delta \operatorname{Ind}$ does not become Ind). Thus we wish to show that for a compact space Y, if $\delta \operatorname{Ind} Y \leqq n$ then dim $Y \leqq n$; and we may suppose this has already been done for $n - 1$.

Let $\{U_i\}$ be any finite open covering of Y, and let $\{V_i\}$ be a strict shrinking of it (i.e. for each i, $V_i^- \subset U_i$). For each i, let W_i be an $(n-1)$-dimensional closed set freeing V_i from $Y - U_i$; that is, $\delta \operatorname{Ind} W_i \leqq n - 1$, so by the inductive hypothesis dim $W_i \leqq n - 1$ also. Then the union W of the W_i has dimension dim $W \leqq n - 1$. Let $\{P_j\}$ be an $(n-1)$-dimensional open covering of a neighborhood N of W which is finer than $\{U_i\}$. Let M be a neighborhood of W whose closure is interior to N. Now since M is a uniform neighborhood of every W_i, $Y - M$ is a union of open-closed subsets H_i containing $V_i - M$ and contained in U_i. Let $Q_1 = H_1$, and define Q_i recursively as $H_i - \bigcup_{j<i} H_j$. Then $Y - M$ is the union of the discrete collection $\{Q_i\}$, which with $\{P_j\}$ forms an open covering of dimension at most n refining $\{U_i\}$, as required.

Next we prove an analogue of the theorem of P. S. Aleksandrov (see [16]) characterizing the dimension dim of normal spaces in terms of sets separating several pairs (A_i, B_i) of disjoint closed sets. Note that it will not be a generalization of the topological theorem, since freeing is weaker than separating even for closed sets in compact metric spaces. Nevertheless the proof will be almost the same.

Given a finite family of pairs (A_i, B_i) of sets, with each A_i far from B_i, we wish to find sets C_i freeing A_i and B_i, such that not only is

$\bigcap C_i$ empty, but even the complements of the sets C_i form a uniform covering.

If such a family $\{C_i\}$ exists, we shall call the system $\{(A_i, B_i)\}$ *solvable.*

1.5. THEOREM. *For a uniform space $\mu\bar{X}$ to have dimension $\delta d\, \mu X \leqq n$, it is necessary and sufficient that every family of $n+1$ pairs of far sets in μX should be solvable.*

Proof. Suppose the pairs (A_i, B_i) for $i = 0, \cdots, n$, form an unsolvable family. Take uniformly continuous functions f_i on μX to $[0, 1]$ with $f_i = 0$ on A_i, $f_i = 1$ on B_i. These are the coordinates of a mapping F of μX into the $(n+1)$-dimensional cube Q^{n+1}, which we shall show to be an essential mapping. Indeed, for the contrary we must have a mapping G of μX into the boundary S^n of Q^{n+1} such that $G(x) = F(x)$ whenever $F(x) \in S^n$. Then let C_i be $\{x: 1/3 \leqq G(x)_i \leqq 2/3\}$; these sets free (and even δ-separate) (A_i, B_i), and their complements form a uniform covering, contradicting the assumption that $\{(A_i, B_i)\}$ is unsolvable.

Conversely, if $\delta d(\mu X) > n$ then there is an essential mapping $F: \mu X \rightarrow Q^{n+1}$. Define A_i as $\{x: F(x)_i = 0\}$, and $B_i = \{x: F(x)_i = 1\}$. Now observe that if $\{(A_i, B_i)\}$ were solvable, we could take small uniform neighborhoods of freeing sets C_i which would δ-separate (A_i, B_i) and leave us a uniform covering $\{P_0, \cdots, P_n, Q_0, \cdots, Q_n\}$, with each $A_i \subset P_i$, $B_i \subset Q_i$, and P_i far from Q_i. The nerve of this covering is dual, and naturally homeomorphic, to the polyhedron S^n consisting of the proper faces of Q^{n+1}. Thus a canonical map into this nerve yields a map $G': \mu X \rightarrow S^n$ which takes each point x in $F^{-1}(S^n)$ to a point of S^n which is not diametrally opposite to $F(x)$. Hence on $F^{-1}(S^n)$, G' and F are homotopic; and G' is homotopic to a mapping $G: \mu X \rightarrow S^n$ coinciding with F on $F^{-1}(S^n)$. The contradiction completes the proof.

REMARKS. One can similarly prove the analogue of Sklyarenko's theorem [16]: if $\delta d\, \mu X \geqq n$ then μX contains an infinite family of far pairs any n of which form an unsolvable subfamily (since Q^n contains such a family).

Also, $\delta \operatorname{Ind} \mu X = 0$ if and only if $\delta d\, \mu X = 0$. This is clear from 1.5. (There is also an easier proof which was indicated by Smirnov [17; Theorem 6].)

1.6. LEMMA. *Suppose μE is a subspace of μX, $\delta \operatorname{Ind} \mu E = 0$, and A and B are far sets in μX. Then A and B are δ-separated by some set far from E.*

Proof. Let C and D be far uniform neighborhoods of A and B

respectively. Decompose E into far sets $F \supset C \cap E$, $G \supset D \cap E$. Now F is disjoint from D, hence far from B; similarly G is far from A. Then $A \cup F$ is far from $B \cup G$. Let U and V be far uniform neighborhoods of these sets; then $X - U - V$ is the set required for the lemma.

1.7. THEOREM. *$\delta \operatorname{Ind} \mu X = 0$ if and only if $\delta d\, \mu X = 0$. $\delta \operatorname{Ind} \mu X \leqq 1$ if and only if any two far sets A_1, B_1 can be freed by a set C_1 such that any two far sets A_2, B_2 can be freed by a set C_2 far from C_1.*

The proof is trivial after the preceding remark and lemma. The theorem suggests a characterization of $\delta \operatorname{Ind}$ paralleling 1.5. I do not know if that characterization is valid. I have an example showing that 1.6 does not generalize for $\delta \operatorname{Ind} \mu E = 1$ (one cannot free A and B by a closed set H whose intersection with E is zero-dimensional), but it does not seem worth including here.

Finally, it should be noted that I do not know any example of strict inequality for either 1.3 or 1.4.

2. Metric spaces.

2.1. LEMMA. *Let M be a metric space with subspaces G and H. Then G contains a set J such that*

(1) *every subset of J which is far from H is uniformly discrete, and*

(2) *every subset of G which is far from J is far from H.*

Proof. To construct J, let U_n denote the intersection of G with the $1/n$ neighborhood of H; let J_n be a maximal set of points of U_n distant at least $1/n$ from each other; let $J = \bigcup J_n$.

2.2. THEOREM.[1] *Let M be a metric space, H a nonempty subset of M, and J a subset of M such that every subset of J which is far from H is zero-dimensional. Then $\delta \operatorname{Ind} J \leqq \delta \operatorname{Ind} H$.*

Proof. Consider the case $\delta \operatorname{Ind} H = 0$. Let A and B be far subsets of J. Let C and D be uniform neighborhoods (in M) of A and B respectively, far from each other. Then $C \cap H$ and $D \cap H$ are freed by the empty set; so H is a union of far sets $E \supset C \cap H$, $F \supset D \cap H$. Now $A \cup E$ and $B \cup F$ are far from each other; let K and L be far uniform neighborhoods of them, and let P and Q be uniform neighborhoods of K and L respectively, which are still far from each other. Now $J - K - L$ is far from H; by the hypothesis, it must be a union of far

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[1] This is stronger than the corresponding theorem announced in [8]. However, Lemma 3 of [8] asserts a similar result for arbitrary uniform spaces; I cannot prove it except for dimension zero.

sets $R \supset (J \cap P) - K$ and $S \supset (J \cap Q) - L$. Then the desired separation is achieved by $(J \cap P) \cup (R - Q)$ and $(J \cap Q) \cup (S - P)$. It is clear from the construction that the first of these sets contains A (since J and P contain A), the second contains B, and the union contains J. Also P is far from Q and R is far from S. To see that $J \cap P$ is far from $S - P$, observe that $(J \cap P) - K \subset R$, while K is far from $S - P$ since it is far from $M - P$. Similarly $J \cap Q$ is far from $R - Q$, and we have this case. Incidentally. we do not need the metric for this case.

Suppose the theorem established for $\delta \operatorname{Ind} H \leqq n - 1$, and consider next the case $\delta \operatorname{Ind} H = n$. For any far subsets A and B of J, again let C and D be far uniform neighborhoods of them, and let E and F be far uniform neighborhoods of C and D respectively. Then $E \cap H$ and $F \cap H$ are freed in H by some subset V with $\delta \operatorname{Ind} V \leqq n - 1$. Applying 2.1 to J and V, we obtain a subset K of J satisfying (1) and (2). Then $W = K - C - D$ also satisfies (1) and (2) (the first a fortiori; the second because a set far from $K - C - D$ is the union of a set far from K and a set contained in any preassigned uniform neighborhood of $C \cup D$). By construction W is far from A and B; by the inductive hypothesis $\delta \operatorname{Ind} W \leqq n - 1$. It remains to show that for any uniform neighborhood U of W disjoint from A and B, $J - U$ decomposes into two far sets respectively containing A and B. Here $J - U$ is far from V; i.e. V has a uniform neighborhood T disjoint from $J - U$. T, of course, δ-separates $E \cap H$ and $F \cap H$ in H, so that $H - T = P \cup Q$ with P far from Q, $E \cap H \subset P$, $F \cap H \subset Q$. Let R and S be far uniform neighborhoods of $P \cup A$ and $Q \cup B$. Then $R \cup S \cup T$ is a uniform neighborhood of H. Let I be a uniform neighborhood of H far from $J - R - S - T$, and split $J - I$ into far sets Y, Z, containing $(J \cap R) - I$ and $(J \cap S) - I$ respectively. One finds that $J - U$ decomposes into its intersections with $R \cup (Y - S)$ and $S \cup (Z - R)$, which are far sets containing A and B respectively. Indeed, just as before, R and S already contain A and B. Those points of $J - U$ which are not in $R \cup S$ are in $J - I$ (since they could not be in T either) and hence in Y or Z; so $R \cup S \cup (Y - S) \cup (Z - R) \supset J - U$. R is far from S, Y is far from Z. $(J - U) \cap R$ is far from $J \cap (Z - R)$; for $(J - U) \cap R \cap I \subset I - S - T$ (far from $J - R$) and $(J \cap R) - I \subset Y$ (far from Z). Likewise $(J - U) \cap S$ is far from $J \cap (Y - S)$. This completes the proof.

2.3. COROLLARY. *For any subspace J of a metric space M,* $\delta \operatorname{Ind} J \leqq \delta \operatorname{Ind} M$. *If J is dense in M then* $\delta \operatorname{Ind} J = \delta \operatorname{Ind} M$.

Proof. Put $H = M$ in 2.2.

Next we prove

2.4. THEOREM. *For any metric space M, $\delta \operatorname{Ind} M \leqq \Delta d\, M$.*

Here we may suppose M is complete; for completion does not change $\delta \operatorname{Ind}$ (by 2.3) nor Δd [6]. Recall that a complete metric space M is *supercomplete* [11]; this means that the space of closed subsets (metrized by Hausdorff distance) is complete, and may be restated as follows. A filter $\mathscr{F}$ is *stable* provided for every uniform covering $\mathscr{U}$ there is $A \in \mathscr{F}$ such that for every $B \in \mathscr{F}$, $St(B, \mathscr{U}) \supset A$. Now if $\mathscr{F}$ is stable in M, it converges to the set H of all cluster points of $\mathscr{F}$, in the sense that every uniform neighborhood of H contains a member of $\mathscr{F}$.

Recall also, from [7], that $\Delta d\, M \leqq n$ implies that every uniform covering of M is refined by some uniform covering $\mathscr{U}$ which is a union of $n+1$ uniformly discrete collections $\mathscr{U}_0, \cdots, \mathscr{U}_n$.

Proof of 2.4. We may assume that M is complete, that $\Delta d\, M = n$, and that the theorem is established for spaces of smaller dimension Δd. Then it will suffice to show that any two far sets A, B, can be freed by a set H such that $\Delta d\, H \leqq n-1$. We shall construct H as the limit of a stable filter with basis $\{S_0, S_1, \cdots\}$. Let C and D be far uniform neighborhoods of A and B, and let $S_0 = M - C - D$. Recursively, suppose S_{j-1} is a subset of S_0 which δ-separates A and B, its complement being a union of far sets C_{j-1}, D_{j-1} containing A and B respectively. Let $\mathscr{U}^j$ be a uniform covering so fine that each of its elements is either far from C_{j-1} or far from D_{j-1}, and which is the union of uniformly discrete collections $\mathscr{U}^j_i$, $0 \leqq i \leqq n$. Also, with respect to some fixed metric, each $\mathscr{U}^j$ must have mesh at most 2^{-j}. Let $\mathscr{V}^j$ be a uniform strict shrinking of $\mathscr{U}^j$; that is, its elements V_α are in a one-to-one correspondence with the elements U_α of $\mathscr{U}^j$ so that for some $t > 0$, each U_α is a t-neighborhood of V_α. Thus $\mathscr{V}^j$ is naturally expressed as a union of uniformly discrete collections $\mathscr{V}^j_i$ corresponding to the $\mathscr{U}^j_i$. Let E_j be the union of all these elements V_α of $\mathscr{V}^j_0$ such that U_α contains a point of S_{j-1} which belongs to no element of $\mathscr{V}^j_0$; let $S_j = S_{j-1} - E_j$.

Since S_j contains all of S_{j-1} except for a uniformly discrete collection of sets none of which reaches from near C_{j-1} to near D_{j-1}, S_j δ-separates A and B. Moreover, S_j has an $(n-1)$-dimensional uniform covering of mesh at most 2^{-j}. For this, note that S_j is a union of two far sets; those members of $\mathscr{V}^j_0$ which meet S_j are distant by at least t from the rest of S_j. Now on one part of S_j the trace of $\mathscr{U}^j_0$ is a 0-dimensional uniform covering; on the rest of S_j the trace of the rest of $\mathscr{U}^j$ is an $(n-1)$-dimensional uniform covering. Finally, by construction, $St(S_j, \mathscr{U}^j) \supset S_{j-1}$. Therefore the sequence $\{S_j\}$ is indeed a basis of a stable filter. Since M is supercomplete, the limit H frees A and B; and $\Delta d\, H \leqq n-1$, as was to be shown.

It is known [7] that for any uniform space μX, if $\Delta d\, \mu X$ is finite then $\delta d\, \mu X = \Delta d\, \mu X$. Combining this with 1.4 and 2.4, we have

2.5. COROLLARY. *If M is a metric space and $\Delta d\, M < \infty$, then $\Delta d\, M = \delta d\, M = \delta\, \mathrm{Ind}\, M$.*

Examples are known [7] of uniform spaces for which Δd is infinite but $\delta\, \mathrm{Ind}$ is finite and equal to δd. No metric example is known, and it seems possible that the three dimension functions coincide for all metric spaces. We do have the following.

2.6. *For a metric space M, if $\delta d\, M = 0$ then $\Delta d\, M = 0$.*

Proof. Fix a metric. From $\delta d\, M = 0$ it follows that for every positive ε there is a positive δ such that any two points distant by ε are separated by some decomposition of M into two sets at distance δ. Assuming the contrary, we should have a sequence of pairs (x_n, y_n) distant by ε such that no infinite subsequence could be simultaneously separated by such a decomposition. If some infinite set of x's or y's has diameter $< \varepsilon/2$, we have a contradiction; otherwise there is an infinite set of indices n for which the x_n and y_n form a uniformly discrete set, and we have another contradiction. But then a routine argument shows that every covering Lebesgue number ε is refined by a 0-dimensional covering having Lebesgue number δ.

3. Dimension of uniformisable spaces.

I believe that the only serious investigation of the dimension theory of nonnormal spaces so far has been the concluding section of Smirnov's paper [17]. There the dimension function dim is defined, as the covering dimension with respect to the family of all finite normal coverings, and the decidedly imperfect analogy with δd is worked out. Of course $\dim X = \delta d\, aX$, where a is the fine uniformity on X. Dowker has given a proof [2] that $\delta d\, aX = \Delta d\, aX$ if X is normal (not using this notation); and I pointed out [7] that the same proof[2] shows that $\delta d\, \mu X = \Delta d\, \mu X$ whenever μ is fine or even locally fine.

The dimension function ind is of course familiar for more general spaces; and it is customary to call a uniformisable space X *zero-dimensional* if $\mathrm{ind}\, X = 0$. It is known [3, 6] that $\mathrm{ind}\, X = 0$ does not imply $\dim X = 0$ (even for normal X); but if $\mathrm{ind}\, X = 0$ then X has a zero-dimensional compactification and with it a zero-dimensional uniformity. Defining $\min \dim X$ as the minimum value of $\Delta d\, \mu X$ over all compatible uniformities μ, we may summarize as follows:

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[2] In presenting this proof in a coures of lectures I found it necessary to rearrange it to fill in what seems to be a gap in the reasoning (page 212, line 19 of [**2**]); but the rearrangement, if it is necessary, is not necessitated by the generalization.

3.1. For any uniformisable space X, min dim $X \leqq$ dim X. Examples of strict inequality are known among normal spaces, but not among completely uniformisable spaces. If ind $X = 0$ then min dim $X = 0$, and conversely; however, ind X may exceed dim X, even for compact X [13].

Let us introduce two more dimension functions: a Ind $X = \delta$ Ind aX, and Ind X, defined as follows. As usual, Ind $X = -1 \leftrightarrow X$ is empty. Ind $X \leqq n$ if every two completely separated subsets of X are topologically separated by some subset H such that Ind $H \leqq n-1$; and finally, Ind $X = n$ means Ind $X \leqq n$ but not Ind $X \leqq n-1$. With these we have

3.2. *For any uniformisable space* X, Ind $X \geqq a$ Ind $X \geqq$ dim $X \geqq$ min dim X. *Inequality may occur anywhere in this chain except perhaps between* a Ind *and* dim.

For the proof, a Ind $\geqq$ dim follows from 1.4. To see that Ind $\geqq a$ Ind it suffices to observe that in a fine space a set which separates two closed sets also frees them. For the examples of Lokucievski [12], Lunc [13], and Mardšić [14] having Ind $X >$ dim X, a Ind X coincides with the smaller number dim X.

Note that min dim X could also be defined as the minimum of dim Y over all spaces Y topologically containing X (since $\Delta d\, \mu X \geqq \dim \beta\mu X$). Of course min dim is monotonic, for arbitrary subspaces. Smirnov has shown [17] that dim is not monotonic for closed subspaces; and as it happens, the same example shows that Ind and a Ind are not monotonic for closed subspaces. Both dim and Ind are monotonic for C^*-embedded [4] normal subspaces. (For dim, [17]; for Ind, an easy exercise.) For a Ind this is an open problem.

The problem is open whether dim is monotonic for topologically complete subspaces, or in other words whether dim $X =$ min dim X when X admits a complete uniformity. We have[3]

3.3. *For Lindelöf spaces* X, dim $X =$ min dim X.

Proof. Suppose X is embedded in Y and dim $Y = n$. Then X is embedded in βY and dim $\beta Y = n$. For any finite open covering $\{U_i\}$ of X, there are open sets V_i of βX such that $V_i \cap X = U_i$. Since each point of X has a neighborhood in βY whose closure is contained in some V_i, and X is Lindelöf, there is a σ-compact set Z containing X and covered by the V_i. Since dim is monotonic for closed sets in compact

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[3] Aleksandrov in [**1**; p. 40] credits Morita with (essentially) a stronger result than this: if $X \subset Y$ and both X and Y have the star-finite property then dim $X \leqq$ dim Y. One can prove this, without the restriction on Y, by modifying the proof of 3.3 here.

Added in proof. Professor Morita has shown me his proof, which is very direct from his published results.

spaces and satisfies the countable sum theorem for closed sets in normal spaces, dim $Z \leqq n$. Then $\{V_i \cap Z\}$ is refined by an n-dimensional open covering of Z; so $\{U_i\}$ is refined by an n-dimensional open covering of X.

Perhaps one could prove that dim is monotonic for closed subspaces of topologically complete spaces. A stronger proposition (in view of [10; 7.2]) would be that dim is lower semi-continuous on inverse limits. As noted in the introduction, it is unknown whether an inverse limit X of discrete spaces can have dim $X > 0$, even if X is completely metrizable (even if the discrete spaces are countable).

From 1.5, which is not a generalization of the corresponding theorem of Aleksandrov, we easily get a generalization of that theorem; for note that in the proof we constructed sets C_i which δ-separate (hence separate) the pairs (A_i, B_i).

3.4. *Aleksandrov's Theorem. A uniformisable space* X *has* dim $X \leqq n$ *if and only if any* $n+1$ *pairs of completely separated sets* (A_i, B_i) *can be separated by sets* C_i *whose complements form a normal covering.*

Similar remarks apply to Sklyarenko's refinement of the theorem; but this result is actually stronger when stated in terms of freeing.

For Ind there is a valid analogue of 1.7, and at least for a moderately extensive class of spaces the characterization generalizes to higher dimensions.

3.5. *For any uniformisable space* X, Ind $X = 0$ *if and only if* dim $X = 0$. *For normal spaces* X, Ind $X \leqq 1$ *if and only if any two disjoint closed sets* A_1, B_1 *can be separated by a closed set* C_1 *such that any two disjoint closed sets* A_2, B_2 *can be separated by a closed set* C_2 *disjoint from* C_1. *For completely normal spaces* X, Ind $X \leqq n$ *is equivalent to following: if any* $n+1$ *pairs of disjoint closed sets* (A_i, B_i) *are successively presented, one can successively determine closed sets* C_i *separating* A_i *and* B_i, *each without knowledge of the later pairs* (A_j, B_j) *for* $j > i$, *such that* $\bigcap C_i = 0$.

Proof. Again the zero-dimensional case follows from Aleksandrov's theorem (here, from 3.4). The 1-dimensional case goes just like 1.6; since X is normal, the disjoint closed sets F, G of the construction can be separated. In the n-dimensional case the subspace $H_i = \bigcap_{j<i} C_i$(Ind $H_i \leqq n - i$) splits into relatively open sets F_i, G_i, separated by H_{i+1}; since X is completely normal, $A_{i+1} \cup F_i$ and $B_{i+1} \cup G_i$ can be separated.

I do not know a uniformisable space failing to satisfy all of 3.5.

Let us conclude with the theorem

3.6. THEOREM. *Every complete metric space is homeomorphic with a closed subset of a countable product of finite-dimensional uniform complexes.*

Note that if "countable" is deleted, the remaining result is known; in fact, "metric" can then be deleted [10]. But countability makes it possible to represent the given X as an inverse limit of complexes K_i with all the coordinate projections π_i: $X \rightarrow K_i$ irreducible [9], and this means, with $\dim K_i \leqq \dim X$ for all X.

3.7. COROLLARY. *Every complete metric space X is homeomorphic with the limit of an inverse mapping system of uniform complexes of dimension at most* $\dim X$.

Proof of 3.6. We are given the space X and a complete metric uniformity, hence a normal sequence of coverings $\mathscr{U}^i$ which do two things:

(a) for any point x and neighborhood U, there is i such that $St(x, \mathscr{U}^i) \subset U$, and

(b) for every nonconvergent filter $\mathscr{F}$ there is i such that $\mathscr{F}$ contains no element of $\mathscr{U}^i$. We need a normal sequence of finite-dimensional coverings $\mathscr{W}^i$ which also does these things. It will suffice to find finite-dimensional open coverings $\mathscr{V}^i$ satisfying (a) and (b); then the $\mathscr{W}^i$ can be constructed by finite intersection and star-refinement. (Every finite-dimensional normal covering has a finite-dimensional normal star-refinement; see e.g. [5].)

We may assume that each $\mathscr{U}^i$ is a countable union of topologically discrete collections $\mathscr{U}^i_j$ [18]. Let A_{ij} denote the union of the elements of $\mathscr{U}^i_j$ and let $\mathscr{A}^i = \{A_{ij}\text{: all } j\}$. Now each countable open covering $\mathscr{A}^i$ has a countable star-finite open refinement $\mathscr{B}^i$ [15]; and we may suppose that $\mathscr{B}^i$ is a star-refinement of $\mathscr{A}^i$ (e.g. by [5; 1.2]). Decompose each $\mathscr{B}^i$ into its "components" $\mathscr{B}^i_j$; precisely, let $\{B_{ij}\}$ be the finest 0-dimensional covering coarser than $\mathscr{B}^i$, and $\mathscr{B}^i_j$ the trace of $\mathscr{B}^i$ on B_{ij}. For each i and j, select an element C^{ij}_1 of $\mathscr{B}^i_j$; define $\mathscr{C}^{ij}_1$ as the unit class $\{C^{ij}_1\}$. Let C^{ij}_0, $\mathscr{C}^{ij}_0$ be empty. Recursively define $\mathscr{C}^{ij}_{k+1}$ as the set of all members of $\mathscr{B}^i_j$ which meet C^{ij}_k but do not meet C^{ij}_{k-1}, and C^{ij}_{k+1} as the union of $\mathscr{C}^{ij}_{k+1}$. Let Γ^i be the 1-dimensional covering consisting of all C^{ij}_k. Let $\mathscr{C}^i$ be a strict shrinking of Γ^i, i.e. a similarly indexed covering $\{C^*_{ijk}\}$ with the closure of each C^*_{ijk} contained in C^{ij}_k. For each i, j, k, let $\mathscr{D}^{ijk}$ be the finite covering consisting of the elements B_{ijkl} of $\mathscr{C}^{ij}_k$ and the set $X - C^*_{ijk}$. Next, for each i and j, there are neighborhoods E^{ij}_α of the closures of the members U^{ij}_α of $\mathscr{U}^i_j$ which still form a discrete collection; let $\mathscr{E}^{ij}$ be the 1-dimensional covering consisting of all E^{ij}_α and the set $X - A_{ij}$.

Then the family of all intersections $\mathscr{C}^i \wedge \mathscr{D}^{ijk} \wedge \mathscr{E}^{il}$ satisfies (a) and (b). For (a), consider any point x and covering $\mathscr{U}^i$. x lies in some C^*_{ijk} and in some B_{ijkm}; and $St(B_{ijkm}, \mathscr{B}^i)$ is contained in some A_{il}. Then we need only $\mathscr{D}^{ijk} \wedge \mathscr{E}^{il}$; any member of this covering containing x must have the form $B_{ijkn} \cap E^{il}_\alpha$. Here B_{ijkn} meets B_{ijkm}, so is contained in A_{il}, and $B_{ijkn} \cap E^{il}_\alpha \subset A_{il} \cap E^{il}_\alpha = U^{il}_\alpha$. For (b), if the filter $\mathscr{F}$ meets every $\mathscr{C}^i$, $\mathscr{D}^{ijk}$, $\mathscr{E}^{il}$ then it contains some C^*_{ijk}, some B_{ijkm}, a fortiori some A_{il}, and finally some U^{il}_α for each i.

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