

# Pacific Journal of Mathematics

**A NOTE ON AUTOMORPHISMS OF LIE ALGEBRAS**

N. JACOBSON

# A NOTE ON AUTOMORPHISMS OF LIE ALGEBRAS

N. JACOBSON

In a beautiful paper which appeared in 1939 ([4]), F. Gantmacher made a thorough study of automorphisms of semi-simple Lie algebras over the field of complex numbers. Among other things, he defined the index  $n(G_i)$  of a connected component  $G_i$  of the automorphism group  $G = G(\mathfrak{L})$  as the minimum multiplicity of the characteristic root 1 for elements of  $G_i$ . The main purpose of this note is the determination of these indices. It is somewhat surprising that this does not appear in Gantmacher's paper since all the methods for deriving the formula for index  $G_i$  are available in his paper. The secondary purpose of this note is to extend Gantmacher's theory to the case of Lie algebras over arbitrary algebraically closed base fields of characteristic 0. This can be done by using algebraic group concepts and techniques which are by now well known. Nevertheless, it seems worthwhile to carry out the program in detail since Gantmacher's results give a real insight into the action of an automorphism in a semi-simple Lie algebra. For example, as we indicate, they can be used to give a new derivation and sharpening of theorems on fixed points which are due to Borel and Mostow ([1]).

**1. Generalities on automorphisms.** Let  $\mathfrak{L}$  be a finite dimensional Lie algebra over an algebraically closed field  $\phi$  of characteristic 0,  $G$  the group of automorphisms of  $\mathfrak{L}$ .  $G$  is an algebraic group and it has a decomposition  $G = G_0 \cup G_1 \cup \cdots \cup G_{r-1}$  where  $G_i$  are the algebraic components and  $G_0$  is the component of the identity element 1. This is an invariant subgroup of finite index  $r$  in  $G$  and is irreducible, which means that the intersection of any two nonvacuous open subsets in  $G_0$  is nonvacuous. The topology referred to here is the Zariski topology in which open sets are the complements of algebraic subsets of  $G$ .

If  $\eta$  is a nonsingular linear transformation in  $\mathfrak{L}$ ,  $\eta$  has a unique decomposition as  $\eta = \eta_s \eta_u = \eta_u \eta_s$  where  $\eta_s$  and  $\eta_u$  are polynomials in  $\eta$ ,  $\eta_s$  is semi-simple and  $\eta_u$  is unipotent, that is,  $\eta_u - 1$  is nilpotent. If  $\eta$  is an automorphism then  $\eta_u$  and  $\eta_s$  are automorphisms. If  $\eta$  is a unipotent automorphism then

$$\log \eta = (\eta - 1) - \frac{1}{2}(\eta - 1)^2 + \frac{1}{3}(\eta - 1)^3 - \cdots$$

---

Received February 27, 1961. This research was supported in part by the United States Air Force through the Air Force Office of Scientific Research and Development Command under Contract No. AF 49 (C38) 515. Reproduction in whole or in part is permitted for any purpose of the United States Government.

is a nilpotent derivation. Conversely, if  $D$  is a nilpotent derivation then  $\eta = \exp D$  is a unipotent automorphism. Any unipotent automorphism is contained in the algebraic component of 1. Hence an automorphism  $\eta$  and its semi-simple part  $\eta_s$  are contained in the same component.

If  $\eta$  is a linear transformation in  $\mathfrak{L}$  we can decompose  $\mathfrak{L} = \mathfrak{L}_\alpha \oplus \mathfrak{L}_\beta \oplus \cdots \oplus \mathfrak{L}_\rho$  where  $\mathfrak{L}_\alpha$  is the characteristic space of  $\eta$  corresponding to the characteristic root  $\alpha$ :  $\mathfrak{L}_\alpha = \{x_\alpha \mid x_\alpha(\eta - \alpha 1)^k = 0 \text{ for some } k\}$ . The semi-simple part  $\eta_s$  is the linear transformation which leaves each  $\mathfrak{L}_\alpha$  invariant and coincides with the scalar multiplication  $\alpha 1$  in  $\mathfrak{L}_\alpha$ . If  $\eta$  is an automorphism then  $[\mathfrak{L}_\alpha \mathfrak{L}_\beta] = 0$  if  $\alpha\beta$  is not a characteristic root and  $[\mathfrak{L}_\alpha \mathfrak{L}_\beta] \subseteq \mathfrak{L}_{\alpha\beta}$  if  $\alpha\beta$  is a characteristic root. Here  $[\mathfrak{L}_\alpha \mathfrak{L}_\beta]$  is the subspace spanned by the Lie products  $[x_\alpha x_\beta]$ ,  $x_\alpha \in \mathfrak{L}_\alpha$ ,  $x_\beta \in \mathfrak{L}_\beta$ . In particular,  $\mathfrak{L}_1$  the space of the characteristic root 1 is a subalgebra and  $[\mathfrak{L}_\alpha \mathfrak{L}_1] \subseteq \mathfrak{L}_\alpha$ . In most considerations the refined decomposition  $\mathfrak{L} = \Sigma \mathfrak{L}_\alpha$  will be replaced by a coarser Fitting type decomposition:  $\mathfrak{L} = \mathfrak{L}_1 \oplus \mathfrak{N}$  where  $\mathfrak{N} = \sum_{\alpha \neq 1} \mathfrak{L}_\alpha$ . These two spaces are invariant under  $\eta$ ,  $\eta$  is unipotent in  $\mathfrak{L}_1$  and  $\eta - 1$  is nonsingular in  $\mathfrak{N}$ . We have  $[\mathfrak{N} \mathfrak{L}_1] \subseteq \mathfrak{N}$ . If  $\eta$  is semi-simple then  $\mathfrak{L}_1$  is the set of fixed points under  $\eta$  and  $\mathfrak{N} = \mathfrak{L}^{\eta-1} = \{x^\eta - x \mid x \in \mathfrak{L}\}$ .

As before, let  $G = G_0 \cup G_1 \cup \cdots \cup G_{r-1}$  be the decomposition of  $G$  into its algebraic irreducible components. We define the *index*  $n(G_i)$  of  $G_i$  as the minimum multiplicity of the characteristic root 1 for the  $\eta \in G_i$ . The multiplicity of the root is the same as the dimensionality of the corresponding characteristic space. An element  $\eta \in G_i$  is called *regular* if  $\dim \mathfrak{L}_1(\eta) = \text{index } G_i$ . Let  $(u_1, \dots, u_n)$  be a basis for  $\mathfrak{L}$  and let  $(\alpha)$  be a matrix of  $\eta$  relative to this basis. Write the characteristic polynomial of  $(\alpha) - 1$  as

$$(1) \quad f_\eta(\lambda) = \lambda^n - \rho_1(\eta)\lambda^{n-1} + \cdots + (-1)^n \rho_n(\eta).$$

The mappings  $\eta \rightarrow \rho_j(\eta)$  are polynomial functions. If the index of  $G_i = l_i$ , then  $\rho_n(\eta) = \cdots = \rho_{n-l_i+1}(\eta) = 0$  for all  $\eta \in G_i$  but  $\rho_{n-l_i}(\eta) \neq 0$  on  $G_i$ . The regular elements of  $G_i$  are those such that  $\rho_{n-l_i}(\eta) \neq 0$ . Hence they form an open set in  $G_i$ .

**2. Lie Algebras of algebraic groups.** We need to recall some notions on linear algebraic groups. The results we shall quote can be found in two books by Chevalley ([3]). We recall first that if  $V$  and  $W$  are finite dimensional vector spaces, a rational mapping  $R$  of  $V$  into  $W$  is a mapping of the form  $x = \Sigma \xi_i e_i \rightarrow y = \Sigma \eta_j f_j$  where  $\eta_j = R_j(\xi) \equiv R_j(\xi_1, \dots, \xi_m)$  are rational functions of the  $\xi$ 's. Here  $(e_1, \dots, e_m)$  is a basis for  $V$  and  $(f_1, \dots, f_n)$  is a basis for  $W$ . The rational mapping is defined on an open subset of  $V$ . An important special case is that in which  $W = \Phi$ . Then  $R$  is a rational function on  $V$ . In the general case, if  $a$  is a point at which  $R$  is defined, the differential  $D_a$  of  $R$  at  $a$  is the linear map-

ping  $x \rightarrow (D_a R)(x)$  where

$$(D_a R)(x) = \sum_1^n \left( \sum_1^m \left( \frac{\partial R_j}{\partial \lambda_k} \right)_{\lambda_k = \alpha_k} \xi_k \right) f_j$$

for  $a = \sum \alpha_i e_i$ . Here  $\partial R_j / \partial \lambda_k$  is the formal partial derivative of the rational expression  $R_j(\lambda_1, \dots, \lambda_m)$  in indeterminates  $\lambda$  with respect to the indeterminate  $\lambda_k$ . Let  $E$  be an irreducible set in  $V$ , that is, the ideal  $\mathfrak{A}$  of polynomials  $P(\lambda_1, \dots, \lambda_m)$  which are 0 at every point of  $E$  is prime. Let  $a$  be a point of  $E$ . We define the tangent space to  $E$  at  $a$  to be the subspace of vectors  $x$  in  $V$  such that  $(D_a P)(x) = 0$  for every  $P$  in the ideal  $\mathfrak{A}$  corresponding to  $E$ . The condition on  $x = \sum \xi_k e_k$  are:  $\sum_k (\partial P / \partial \lambda_k)_{\lambda_k = \alpha_k} \xi_k = 0$  for all  $P \in \mathfrak{A}$ . It suffices to have these conditions satisfied for a set of generators  $(P_1, \dots, P_q)$  of  $\mathfrak{A}$ . If  $R$  is a rational mapping defined on  $E$  then  $R(E)$  is irreducible and  $D_a R$  maps the tangent space to  $E$  at  $a \in E$  into a subspace of the tangent space at  $R(a)$  of  $R(E)$ .

If  $G$  is an irreducible algebraic linear group the tangent space  $\mathfrak{L}$  at 1 for  $G$  can be made a Lie algebra by identifying its elements with derivations in the algebra of polynomial functions on  $G$ . The dimensionality of  $\mathfrak{L}$  is the same as that of  $G$ . If  $\mathfrak{L}$  is any Lie algebra and  $G_0$  is the component of 1 of the group of automorphisms then the Lie algebra of  $G_0$  is the Lie algebra of derivations of  $\mathfrak{L}$ . If  $\mathfrak{L}$  is semi-simple then all the derivations are inner and the Lie algebra of  $G_0$  is  $ad \mathfrak{L}$  the set of adjoint mappings  $ad a: x \rightarrow [xa]$  in  $\mathfrak{L}$ .

**3. Regular automorphisms of semi-simple Lie algebras.** We shall now derive the purely algebraic form of Gantmacher's results on automorphisms of semi-simple Lie algebras.

Let  $\mathfrak{L}$  be semi-simple and let  $\eta$  be an automorphism of  $\mathfrak{L}$ ,  $\mathfrak{L} = \mathfrak{L}_1 \oplus \mathfrak{N}$  the Fitting decomposition of  $\mathfrak{L}$  relative to  $\eta$ . Thus  $\mathfrak{L}_1$  is the space of the characteristic root 1,  $\mathfrak{N}$  the sum of the other characteristic spaces,  $\mathfrak{L}_1$  is a subalgebra and  $[\mathfrak{N}, \mathfrak{L}_1] \subseteq \mathfrak{N}$ . We note first

**THEOREM 1.**  $\mathfrak{L}_1$  is a reductive subalgebra of  $\mathfrak{L}$ .

*Proof.* The assertion is that the subalgebra  $ad_{\mathfrak{L}} \mathfrak{L}_1$  of the Lie algebra  $ad_{\mathfrak{L}} \mathfrak{L} = ad \mathfrak{L}$  is completely reducible. This property holds for  $ad \mathfrak{L}$  since  $\mathfrak{L}$  is semi-simple. If  $\eta_s$  is the semi-simple part of  $\eta$  then  $\mathfrak{L}_1 = \{l_1 | l_1^s = l_1\}$ . This is equivalent to  $[\eta_s, ad l_1] = 0$  and this implies that  $ad_{\mathfrak{L}} \mathfrak{L}_1$  is a split-table Lie algebra of linear transformations in the sense of Malcev. Also we have  $ad_{\mathfrak{L}} \mathfrak{L} = ad_{\mathfrak{L}} \mathfrak{L}_1 \oplus ad_{\mathfrak{L}} \mathfrak{N}$  and  $[ad_{\mathfrak{L}} \mathfrak{N}, ad_{\mathfrak{L}} \mathfrak{L}_1] \subseteq ad_{\mathfrak{L}} \mathfrak{N}$ . These two properties imply that  $ad_{\mathfrak{L}} \mathfrak{L}_1$  is completely reducible ([6] p. 109).

Since  $\mathfrak{L}_1$  is reductive we have  $\mathfrak{L}_1 = \mathfrak{L}'_1 \oplus \mathfrak{C}$  where  $\mathfrak{L}'_1$  is the derived

algebra of  $\mathfrak{L}_1$  and is semi-simple;  $\mathfrak{C}$  is the center of  $\mathfrak{L}_1$  and  $ad_{\mathfrak{L}_1}c$  is semi-simple for every  $c$  in  $\mathfrak{C}$  ([6] p. 106).

**THEOREM 2.** *If  $\eta$  is regular  $\mathfrak{L}_1$  is abelian and  $\eta$  is semi-simple.*

*Proof.* Suppose  $\mathfrak{L}_1$  is not abelian so that  $\mathfrak{L}'_1$  is a nonzero semi-simple Lie algebra. There exist elements  $z$  and  $w$  in  $\mathfrak{L}'_1$  such that  $ad_{\mathfrak{L}'_1}z$  and  $ad_{\mathfrak{L}'_1}w$  are nilpotent derivations and  $(\exp ad_{\mathfrak{L}'_1}z)(\exp ad_{\mathfrak{L}'_1}w)$  is an automorphism of  $\mathfrak{L}'_1$  which is not unipotent.<sup>1</sup> Since  $ad_{\mathfrak{L}'_1}z$  and  $ad_{\mathfrak{L}'_1}w$  are nilpotent it is known that  $adz \equiv ad_{\mathfrak{L}_1}z$  and  $adw$  are nilpotent. If  $\alpha, \beta \in \mathcal{O}$  we can form the automorphism  $(\exp \alpha ad z)(\exp \beta ad w)$  which is in the component  $G_0$  of 1 of the automorphism group of  $\mathfrak{L}$ . Since  $[\mathfrak{L}_1\mathfrak{L}_1] \subseteq \mathfrak{L}_1$  and  $[\mathfrak{N}\mathfrak{L}_1] \subseteq \mathfrak{N}$  it follows that the automorphism

$$\zeta(\alpha, \beta) = \eta_s(\exp \alpha ad z)(\exp \beta ad w)$$

satisfies  $\mathfrak{L}_1^{\zeta(\alpha, \beta)} \subseteq \mathfrak{L}_1$ ,  $\mathfrak{N}^{\zeta(\alpha, \beta)} \subseteq \mathfrak{N}$ . Since  $\eta_s = 1$  in  $\mathfrak{L}_1$  the restriction of  $\zeta(1, 1)$  to  $\mathfrak{L}_1$  is not unipotent. Also the restriction of  $\zeta(0, 0) - 1 = \eta_s - 1$  to  $\mathfrak{N}$  is nonsingular. It follows by a standard argument that  $\alpha, \beta$  can be chosen so that the multiplicity of the characteristic root 1 for  $\zeta(\alpha, \beta)$  is less than  $\dim \mathfrak{L}_1$ . Since  $\zeta(\alpha, \beta)$  is in the same component as  $\eta$  this means that  $\eta$  is not regular contrary to assumption. Hence  $\mathfrak{L}_1$  is abelian. Then every  $ad_{\mathfrak{L}_1}l_1$ ,  $l_1 \in \mathfrak{L}_1$  is semi-simple. On the other hand,  $\log \eta_u$  is a nilpotent derivation in  $\mathfrak{L}$ . Since the derivations of  $\mathfrak{L}$  are all inner,  $\log \eta_u = adz$ ,  $z \in \mathfrak{L}$ . Since  $[\eta, \log \eta_u] = 0$ ,  $z \in \mathfrak{L}_1$ . Hence  $adz$  is nilpotent and semi-simple. Then  $adz = 0$  which implies that  $\eta_u = 1$  and  $\eta = \eta_s$  is semi-simple.

We wish to prove the converse of Theorem 2. For this we shall need the following

**LEMMA.** *Let  $\eta$  be an automorphism of a semi-simple Lie algebra  $\mathfrak{L}$  such that the subalgebra  $\mathfrak{L}_1$  is abelian. Let  $H$  be the subgroup of the automorphisms group  $G$  of  $\mathfrak{L}$  of elements commuting with  $\eta$ ,  $H_0$  the algebraic component of 1 of  $H$ ,  $G_0$  the algebraic component of 1 of  $G$ . Then*

$$(2) \quad K = \{\tau\eta\zeta\tau^{-1} \mid \tau \in G_0, \zeta \in H_0\}$$

*contains a nonvacuous open subset of  $\eta G_0$ .*

*Proof.* We note first that the proof of Theorem 2 shows that every  $\zeta \in H_0$  is semi-simple; hence every  $\eta' = \eta\zeta = \zeta\eta$  is semi-simple. The Lie algebra of  $G_0$  is  $ad \mathfrak{L}$  and the Lie algebra of  $H_0$  is  $ad_{\mathfrak{L}_1}\mathfrak{L}_1$  since  $\mathfrak{L}_1$  is the

<sup>1</sup> For example, we can take  $z, w$  to satisfy  $[zw] = h \neq 0$ ,  $[zh] = 2z$ ,  $[wh] = -2w$ . See the proof of the lemma to Th. 5.

set of fixed points under  $\gamma$ . If  $\xi, \zeta \in H_0$  then the mapping  $\xi \rightarrow \xi^{-1}\xi\zeta$  is the identity; hence the induced mapping  $x \rightarrow \zeta^{-1} \times \zeta$  is the identity in  $ad_{\mathfrak{L}}\mathfrak{L}_1$ . Since  $\zeta^{-1}(adl_1)\zeta = adl_1^\zeta$  it follows that  $l_1^\zeta = l_1$ . This implies that the space of fixed points  $\mathfrak{L}_1(\gamma') \supseteq \mathfrak{L}_1(\gamma)$  for any  $\gamma' = \eta\zeta$ . We note next that  $K$  is the orbit of  $\gamma H_0$  under the group of mappings  $\xi \rightarrow \tau\xi\tau^{-1}$ ,  $\tau \in G_0$ . This implies that  $K$  is *épais* in the sense of Chevalley:  $K$  is irreducible and  $K$  contains a nonvacuous open subset of its closure (see Chevalley [3] tome III, p. 193). The result we wish to prove will now follow by showing that  $K$  is dense in  $\gamma G_0$ . (This is all which will be needed for the proof of Theorem 3.) Let  $\gamma' = \eta\zeta$  be any element of  $\eta H_0$ . Then the tangent space  $\mathfrak{T}(K; \gamma')$  of  $K$  at  $\gamma'$  contains the tangent space  $\mathfrak{T}(\gamma H_0; \gamma')$  of  $\gamma H_0$  at  $\gamma'$  as well as the image of  $\gamma'$  under the Lie algebra of mappings  $x \rightarrow [x, adl]$ ,  $l \in L$  (Chevalley, loc. cit. p. 192). This is the set of mappings  $[\gamma', adl]$ ,  $l \in \mathfrak{L}$ . Now  $\mathfrak{T}(H_0; 1)$  is the Lie algebra  $ad_{\mathfrak{L}}\mathfrak{L}_1$  so  $\mathfrak{T}(H_0, \gamma') = \mathfrak{T}(\gamma' H_0; \gamma') = \gamma' ad_{\mathfrak{L}}\mathfrak{L}_1$ . We wish to show that  $\mathfrak{T}(K; \gamma') = \mathfrak{T}(\gamma G_0; \gamma') = \gamma' ad_{\mathfrak{L}}\mathfrak{L}$ . Since  $\mathfrak{T}(K, \gamma')$  contains  $\gamma' ad_{\mathfrak{L}}\mathfrak{L}_1$  and  $[\gamma', ad\mathfrak{L}] = \{[\gamma', adl], l \in \mathfrak{L}\}$ , it is enough to show that  $ad\mathfrak{L} = ad_{\mathfrak{L}}\mathfrak{L}_1 + (\gamma')^{-1}[\gamma', ad\mathfrak{L}]$ . Since  $(\gamma')^{-1}[\gamma', adl] = adl - adl^{\gamma'} = ad(l - l^{\gamma'})$ , it suffices to show that  $\mathfrak{L} = \mathfrak{L}_1 + \mathfrak{L}^{1-\gamma'}$ . But this is clear since  $\gamma'$  is semi-simple. We have therefore proved that the tangent spaces to  $K$  and to  $\gamma G_0$  at any point  $\gamma' = \eta\zeta$  coincide. Since  $K$  is the orbit of  $\gamma H_0$  under the set of mappings  $\xi \rightarrow \tau\xi\tau^{-1}$  it follows that the tangent space to  $K$  and to  $\gamma G_0$  at any point of  $K$  coincide. This implies that  $K$  is dense in  $\gamma G_0$ .

**THEOREM 3.** *If  $\mathfrak{L}_1$  is abelian then  $\eta$  is regular.*

*Proof.* Let  $K$  be the set defined by (2). Then the lemma implies that  $K$  contains a regular element  $\tau\eta\zeta\tau^{-1}$ ,  $\zeta \in H_0$ ,  $\tau \in G_0$ . Then  $\eta\zeta$  is a regular element contained in the component  $G_i = \gamma G_0$  containing  $\eta$ . The foregoing proof shows also that  $\mathfrak{L}_1(\eta\zeta) \supseteq \mathfrak{L}_1(\eta)$ . Since  $\eta\zeta$  is regular we have  $\mathfrak{L}_1(\eta\zeta) = \mathfrak{L}_1(\eta)$  so  $\eta$  is regular also.

Let  $\eta$  be a regular element and, as before, let  $H_0(\eta)$  be the algebraic component of 1 in the subgroup  $H(\eta)$  of  $G$  of elements commuting with  $\eta$ . Then  $H_0(\eta)\eta = \eta H_0(\eta)$  is the component of  $\eta$  in  $H(\eta)$ . As we have seen in the preceding proof, the Lie algebra  $H_0(\eta)$  is  $ad_{\mathfrak{L}}\mathfrak{L}_1(\eta)$  where  $\mathfrak{L}_1(\eta)$  is the set of fixed points under  $\eta$ . Moreover,  $\mathfrak{L}_1(\eta)$  and  $H_0(\eta)$  are abelian. The fact that  $H_0(\eta)$  is abelian implies that  $H_0(\eta) \subseteq H_0(\eta^\zeta)$  for any  $\zeta \in H_0(\eta)$ . If  $\eta\zeta$  is regular also then we have  $H_0(\eta\zeta) = H_0(\eta)$ . In a similar manner, we see that the elements of  $\mathfrak{L}_1(\eta)$  are fixed under  $\eta\zeta$ . If  $\eta\zeta$  is regular also then  $\mathfrak{L}_1(\eta) = \mathfrak{L}_1(\eta\zeta)$ ,  $\zeta \in H_0(\eta)$ . We note also that the argument used in the proof of Theorem 2 shows that every element of  $H_0(\eta)$  and hence of  $H_0(\eta)\eta$  is semi-simple.

We now use the full force of the lemma to prove the following

**THEOREM 4.** *Let  $\eta_1$  and  $\eta_2$  be two regular automorphisms contained in the same component of the automorphism group. Let  $\mathfrak{L}_1(\eta_j)$  be the subalgebra of  $\mathfrak{L}$  of fixed elements under  $\eta_j$ ,  $j = 1, 2$ , and let  $H_0(\eta_j)$  be the algebraic component of 1 in the subgroup  $H(\eta_j)$  of automorphisms commuting with  $\eta_j$ . Then there exists a  $\tau \in G_0$  such that  $\mathfrak{L}_1(\eta_1)^\tau = \mathfrak{L}_1(\eta_2)$  and  $\eta_2 H_0(\eta_2) = \tau^{-1}(\eta_1 H_0(\eta_1))\tau$ .*

*Proof.* Let  $K_1$  and  $K_2$  be the sets (2) defined by  $\eta_1$  and  $\eta_2$  respectively. Then  $K_1$  and  $K_2$  contain open subsets of the component  $G_i$  containing  $\eta_1$  and  $\eta_2$ . It follows that there exists a regular element in  $K_1 \cap K_2$ . Hence there exist  $\tau_j \in G_0$ ,  $\zeta_j \in H_0(\eta_j)$  such that

$$\eta = \tau_1(\eta_1 \zeta_1) \tau_1^{-1} = \tau_2(\eta_2 \zeta_2) \tau_2^{-1}$$

is regular. Then  $\eta_1 \zeta_1$  and  $\eta_2 \zeta_2$  are regular elements of  $G_i$  and  $\eta_2 \zeta_2 = \tau^{-1}(\eta_1 \zeta_1) \tau$  where  $\tau = \tau_1^{-1} \tau_2$ . Then  $\mathfrak{L}_1(\eta_2) = \mathfrak{L}_1(\eta_2 \zeta_2) = \mathfrak{L}_1(\eta_1 \zeta_1)^\tau = \mathfrak{L}_1(\eta_1)^\tau$ . In a similar manner we see that  $\tau^{-1}(\eta_1 H_0(\eta_1)) \tau = \eta_2 H_0(\eta_2)$ .

We have noted that if  $\eta$  is regular every element of  $\eta H_0(\eta)$  is semi-simple. We wish to prove that conversely any semi-simple automorphism belongs to an  $\eta H_0(\eta)$  where  $\eta$  is regular. The proof of this result in the complex case given by Gantmacher is based on the use of exponentials of elements  $ad l$ ,  $l \in \mathfrak{L}_1$ . These are not available in the algebraic case. However, a suitable substitute for these has been found by Seligman and we shall use these.

Let  $\mathfrak{R}$  be a semi-simple subalgebra of a Lie algebra  $\mathfrak{L}$ ,  $\mathfrak{S}$  a Cartan subalgebra of  $\mathfrak{R}$ . Let  $\alpha_i$ ,  $i = 1, 2, \dots, l$  be a simple system of roots for  $\mathfrak{R}$  relative to  $\mathfrak{S}$ ,  $\mathfrak{R}_{\alpha_i}$  the corresponding one dimensional root spaces. Then there exists a canonical set of generators  $e_i, f_i, h_i$ ,  $i = 1, 2, \dots, l$ , for  $\mathfrak{R}$  such that the  $h_i$  form a basis for  $\mathfrak{S}$ ,  $e_i \in \mathfrak{R}_{\alpha_i}$ ,  $f_i \in \mathfrak{R}_{-\alpha_i}$  and the following relations hold:

$$\begin{aligned} [h_i, h_j] &= 0 \\ [e_i, f_j] &= \delta_{ij} h_i \\ [e_i, h_j] &= A_{ji} e_i \\ [f_i, h_j] &= -A_{ji} f_i \end{aligned} \quad (3)$$

where  $(A_{ij})$  is the Cartan matrix of the simple system  $\alpha_i$ . We have  $A_{ii} = 2$  and  $A_{ij}$  is a nonpositive integer if  $i \neq j$ . It is known that the mappings  $ade_i$  and  $adf_i$  are nilpotent (in  $\mathfrak{L}$ ). Following Seligman we introduce the automorphism in  $\mathfrak{L}$ :

$$(4) \quad \sigma_i(\xi) = (\exp ad_{\xi} e_i)(\exp ad_{\xi^{-1}} f_i)(\exp ad_{\xi} e_i)$$

where  $\xi$  is any nonzero element of  $\Phi$  and  $i = 1, 2, \dots, l$ . Also we set

$$(5) \quad \omega_i(\xi) = \sigma_i(\xi)\sigma_i(-1)$$

and we let  $H$  be the group of automorphisms of  $\mathfrak{L}$  generated by the  $\omega_i(\xi)$ ,  $\xi \neq 0$  in  $\Phi$ ,  $i = 1, 2, \dots, l$ . Clearly the  $\omega_i(\xi)$  map  $\mathfrak{R}$  into itself. It has been shown by Seligman that the restriction  $\tilde{H}$  of  $H$  to  $\mathfrak{R}$  coincides with the group of automorphism in  $\mathfrak{R}$  such that

$$(6) \quad e_i \rightarrow \mu_i e_i, f_i \rightarrow \mu_i^{-1} f_i, h_i \rightarrow h_i$$

where the  $\mu_i$  are arbitrary nonzero elements.<sup>2</sup> Then  $\tilde{H}$  is an irreducible abelian algebraic group of automorphisms in  $\mathfrak{R}$  (an  $l$ -dimensional torus). We shall now prove the

LEMMA.  $H$  is an abelian group.

*Proof.* If we recall the form of the irreducible  $\mathfrak{R}$ -modules we see that  $\mathfrak{L}$  is generated by the  $f$  and a set  $X$  of elements  $x$  such that  $[xe_i] = 0$ ,  $i = 1, 2, \dots, l$  and  $[xh] = \lambda(h)x$ ,  $\lambda(h) \in \Phi$  ([5] p. 44). We fix  $i$  and write  $F = \text{ad } f_i$ ,  $E = \text{ad } e_i$ . Then if  $x_0 = x \in X$  and we define  $x_j = x_0 F^j$ , it is known that  $x = x_0, x_1, \dots, x_m$  are linearly independent and satisfy:

$$(7) \quad \begin{aligned} x_j F &= x_{j+1}, & j &= 0, \dots, m-1, & x_m F &= 0 \\ x_0 E &= 0, & x_j E &= -j(m-j+1)x_j, & j &= 1, \dots, m. \end{aligned}$$

Hence  $x_0 \exp \xi E = x_0$ ,

$$\begin{aligned} x_0 \exp \xi^{-1} F &= \sum_0^m \frac{\xi^{-j}}{j!} x_j \\ x_j (\exp \xi E) &= \sum_{l=0}^j (-1)^l \prod_{k=1}^l (m-j+k) \binom{j}{l} \xi^l x_{j-l}. \end{aligned}$$

Hence  $x_0 (\exp \xi E) (\exp \xi^{-1} F) (\exp \xi E) = \sum_{r=0}^m a_r \xi^{-r} x_r$  where

$$\begin{aligned} a_r &= \sum_{l=0}^{m-r} (-1)^l \binom{r+l}{l} \prod_{k=1}^l (m-r-l+k) \frac{1}{(r+l)!} \\ &= \frac{1}{r!} \sum_{l=1}^{m-r} (-1)^l \binom{m-r}{l}. \end{aligned}$$

It follows that  $a_r = 0$  if  $r \neq m$  and  $a_m = 1/m!$ . Hence

$$x_0 (\exp \xi E) (\exp \xi^{-1} F) (\exp \xi E) = (1/m!) \xi^{-m} x_m.$$

A similar calculation shows that

$$x_m (\exp \xi E) (\exp \xi^{-1} F) (\exp \xi E) = (-1)^m m! \xi^m x_0.$$

Hence

<sup>2</sup> [8] p. 446. A simpler proof will be given in a forthcoming book on Lie algebras by the author.



$$(8) \quad x\omega_i(\xi) = \xi^{-m}x$$

where  $m$  is a nonnegative integer. This implies that  $x\omega_i(\xi)\omega_i(\xi') = x\omega_i(\xi')\omega_i(\xi)$  if  $x \in X$ . In view of (6),  $\omega_i(\xi)$  and  $\omega_i(\xi')$  commute also in their action on the  $f_j$ . It follows that the group  $H$  generated by the  $\omega_i(\xi)$  is abelian.

Now let  $\rho$  be a semi-simple automorphism of the semi-simple Lie algebra  $\mathfrak{L}$  and let  $\mathfrak{L}_1(\rho)$  be the set of  $\rho$ -fixed elements. We have  $\mathfrak{L}_1(\rho) = \mathfrak{K} \oplus \mathfrak{C}$  where  $\mathfrak{K}$  is the derived algebra and  $\mathfrak{C}$  is the center. Then  $\mathfrak{K}$  is semi-simple and we can apply the above considerations. Then let  $\mathfrak{S}$  be a Cartan subalgebra of  $\mathfrak{K}$ ,  $e_i, f_i, h_i$  canonical generators of the type indicated such that the  $h_i$  form a basis for  $\mathfrak{S}$ . Let  $H$  be the abelian group of automorphisms of  $\mathfrak{L}$  generated by the  $\omega_i(\xi)$ ,  $\tilde{H}$  its restriction to  $\mathfrak{K}$ . Then if  $\mu_1, \mu_2, \dots, \mu_l$  are arbitrary nonzero elements of  $\Phi$ , the automorphism  $\zeta(\mu_1, \dots, \mu_l)$  of  $\mathfrak{K}$  such that  $e_i \rightarrow \mu_i e_i, f_i \rightarrow \mu_i^{-1} f_i$  belongs to  $\tilde{H}$ . It is known that  $\mathfrak{K}$  has a basis consisting of the  $h_i$ , certain products  $[\dots [e_{i_1} e_{i_2}] \dots e_{i_r}]$  and certain products  $[\dots [f_{i_1} f_{i_2}] \dots f_{i_r}]$ . The first of these is a characteristic vector of  $\zeta(\mu_1, \dots, \mu_l)$  belonging to  $\mu_{i_1} \mu_{i_2} \dots \mu_{i_r}$  and the second belongs to the root  $(\mu_{i_1} \mu_{i_2} \dots \mu_{i_r})^{-1}$ . It follows that the  $\mu$ 's can be chosen so that  $\mathfrak{S}$  is the characteristic space of the root 1 of  $\zeta(\mu_1, \dots, \mu_l)$ . It is clear that any  $\zeta \in H$  commutes with  $\rho$  and so it respects the decomposition:  $\mathfrak{L} = \mathfrak{L}_1(\rho) \oplus \mathfrak{L}^{1-\rho}, \mathfrak{L}^{1-\rho} = \{x - x^\rho\}$ . It follows by the standard specialization argument that there exists an automorphism  $\eta = \rho\zeta$ ,  $\zeta \in H$ , such that  $\mathfrak{L}_1(\eta) = \mathfrak{S} \oplus \mathfrak{C}$ . Since  $\mathfrak{L}_1(\eta)$  is abelian we see that  $\eta$  is regular.

Let  $H_0(\eta)$  be the algebraic component of 1 in the group of automorphisms of  $\mathfrak{L}$  commuting with  $\eta$ . Since  $H$  is abelian,  $\omega_i(\xi)$  commutes with  $\eta$ . It follows from (8) that for fixed  $i, \xi \neq 0$  in  $\Phi$ , the  $\omega_i(\xi)$  form an irreducible algebraic group. Hence  $\omega_i(\xi) \in H_0(\eta)$  and  $H \subseteq H_0(\eta)$ . Then  $\zeta^{-1} \in H_0(\eta)$  and  $\rho = \eta\zeta^{-1}$ .

We have therefore proved

**THEOREM 5.** *If  $\rho$  is a semi-simple automorphism of a semi-simple Lie algebra then  $\rho$  has the form  $\eta\zeta$  where  $\eta$  is regular and  $\zeta \in H_0(\eta)$  the component of 1 of the group of automorphisms commuting with  $\eta$ .*

**4. Determination of the indices.** Let  $\mathfrak{L}$  be semisimple,  $\mathfrak{S}$  a Cartan subalgebra,  $e_i, f_i, h_i, i = 1, 2, \dots, l$  canonical generators such that the  $h_i$  form a basis for  $\mathfrak{S}$  and (3) hold. We define the *group of automorphisms of the Cartan matrix*  $(A_{ij})$  to be the subgroup of the symmetric group  $S_l$  on  $1, 2, \dots, l$  of the permutations  $s$  such that  $A_{s(i), s(j)} = A_{i, j}$ ,  $i, j = 1, 2, \dots, l$ . If  $s$  is in this group there is a unique automorphism  $\sigma$  of  $\mathfrak{L}$  such that  $e_i^\sigma = e_{s(i)}, f_i^\sigma = f_{s(i)}$ . The set of these automorphisms is a finite group  $F$  isomorphic to the group of automorphisms of the

Cartan matrix. It is known that  $G = G_0 F = F G_0$ .<sup>3)</sup>

Let  $\sigma \in F$ ,  $s$  the corresponding automorphism of the Cartan matrix. If  $\mu_i, i = 1, 2, \dots, l$ , are arbitrary nonzero elements of  $\phi$  then there exists an automorphism  $\zeta$  of  $\mathfrak{L}$  such that

$$(9) \quad e_i^\zeta = \mu_i e_i, \quad f_i^\zeta = \mu_i^{-1} f_i.$$

The argument used in proving Theorem 5 shows that  $\zeta \in G_0$ . We choose  $\mu_i$  so that

$$(10) \quad \mu_{s(i)} = \mu_i$$

for every  $i$ , and for every positive root  $\alpha = \sum \kappa_i \alpha_i$ ,  $\kappa_i$  nonnegative integral,

$$(11) \quad (\mu_1^{\kappa_1} \mu_2^{\kappa_2} \dots \mu_l^{\kappa_l})^m \neq 1,$$

$m$  the order of  $s$ . Clearly such a choice of the  $\mu_i$  can be made. Also it is evident that  $\zeta$  is semi-simple and  $\mathfrak{L}_1(\zeta^m) = \mathfrak{L}$ . We have  $\sigma\zeta = \zeta\sigma$  and  $\sigma^m = 1$  so that  $\sigma$  is semi-simple. Hence  $\eta = \sigma\zeta$  is semi-simple and  $\mathfrak{L}_1(\eta)$  is the set of  $\eta$ -fixed points. If  $x \in \mathfrak{L}_1(\eta)$ ,  $x^{\eta^m} = x^{\zeta^m} = x$  so  $x \in \mathfrak{L}$ . Since  $x^\zeta = x$  for  $x \in \mathfrak{L}$  we have  $x^\sigma = x^{\zeta\sigma} = x^\eta = x$ . Conversely, if  $x \in \mathfrak{L}$  and  $x^\sigma = x$  then  $x \in \mathfrak{L}_1(\eta)$ . Hence  $\mathfrak{L}_1(\eta)$  is the subspace of  $\mathfrak{L}$  of  $\sigma$ -fixed elements. We have  $h_i^\sigma = h_{s(i)}$  for the basis  $(h_1, h_2, \dots, h_l)$  of  $\mathfrak{L}$ . Let

$$(12) \quad s = (i_1 \dots i_{m_1})(j_1 \dots j_{m_2}) \dots (u_1 \dots u_{m_p})$$

be the decomposition of the permutation  $s$  into disjoint cycles of length  $\geq 1$  ( $m_1 + m_2 + \dots + m_p = l$ ). Then it is clear that the elements

$$(13) \quad g_1 = h_{i_1} + \dots + h_{i_{m_1}}, \dots, g_p = h_{u_1} + \dots + h_{u_{m_p}}$$

constitute a basis for the subspace of  $\mathfrak{L}$  of  $\sigma$ -fixed points. Hence  $\dim \mathfrak{L}_1(\eta) = p$ , the number of cycles in the decomposition of  $s$ . Since  $\zeta \in G_0$ ,  $\eta$  and  $\sigma$  are in the same component of the automorphism group. Since  $\mathfrak{L}_1(\eta) \subseteq \mathfrak{L}$ ,  $\mathfrak{L}_1(\eta)$  is abelian and so  $\eta$  is a regular element in the same component as  $\sigma$ . We can therefore state

**THEOREM 6.** *Let  $G_i = \sigma_i G_0$  be a component of the group of automorphisms where  $\sigma_i \in F$  and corresponds to the automorphism  $s_i$  of the Cartan matrix. Then the index of  $G_i$  is the number of cycles in the decomposition of  $s_i$  into disjoint cycles of lengths  $\geq 1$ .*

For  $G_0$  we may take  $\sigma_0 = 1$  and we obtain that the index of  $G_0$  is  $l$ , the dimensionality of the Cartan subalgebra  $\mathfrak{L}$ . On the other hand, it is clear that if  $\sigma_i \neq 1$  then the index of  $\sigma_i G_0$  is positive but is less than  $l$ . Hence  $\sigma_i G_0 \neq G_0$  and the decomposition  $G = F G_0$  is semidirect.

<sup>3</sup> The arguments of [8] can be used to prove this. A detailed discussion will be given in Chapter IX of the author's forthcoming book.

**COROLLARY.** *The index of  $G_0$  is  $l$  and the index of any  $G_i \neq G_0$  is positive and less than  $l$ . The decomposition  $G = FG_0$  is semi-direct.*

Our analysis shows also that any Cartan subalgebra is the space  $\mathfrak{L}_1(\eta)$  for a regular automorphism belonging to  $G_0$ . Hence Theorem 4 proves again the conjugacy theorem for Cartan subalgebras by means of an element of  $G_0$ . Theorem 4 can be considered as a natural generalization of the classical conjugacy theorem.

Let  $\eta = \sigma\zeta$  be the automorphism which we constructed for the proof of Theorem 6. If  $\nu_1, \nu_2, \dots, \nu_l$  are nonzero elements of  $\Phi$  such that  $\nu_{s(i)} = \nu_i, i = 1, 2, \dots, l$ , then it is clear that the automorphism  $\zeta(\nu_1, \nu_2, \dots, \nu_l)$  such that

$$(14) \quad e_i^{\zeta(\nu_1, \dots, \nu_l)} = \nu_i e_i, \quad f_i^{\zeta(\nu_1, \dots, \nu_l)} = \nu_i^{-1} f_i$$

commutes with  $\eta$ . The set of these automorphisms is an irreducible algebraic group of  $p$  dimensions where  $p$  is the index of  $\eta G_0$ . It follows that this group coincides with  $H_0(\eta)$  the component of 1 in the group of automorphisms commuting with  $\eta$ . Theorem 4 therefore implies that if  $\rho$  is any semi-simple automorphism then  $\rho$  has the form  $\tau(\sigma_i \xi) \tau^{-1}$  where  $\tau \in G_0$ ,  $\sigma_i \in F$  and  $\xi$  is of the form  $\zeta(\nu_1, \dots, \nu_l)$  as in (14). This is Gantmacher's "canonical form" for the semi-simple automorphism  $\rho$ . It is clear from the definition of  $\sigma_i$  that  $\mathfrak{H}^{\sigma_i} \subseteq \mathfrak{H}$ . In fact, if we choose the basis for  $\mathfrak{H}$  as before, then  $h_j^{\sigma_i} = h_{s_i(j)}$  where  $s_i$  is the permutation of  $1, 2, \dots, l$  associated with  $\sigma_i$ . It is clear that the restriction of  $\sigma_i$  to  $\mathfrak{H}$  is periodic and the subspace of  $\sigma_i$ -fixed points of  $\mathfrak{H}$  has  $l_i$ -dimensions where  $l_i$  is the index of the component  $G_0^{s_i}$ . Since  $h^i = h$  for every  $h \in \mathfrak{H}$  these results hold also for  $\sigma_i \xi$ . Since  $\rho = \tau(\sigma_i \xi) \tau^{-1}$  we have the following

**THEOREM 7.** *If  $\rho$  is a semi-simple automorphism of a semi-simple Lie algebra, then there exists a Cartan subalgebra  $\mathfrak{H}$  such that  $\mathfrak{H}^\rho \subseteq \mathfrak{H}$ , the restriction of  $\rho$  to  $\mathfrak{H}$  is periodic and the subspace of  $\rho$ -fixed points of  $\mathfrak{H}$  has dimensionality equal to the index of the component  $\rho G_0$ .*

We look next at the indices of the components  $G_i \neq G_0$  and for the sake of simplicity we confine our attention to the simple algebras. Outer automorphisms (that is, automorphisms not in  $G_0$ ) exist in the following cases:  $A_l, l > 1$ ,  $D_l, l \geq 4$  and  $E_6$  and only in these cases. The group of automorphisms of the Cartan matrix can be identified with the group of automorphisms of the associated Dynkin diagram. For  $A_l$  the automorphism  $\neq 1$  of the Dynkin diagram (suitably labelled) is  $i \rightarrow l + 1 - i$ . If  $l$  is even the cycle decomposition is  $(1l)(2, l-1) \dots (l/2, l/2+1)$  and for odd  $l$  it is  $(1l)(2, l-1) \dots ((l-1)/2, (l-1)/2+2)((l-1)/2+1)$ . In

both cases, if  $G_1 \neq G_0$ , then  $\text{index } G_1 = [(l+1)/2]$ . For  $D_l$ ,  $l > 4$ ,  $G = G_0 \cup G_1$ ,  $G_1 \neq G_0$  and the permutation associated with  $G_1$  is  $i \rightarrow i$  if  $i \leq l-2$ ,  $l \rightarrow l-1$ ,  $l-1 \rightarrow l$ . The cycle decomposition is  $(1) \cdots (l-2) (l-1, l)$ . Hence  $\text{index } G_1 = l-1$ . For  $D_4$  the group of automorphisms of the Dynkin diagram is the symmetric group on 1, 3, 4 if  $\alpha_1, \alpha_3, \alpha_4$  are the end points of the diagram. If the permutation associated with  $G_i$  is of order two then  $\text{index } G_i = 3$ . If the permutation is of order three then  $\text{index } G_i = 2$ . For a suitable ordering of the vertices the automorphism  $\neq 1$  of the Dynkin diagram of  $E_6$  is (15) (24) (3) (6). Hence  $\text{index } G_1 = 4$ .

**THEOREM 8.** *For  $A_l$ ,  $l > 1$ ,  $G = G_0 \cup G_1$  and  $\text{index } G_1 = [(l+1)/2]$ . For  $D_l$ ,  $l > 4$ ,  $G = G_0 \cup G_1$  and  $\text{index } G_1 = l-1$ . For  $D_4$ ,  $G/G_0$  is the symmetric group  $S_3$  and  $\text{index } G_i = 2$  if the coset of  $G_i$  is of order 3 and  $\text{index } G_i = 3$  if the coset of  $G_i$  is of order 2. For  $E_6$ ,  $G = G_0 \cup G_1$  and  $\text{index } G_1 = 4$ .*

**5. Application to fixed points.** In the applications to fixed points we can for the most part relax the assumption on the base field  $\phi$  and suppose only that  $\phi$  is of characteristic 0. If  $\eta$  is an automorphism in  $\mathfrak{L}$  over  $\phi$  and  $P$  is an extension field of  $\phi$  then  $\eta$  has a unique extension to an automorphism  $\eta$  of  $\mathfrak{L}_P$  and the space of  $\eta$ -fixed points of  $\mathfrak{L}_P$  has the form  $\mathfrak{F}(\eta)_P$  where  $\mathfrak{F}(\eta)$  is the space of  $\eta$ -fixed points of  $\mathfrak{L}$ . This remark reduces most considerations of fixed points to the algebraically closed case.

The following result is due to Borel and Mostow ([1] p 398) for semi-simple automorphisms.

**THEOREM 9.** *If  $\mathfrak{L}$  is a nonsolvable Lie algebra over a field of characteristic 0 then any automorphism  $\eta$  of  $\mathfrak{L}$  has a fixed point.*

*Proof.* It suffices to assume the base field is algebraically closed. Let  $\mathfrak{R}$  be the radical of  $\mathfrak{L}$ . Then  $\eta$  induces an automorphism  $\bar{\eta}$  in the semi-simple Lie algebra  $\bar{\mathfrak{L}} = \mathfrak{L}/\mathfrak{R}$ . By the Corollary to Th. 6,  $\dim \bar{\mathfrak{L}}_1(\bar{\eta}) \geq 1$ . This means that 1 is a characteristic root of  $\bar{\eta}$ . Hence 1 is a characteristic root of  $\eta$  and there exists a nonzero fixed point.

It is convenient at this point to introduce another type of index  $m(G_i)$  of a component  $G_i$  of the automorphism group in the algebraically closed case. We set  $m(G_i) = \text{minimum of } \dim \mathfrak{F}(\eta) \text{ for } \eta \in G_i$  where  $\mathfrak{F}(\eta)$  is the space of  $\eta$ -fixed points. If  $(\alpha)$  is a matrix of  $\eta$  then  $\dim \mathfrak{F}(\eta) = n - \text{rank } ((\alpha) - 1)$ ,  $n = \dim \mathfrak{L}$ . Hence  $m(G_i) = k_i$  means that for every  $\eta \in G_i$  every minor of order  $n - k_i + 1$  of  $(\alpha) - 1$  vanishes but there exists an  $\eta \in G_i$  such that  $(\alpha) - 1$  has a nonvanishing minor of order  $n - k_i$ . It is clear from this that the elements  $\eta$  of  $G_i$  such that

$\dim \mathfrak{F}(\eta) = k_i \equiv m(G_i)$  form an open set in  $G_i$ . Hence this set contains a regular element  $\eta$ . If  $\mathfrak{L}$  is semisimple it follows that such an  $\eta$  is semi-simple. Then  $\mathfrak{F}(\eta) = \mathfrak{L}_1(\eta)$  and consequently  $m(G_i) = n(G_i)$ . We can state this result in the following way.

**THEOREM 10.** *Let  $\mathfrak{L}$  be a semi-simple Lie algebra over an algebraically closed field of characteristic 0 and let  $l_i$  be the index of the component  $G_i$  of the group of automorphisms of  $\mathfrak{L}$ . Let  $\eta \in G_i$  and let  $\mathfrak{F}(\eta)$  be the space of fixed points under  $\eta$ . Then  $\dim \mathfrak{F}(\eta) \geq l_i$  and there exists  $\eta \in G_i$  such that  $\dim \mathfrak{F}(\eta) = l_i$ .*

This result can also be applied to the case of an arbitrary base field of characteristic 0. A given automorphism  $\eta$  has its extension to an automorphism  $\eta$  of  $\mathfrak{L}_o, \Omega$  the algebraic closure of  $\Phi$ . Also  $\mathfrak{L}_o$  is semi-simple if  $\mathfrak{L}_o$  is semi-simple. The result just proved gives a lower bound for  $\dim \mathfrak{F}(\eta)$  once the component of  $\eta$  in the group of automorphisms of  $\mathfrak{L}_o$  is known. Even without this information we can say that  $\dim \mathfrak{F}(\eta) \geq m$  where  $m$  is the minimum of the indices of the components  $G_i$  of the group of automorphisms of  $\mathfrak{L}_o$ .

Again let  $\mathfrak{L}$  be semi-simple over any field of characteristic 0 and let  $\eta$  be semi-simple in the sense that its minimum polynomial is a product of distinct prime factors. Since the base field is perfect this property is preserved under field extension. It follows from this that  $\mathfrak{L}_1(\eta) = \mathfrak{F}(\eta)$  is reductive. This implies that any Cartan subalgebra  $\mathfrak{H}_1$  of  $\mathfrak{L}_1(\eta)$  is abelian and reductive in  $\mathfrak{L}$ . Moreover, any reductive abelian subalgebra of  $\mathfrak{L}_1(\eta)$  can be imbedded in a Cartan subalgebra and any two Cartan subalgebras of  $\mathfrak{L}_1(\eta)$  have the same dimensionality. It follows by a field extension argument and Th. 7 that the dimensionality of  $\mathfrak{H}_1$  is not less than the index of the component of  $\eta$  in the group of automorphisms of  $\mathfrak{L}_o, \Omega$  the algebraic closure of the base field.

The result just indicated holds also for arbitrary  $\mathfrak{L}$  and semi-simple  $\eta$  by virtue of a result of Mostow's that there exists a Levi-decomposition  $\mathfrak{L} = \mathfrak{R} + \mathfrak{R}$  where  $\mathfrak{R}$  is the radical and  $\mathfrak{R}$  is a semi-simple subalgebra invariant under  $\eta$  ([7]). It is known that if  $a \in \mathfrak{R}$  and  $ad_{\mathfrak{R}} a$  is semi-simple then  $ad_{\mathfrak{L}} a$  is semi-simple. We can therefore state the following extension of a theorem of de Siebenthal-Borel-Mostow ([1] p. 498).

**THEOREM 11.** *Let  $\eta$  be a semi-simple automorphism of a Lie algebra  $\mathfrak{L}$  over a field of characteristic 0. Let  $\mathfrak{R}$  be the radical,  $\mathfrak{L} = \mathfrak{L}/\mathfrak{R}$  and let  $m$  be the minimal index of the components of the group of automorphism of  $\mathfrak{L}_o, \Omega$  the algebraic closure of the base field. Then  $m \geq 1$  and there exists an abelian reductive subalgebra of  $m$  dimensions whose elements are fixed under  $\eta$ .*

## BIBLIOGRAPHY

1. A. Borel and G. D. Mostow, *On semi-simple automorphisms of Lie algebras*, Annals of Math., **61** (1955), 389-405.
2. H. Cartan and C. Chevalley, *Séminaire École Normale Supérieure*, Géométrie Algébrique, 1956.
3. C. Chevalley, *Théorie des Groupes de Lie*, Tome II (1951) and Tome III (1955).
4. F. Gantmacher, *Canonical representation of automorphisms of a complex semi-simple Lie algebra*, Mat. Sbornik 5(47) (1939), 101-146.
5. Harish-Chandra, *On some applications of the universal enveloping algebra of a semi-simple Lie algebra*, Trans. Amer. Math. Soc., **70** (1951), 28-96.
6. N. Jacobson, *Completely reducible Lie algebras of linear transformations*, Proc. Amer. Math. Soc., **2** (1951), 105-113.
7. G. D. Mostow, *Fully reducible subgroups of algebraic groups*, Amer. J. Math., **78** (1956), 200-221.
8. G. B. Seligman, *On automorphisms of Lie algebras of classical type*, Trans. Amer. Math. Soc., **92** (1959), 430-448.

YALE UNIVERSITY



# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

RALPH S. PHILLIPS

Stanford University  
Stanford, California

M. G. ARSOVE

University of Washington  
Seattle 5, Washington

A. L. WHITEMAN

University of Southern California  
Los Angeles 7, California

LOWELL J. PAIGE

University of California  
Los Angeles 24, California

## ASSOCIATE EDITORS

E. F. BECKENBACH

T. M. CHERRY

D. DERRY

M. OHTSUKA

H. L. ROYDEN

E. SPANIER

E. G. STRAUS

F. WOLF

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA

CALIFORNIA INSTITUTE OF TECHNOLOGY

UNIVERSITY OF CALIFORNIA

MONTANA STATE UNIVERSITY

UNIVERSITY OF NEVADA

NEW MEXICO STATE UNIVERSITY

OREGON STATE UNIVERSITY

UNIVERSITY OF OREGON

OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY

UNIVERSITY OF TOKYO

UNIVERSITY OF UTAH

WASHINGTON STATE UNIVERSITY

UNIVERSITY OF WASHINGTON

\* \* \*

AMERICAN MATHEMATICAL SOCIETY

CALIFORNIA RESEARCH CORPORATION

SPACE TECHNOLOGY LABORATORIES

NAVAL ORDNANCE TEST STATION

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan



# Pacific Journal of Mathematics

Vol. 12, No. 1

January, 1962

Jonathan L. Alperin, <i>Groups with finitely many automorphisms</i> .....	1
Martin Arthur Arkowitz, <i>The generalized Whitehead product</i> .....	7
John D. Baum, <i>Instability and asymptoticity in topological dynamics</i> .....	25
William Aaron Beyer, <i>Hausdorff dimension of level sets of some Rademacher series</i> .....	35
Frank Herbert Brownell, III, <i>A note on Cook's wave-matrix theorem</i> .....	47
Gulbank D. Chakerian, <i>An inequality for closed space curves</i> .....	53
Inge Futtrup Christensen, <i>Some further extensions of a theorem of Marcinkiewicz</i> .....	59
Charles Vernon Coffman, <i>Linear differential equations on cones in Banach spaces</i> .....	69
Eckford Cohen, <i>Arithmetical notes. III. Certain equally distributed sets of integers</i> .....	77
John Irving Derr and Angus E. Taylor, <i>Operators of meromorphic type with multiple poles of the resolvent</i> .....	85
Jacob Feldman, <i>On measurability of stochastic processes in products space</i> .....	113
Robert S. Freeman, <i>Closed extensions of the Laplace operator determined by a general class of boundary conditions, for unbounded regions</i> .....	121
Robert E. Fullerton, <i>Geometric structure of absolute basis systems in a linear topological space</i> .....	137
Dieter Gaier, <i>On conformal mapping of nearly circular regions</i> .....	149
Andrew Mattei Gleason and Hassler Whitney, <i>The extension of linear functionals defined on <math>H^\infty</math></i> .....	163
Seymour Goldberg, <i>Closed linear operators and associated continuous linear operators</i> .....	183
Basil Gordon, Aviezri Siegmund Fraenkel and Ernst Gabor Straus, <i>On the determination of sets by the sets of sums of a certain order</i> .....	187
Branko Grünbaum, <i>The dimension of intersections of convex sets</i> .....	197
Paul Daniel Hill, <i>On the number of pure subgroups</i> .....	203
Robert Peter Holten, <i>Generalized Goursat problem</i> .....	207
Alfred Horn, <i>Eigenvalues of sums of Hermitian matrices</i> .....	225
Henry C. Howard, <i>Oscillation and nonoscillation criteria for <math>y''(x) + f(y(x))p(x) = 0</math></i> .....	243
Taqdir Husain, <i>S-spaces and the open mapping theorem</i> .....	253
Richard Eugene Isaac, <i>Markov processes and unique stationary probability measures</i> .....	273
John Rolfe Isbell, <i>Supercomplete spaces</i> .....	287
John Rolfe Isbell, <i>On finite-dimensional uniform spaces. II</i> .....	291
N. Jacobson, <i>A note on automorphisms of Lie algebras</i> .....	303
Antoni A. Kosinski, <i>A theorem on families of acyclic sets and its applications</i> .....	317
Marvin David Marcus and H. Minc, <i>The invariance of symmetric functions of singular values</i> .....	327
Ralph David McWilliams, <i>A note on weak sequential convergence</i> .....	333
John W. Milnor, <i>On axiomatic homology theory</i> .....	337
Victor Julius Mizel and Malempati Madhusudana Rao, <i>Nonsymmetric projections in Hilbert space</i> .....	343
Calvin Cooper Moore, <i>On the Frobenius reciprocity theorem for locally compact groups</i> .....	359
Donald J. Newman, <i>The Gibbs phenomenon for Hausdorff means</i> .....	367
Jack Segal, <i>Convergence of inverse systems</i> .....	371
Józef Siciak, <i>On function families with boundary</i> .....	375
Hyman Joseph Zimmerberg, <i>Two-point boundary conditions linear in a parameter</i> .....	385