THE INVARIANCE OF SYMMETRIC FUNCTIONS OF SINGULAR VALUES

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Let $M_{m,n}$ denote the vector space of all $m \times n$ matrices over the complex numbers. A general problem that has been considered in many forms is the following: suppose $\mathcal{A}$ is a subset (usually subspace) of $M_{m,n}$ and let $f$ be a scalar valued function defined on $\mathcal{A}$. Determine the structure of the set $\mathcal{A}_f$ of all linear transformations $T$ that satisfy

\[ f(T(A)) = f(A) \quad \text{for all } A \in \mathcal{A}. \]

The most interesting choices for $f$ are the classical invariants such as rank $[3, 4, 7]$ determinant $[1, 2, 3, 5, 10]$ and more general symmetric functions of the characteristic roots $[6, 8]$. In case $\mathcal{A}$ is the set of $n$-square real skew-symmetric matrices ($m = n$) and $f(A)$ is the Hilbert norm of $A$ then Morita [9] proved the following interesting result: $\mathcal{A}_f$ consists of transformations $T$ of the form

\[
\begin{align*}
T(A) &= U'AU \quad \text{for } n \neq 4, \\
T(A) &= U'AU \text{ or } T(A) = U'A^+U \quad \text{for } n = 4
\end{align*}
\]

where $U$ is a fixed real orthogonal matrix and $A^+$ is the matrix obtained from $A$ by interchanging its $(1, 4)$ and $(2, 3)$ elements.

Recall that the Hilbert norm of $A$ is just the largest singular value of $A$ (i.e., the largest characteristic root of the nonnegative Hermitian square root of $A^*A$).

In the present paper we determine $\mathcal{A}_f$ when $\mathcal{A}$ is all of $M_{m,n}$ and $f$ is some particular elementary symmetric function of the squares of the singular values. We first introduce a bit of notation to make this statement precise. If $A \in M_{n,n}$ then $\lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A))$ will denote the $n$-tuple of characteristic roots of $A$ in some order. The $r$th elementary symmetric function of the numbers $\lambda(A)$ will be denoted by $E_r[\lambda(A)]$; this is, of course, the same as the sum of all $r$-square principal subdeterminants of $A$. We also denote by $\rho(A)$ the rank of $A$.

**Theorem.** A linear transformation $T$ of the space $M_{m,n}$ leaves invariant the $r$th elementary symmetric function of the squares of the singular values of each $A \in M_{m,n}$, for some fixed $r$, $1 < r \leq n$, if and only if there exist unitary matrices $U$ and $V$ in $M_{m,m}$ and $M_{n,n}$ respectively such that

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The sufficiency of (2) and (3) is clear and we prove the necessity in a sequence of lemmas some of which may be of interest in themselves. Assume without loss of generality that \( m \geq n \).

**Lemma 1.** Let \( A, B \in M_{m,n} \) and let \( \varphi_B(x) = E_r[\lambda((xA + B)^* (xA + B))] \) where \( x \) is a real indeterminate. Then

(4) \( \deg \varphi_B(x) \leq 2 \) for all \( B \in M_{m,n} \)

if and only if

(5) \( \rho(A) \leq 1 \).

**Proof.** We first remark that \( \varphi_B(x) \) is actually a polynomial in \( x \) since it is the sum of all \( \square \) \( r \)-square principal subdeterminants of \( (xA + B)^* (xA + B) \). The matrix \( A \) can be written, by a slight extension of the polar factorization theorem to rectangular matrices, in the form \( A = UH \) where \( H \) is \( n \)-square hermitian positive semi-definite and \( U \in M_{m,n} \) satisfies \( U^*U = I_n \), the \( n \)-square identity matrix. Then

\[
\varphi_B(x) = E_r[\lambda((xUH + B)^* (xUH + B))] = E_r[\lambda((xH + U^*B)^* (xH + U^*B))] .
\]

Now let \( H = V^*DV \) where \( V \) is unitary and \( D \) is diagonal. Then

\[
\varphi_B(x) = E_r[\lambda(V^*(xD + VU^*BV^*)^*VV^*(xD + VU^*BV^*)V) = E_r[\lambda((xD + B_1)^* (xD + B_1))] .
\]

where \( B_1 = VU^*BV^* \). Now suppose \( \rho(A) = \rho(D) = 1 \). Then \( D \) has exactly one nonzero entry which we may clearly assume to be in the \((1,1)\) position. It follows that \( (xD + B_1)^* (xD + B_1) \) has a quadratic polynomial in \( x \) in the \((1,1)\) position, first degree polynomials in the other first row and first column positions and constants elsewhere. Therefore, every principal subdeterminant of this matrix is a polynomial in \( x \) of degree at most 2.

On the other hand, if (4) holds then in particular for \( B = 0 \)

\[
\varphi_B(x) = E_r[\lambda(xD^*D)]
\]

and \( \deg \varphi_B(x) \leq 2 \); this implies that the diagonal matrix \( D^*D \) can have at most one nonzero entry. But then \( 1 \geq \rho(D^*D) = \rho(D) = \rho(A) \).

**Lemma 2.** Let \( f(t_1, \cdots, t_n) \) be a monotone strictly increasing function of each \( t_i \) for \( t_i > 0 \). If \( T \) is a linear map of \( M_{m,n} \) into itself satisfying

(2) \( T(A) = UAV \) if \( m \neq n \) and

(3) \( T(A) = UAV \) or \( T(A) = UA'V \) if \( m = n \).
\[
f(\lambda(A^*A)) = f(\lambda((T(A))^*T(A))), \quad A \in M_{m,n}
\]
then \( T \) is nonsingular.

**Proof.** Suppose \( T(A) = 0 \). Then

\[
f(\lambda(X^*X)) = f(\lambda((T(X))^*T(X)))
\]

\[
= f(\lambda(((T(A + X))^*T(A + X)))
\]

\[
= f(\lambda((A + X)^*(A + X))) .
\]

Let \( A = UH \) where \( U^*U = I_n \) and \( H \) is nonnegative Hermitian. Then taking \( H = V^*DV \) where \( D \) is diagonal and \( V \) is unitary we find as in Lemma 1 that

\[
f(\lambda(X^*X)) = f(\lambda((D + Y)^*(D + Y))),
\]

\( Y = VU^*XV^* \). Now as \( X \) runs over \( M_{m,n} \) \( Y \) runs over \( M_{n,n} \) and moreover

\[
\lambda(X^*X) = \lambda(V^*Y^*VU^*UV^*YV) = \lambda(Y^*Y).
\]

Hence

(6) \[
f(\lambda(Y^*Y)) = f(\lambda((D + Y)^*(D + Y)))
\]

for all \( Y \in M_{n,n} \). Let \( Y \) be a real diagonal matrix with diagonal elements \( y_1, \ldots, y_n \). Then if \( D \) has diagonal elements \( d_1, \ldots, d_n \) we conclude from (6) that

\[
f(y^*_1, \ldots, y^*_n) = f(d_1^2 + y_1^2, \ldots, d_n^2 + y_n^2).
\]

Thus \( D = 0, A = 0 \) and \( T \) is nonsingular.

We remark at this point that the elementary symmetric functions satisfy the conditions of Lemma 2 and hence the \( T \) of the theorem is nonsingular.

**Lemma 3.** If \( \rho(A) = 1 \) then \( \rho(T(A)) = 1 \).

**Proof.** If \( \rho(A) = 1 \) then, by Lemma 1, \( \deg \varphi_a(x) \leq 2 \). Now

\[
\varphi_a(x) = E,[(\lambda((xA + B)^*(xA + B))]
\]

\[
= E,[(\lambda((TxA + B))^*T(xA + B))]
\]

\[
= E,[(\lambda((xT(A) + T(B))^*(xT(A) + T(B))))] .
\]

By Lemma 2 \( T \) is nonsingular so \( T(B) \) ranges over \( M_{m,n} \) as \( B \) does. Hence, by Lemma 1, \( \rho(T(A)) \leq 1 \). But \( T(A) \neq 0 \) since \( \rho(A) = 1 \). Thus \( \rho(T(A)) = 1 \).

At this point we invoke [7: p. 1219] that tells us that a linear transformation on \( M_{m,n} \) which preserves rank 1 has the desired form:
\[ T(A) = UAV \text{ for all } A \in M_{m,n} \]

or

\[ T(A) = UA'V \text{ for all } A \in M_{m,n}, \]

where \( U \) and \( V \) are nonsingular \( m \)-square and \( n \)-square matrices respectively and the second eventuality occurs only if \( m = n \). The proof of the theorem will be complete if we show

**Lemma 4.** \( U \) and \( V \) may be chosen to be unitary.

**Proof.** We show this when \( T \) has the form (2); if \( T \) has the form (3) the argument is essentially the same. Let \( V = HP \) and \( U = QK \) where \( H \) and \( K \) are positive definite Hermitian and \( P \) and \( Q \) are unitary. Then

\[
E_r[\lambda(A^*A)] = E_r[\lambda((UAV)^*(UAV))]
\]

\[
= E_r[\lambda(V^*A^*U^*UAV)]
\]

\[
= E_r[\lambda(P^*HA^*K^2AHP)]
\]

\[
= E_r[\lambda(HA^*K^2AH)]
\]

\[
= E_r[\lambda(H^2A^*K^2A)]
\]

for all \( A \). Let \( H = XDX^*, K = YGY^*, X \) and \( Y \) unitary, \( D \) and \( G \) diagonal matrices with main diagonals \( d_1, \ldots, d_n \) and \( g_1, \ldots, g_n \) respectively. Then

\[
E_r[\lambda(A^*A)] = E_r[\lambda(XD^2X^*A^*YG^2Y^*A)]
\]

\[
= E_r[\lambda(D^2B^*G^2B)]
\]

where \( B = Y^*AX \). Now

\[
\lambda(A^*A) = \lambda(XB^*Y^*YBX^*) = \lambda(B^*B)
\]

and hence

\[
E_r[\lambda(B^*B)] = E_r[\lambda(D^2B^*G^2B)]
\]

for all \( B \). Choose \( B \) as follows:

\[
B = \begin{bmatrix}
0 & 1 & & \\
& \ddots & \ddots & \\
& & 0 & \\
0 & & 1 & 0 \\
0 & & & 0
\end{bmatrix}
\]
in which the upper left block is the indicated $r$-square permutation matrix. Then clearly $E_r[\lambda(B^*B)] = 1$ and

$$D^2B^*G^2B = \begin{bmatrix} d_1^2g_1^2 & & & & \\
 & d_2^2g_2^2 & & & \\
 & & \ddots & & \\
 & & & d_{r-1}^2g_{r-1}^2 & \\
 & & & & d_r^2g_r^2 \end{bmatrix}$$

Thus

$$1 = E_r[\lambda(B^*B)] = \prod_{j=1}^{r} d_j^2g_j^2.$$  

Now set $D^2 = RD^2R$ where $R$ is an $n$-square permutation matrix and $D^2_\sigma$ is a diagonal matrix obtained from $D^2$ by a permutation $\sigma$ of the diagonal elements of $D^2$. Then

$$\lambda(D^2B^*G^2B) = \lambda(RD^2R^*B^*G^2B)$$

$$= \lambda(D^2_\sigma(BR)^*G^2(BR))$$

$$= \lambda(D^2_\sigma C^*G^2C),$$

where $C = BR$, and

$$\lambda(B^*B) = \lambda(R^*B^*BR) = \lambda(C^*C).$$

Therefore

$$E_r[\lambda(C^*C)] = E_r[\lambda(D^2_\sigma C^*G^2C)]$$

for all $C$. It follows that

$$\prod_{i=1}^{n} d_{\sigma(i)}^2g_i^2 = 1$$

for any permutation $\sigma$ of 1, $\cdots$, $n$. From this we conclude that

$$d_1^2 = \cdots = d_n^2 = d^2$$

and similarly

$$g_1^2 = \cdots = g_n^2 = g^2.$$

Then $G = gI$, $D = dI$ and $U = gQ$, $V = dP$, i.e. $U$, $V$ are scalar multiples of unitary matrices. Now,
\[ E_1[\lambda(A^*A)] = \text{tr}(A^*A) = \sum_{(i,j) \in [1,n]} |a_{ij}|^2 , \]

Hence \(|gd|^r = 1\) and we can choose \(U\) and \(V\) to be \(gdQ\) and \(P\) which are unitary. This completes the proof.

We remark that in case \(r = 1\) \(T\) does not necessarily have the form indicated in (2) and (3). For

\[ E_1[\lambda(A^*A)] = \text{tr}(A^*A) = \sum_{(i,j) \in [1,n]} |a_{ij}|^2 , \]

and if \(T\) is merely a unitary operator on \(M_{m,n}\)

\[ E_1[\lambda((T(A))^*(T(A))) = E_1[\lambda(A^*A)] . \]

For example \(T\) can be the operator that interchanges the (1, 2) and (2, 1) elements of every \(A \in M_{m,n}\) (assume \(m, n > 2\)) and this cannot be accomplished by any pre- and post-multiplication as in (2) or (3).

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