Pacific Journal of Mathematics

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Vol. 12, No. 1 January 1962

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Let $M_{m,n}$ denote the vector space of all $m \times n$ matrices over the complex numbers. A general problem that has been considered in many forms is the following: suppose $\mathfrak{A}$ is a subset (usually subspace) of $M_{m,n}$ and let f be a scalar valued function defined on $\mathfrak{A}$. *Determine the structure of the set* $\mathfrak{A}_f$ *of all linear transformations* T *that satisfy*

$$f(T(A)) = f(A) \text{ for all } A \in \mathfrak{A} . \tag{1}$$

The most interesting choices for f are the classical invariants such as rank [3, 4, 7] determinant [1, 2, 3, 5, 10] and more general symmetric functions of the characteristic roots [6, 8]. In case $\mathfrak{A}$ is the set of n-square real skew-symmetric matrices ($m = n$) and $f(A)$ is the Hilbert norm of A then Morita [9] proved the following interesting result: $\mathfrak{A}_f$ *consists of transformations* T *of the form*

$$T(A) = U'AU \text{ for } n \neq 4 ,$$
$$T(A) = U'AU \text{ or } T(A) = U'A^{+}U \text{ for } n = 4$$

where U *is a fixed real orthogonal matrix and* A^{+} *is the matrix obtained from* A *by interchanging its* $(1, 4)$ *and* $(2, 3)$ *elements.*

Recall that the Hilbert norm of A is just the largest *singular value* of A (i.e., the largest characteristic root of the nonnegative Hermitian square root of A^*A).

In the present paper we determine $\mathfrak{A}_f$ when $\mathfrak{A}$ is all of $M_{m,n}$ and f is some particular elementary symmetric function of the squares of the singular values. We first introduce a bit of notation to make this statement precise. If $A \in M_{n,n}$ then $\lambda(A) = (\lambda_1(A), \cdots, \lambda_n(A))$ will denote the n-tuple of characteristic roots of A in some order. The rth elementary symmetric function of the numbers $\lambda(A)$ will be denoted by $E_r[\lambda(A)]$; this is, of course, the same as the sum of all r-square principal subdeterminants of A. We also denote by $\rho(A)$ the rank of A.

THEOREM. *A linear transformation* T *of the space* $M_{m,n}$ *leaves invariant the rth elementary symmetric function of the squares of the singular values of each* $A \in M_{m,n}$, *for some fixed* r, $1 < r \leqq n$, *if and only if there exist unitary matrices* U *and* V *in* $M_{m,m}$ *and* $M_{n,n}$ *respectively such that*

Received January 16, 1961. The work of the first author was supported in part by the the Office of Naval Research.

(2) $$T(A) = UAV \textit{ if } m \neq n \textit{ and}$$

(3) $$T(A) = UAV \textit{ or } T(A) = UA'V \textit{ if } m = n\ .$$

The sufficiency of (2) and (3) is clear and we prove the necessity in a sequence of lemmas some of which may be of interest in themselves. Assume without loss of generality that $m \geqq n$.

LEMMA 1. *Let* $A, B \in M_{m,n}$ *and let* $\varphi_B(x) = E_r[\lambda((xA + B)^* (xA + B))]$ *where* x *is a real indeterminate. Then*

(4) $$\deg \varphi_B(x) \leqq 2 \textit{ for all } B \in M_{m,n}$$

if and only if

(5) $$\rho(A) \leqq 1\ .$$

Proof. We first remark that $\varphi_B(x)$ is actually a polynomial in x since it is the sum of all $\binom{n}{r}$ r-square principal subdeterminants of $(xA + B)^* (xA + B)$. The matrix A can be written, by a slight extension of the polar factorization theorem to rectangular matrices, in the form $A = UH$ where H is n-square hermitian positive semi-definite and $U \in M_{m,n}$ satisfies $U^*U = I_n$, the n-square identity matrix. Then

$$\begin{aligned}\varphi_B(x) &= E_r[\lambda((xUH + B)^* (xUH + B))] \\ &= E_r[\lambda((xH + U^*B)^* (xH + U^*B))]\ .\end{aligned}$$

Now let $H = V^*DV$ where V is unitary and D is diagonal. Then

$$\begin{aligned}\varphi_B(x) &= E_r[\lambda(V^*(xD + VU^*BV^*)^* VV^*(xD + VU^*BV^*)V)] \\ &= E_r[\lambda((xD + B_1)^* (xD + B_1))]\end{aligned}$$

where $B_1 = VU^*BV^*$. Now suppose $\rho(A) = \rho(D) = 1$. Then D has exactly one nonzero entry which we may clearly assume to be in the $(1, 1)$ position. It follows that $(xD + B_1)^* (xD + B_1)$ has a quadratic polynomial in x in the $(1, 1)$ position, first degree polynomials in the other first row and first column positions and constants elsewhere. Therefore, every principal subdeterminant of this matrix is a polynomial in x of degree at most 2.

On the other hand, if (4) holds then in particular for $B = 0$

$$\varphi_0(x) = E_r[\lambda(x^2D^*D)]$$

and $\deg \varphi_0(x) \leqq 2$; this implies that the diagonal matrix D^*D can have at most one nonzero entry. But then $1 \geqq \rho(D^*D) = \rho(D) = \rho(\mathrm{A})$.

LEMMA 2. *Let* $f(t_1, \cdots, t_n)$ *be a monotone strictly increasing function of each* t_j *for* $t_j > 0$. *If* T *is a linear map of* $M_{m,n}$ *into itself satisfying*

$$f(\lambda(A^*A)) = f(\lambda((T(A))^* T(A))), \quad A \in M_{m,n}$$

then T is nonsingular.

Proof. Suppose $T(A) = 0$. Then

$$\begin{aligned} f(\lambda(X^*X)) &= f(\lambda((T(X))^* T(X))) \\ &= f(\lambda((T(A+X))^* T(A+X))) \\ &= f(\lambda((A+X)^*(A+X))) . \end{aligned}$$

Let $A = UH$ where $U^*U = I_n$ and H is nonnegative Hermitian. Then taking $H = V^*DV$ where D is diagonal and V is unitary we find as in Lemma 1 that

$$f(\lambda(X^*X)) = f(\lambda((D+Y)^*(D+Y))) ,$$

$Y = VU^*XV^*$. Now as X runs over $M_{m,n}$ Y runs over $M_{n,n}$ and moreover

$$\lambda(X^*X) = \lambda(V^*Y^*VU^*UV^*YV) = \lambda(Y^*Y) .$$

Hence

$$f(\lambda(Y^*Y)) = f(\lambda((D+Y)^*(D+Y))) \tag{6}$$

for all $Y \in M_{n,n}$. Let Y be a real diagonal matrix with diagonal elements $y_1, \cdots, y_n$. Then if D has diagonal elements $d_1, \cdots, d_n$ we conclude from (6) that

$$f(y_1^2, \cdots, y_n^2) = f(d_1^2 + y_1^2, \cdots, d_n^2 + y_n^2) .$$

Thus $D = 0, A = 0$ and T is nonsingular.

We remark at this point that the elementary symmetric functions satisfy the conditions of Lemma 2 and hence the T of the theorem is nonsingular.

LEMMA 3. *If $\rho(A) = 1$ then $\rho(T(A)) = 1$.*

Proof. If $\rho(A) = 1$ then, by Lemma 1, deg $\varphi_B(x) \leqq 2$. Now

$$\begin{aligned} \varphi_B(x) &= E_r[\lambda((xA+B)^*(xA+B))] \\ &= E_r[\lambda((T(xA+B))^* T(xA+B))] \\ &= E_r[\lambda((xT(A)+T(B))^*(xT(A)+T(B)))] . \end{aligned}$$

By Lemma 2 T is nonsingular so $T(B)$ ranges over $M_{m,n}$ as B does. Hence, by Lemma 1, $\rho(T(A)) \leqq 1$. But $T(A) \neq 0$ since $\rho(A) = 1$. Thus $\rho(T(A)) = 1$.

At this point we invoke [7: p. 1219] that tells us that a linear transformation on $M_{m,n}$ which preserves rank 1 has the desired form:

$$T(A) = UAV \text{ for all } A \in M_{m,n}$$

or

$$T(A) = UA'V \text{ for all } A \in M_{m,n} \,,$$

where U and V are nonsingular m-square and n-square matrices respectively and the second eventuality occurs only if $m = n$. The proof of the theorem will be complete if we show

LEMMA 4. *U and V may be chosen to be unitary.*

Proof. We show this when T has the form (2); if T has the form (3) the argument is essentially the same. Let $V = HP$ and $U = QK$ where H and K are positive definite Hermitian and P and Q are unitary. Then

$$\begin{aligned} E_r[\lambda(A^*A)] &= E_r[\lambda((UAV)^*(UAV))] \\ &= E_r[\lambda(V^*A^*U^*UAV)] \\ &= E_r[\lambda(P^*HA^*K^2AHP)] \\ &= E_r[\lambda(HA^*K^2AH)] \\ &= E_r[\lambda(H^2A^*K^2A)] \end{aligned}$$

for all A. Let $H = XDX^*$, $K = YGY^*$, X and Y unitary, D and G diagonal matrices with main diagonals $d_1, \cdots, d_n$ and $g_1, \cdots, g_n$ respectively. Then

$$\begin{aligned} E_r[\lambda(A^*A)] &= E_r[\lambda(XD^2X^*A^*YG^2Y^*A)] \\ &= E_r[\lambda(D^2B^*G^2B)] \end{aligned}$$

where $B = Y^*AX$. Now

$$\lambda(A^*A) = \lambda(XB^*Y^*YBX^*) = \lambda(B^*B)$$

and hence

$$E_r[\lambda(B^*B)] = E_r]\lambda(D^2B^*G^2B)]$$

for all B. Choose B as follows:

$$B = \left[\begin{array}{ccc|c} 0 & & 1 & \\ & & \cdot & \\ & \cdot & & 0 \\ \cdot & & & \\ 1 & & 0 & \\ \hline & 0 & & 0 \end{array}\right]$$

in which the upper left block is the indicated r-square permutation matrix. Then clearly $E_r[\lambda(B^*B)] = 1$ and

$$D^2B^*G^2B = \left[\begin{array}{ccccc|c} d_1^2g_r^2 & & & & & \\ & d_2^2g_{r-1}^2 & & 0 & & \\ & & \ddots & & & 0 \\ & 0 & & d_{r-1}^2g_2^2 & & \\ & & & & d_r^2g_1^2 & \\ \hline & & 0 & & & 0 \end{array}\right]$$

Thus

$$1 = E_r[\lambda(B^*B)] = \prod_{j=1}^{r} d_j^2g_j^2 .$$

Now set $D^2 = RD_\sigma^2R$ where R is an n-square permutation matrix and D_σ^2 is a diagonal matrix obtained from D^2 by a permutation σ of the diagonal elements of D^2. Then

$$\begin{aligned}\lambda(D^2B^*G^2B) &= \lambda(RD_\sigma^2R^*B^*G^2B) \\ &= \lambda(D_\sigma^2(BR)^*G^2(BR)) \\ &= \lambda(D_\sigma^2C^*G^2C) ,\end{aligned}$$

where $C = BR$, and

$$\lambda(B^*B) = \lambda(R^*B^*BR) = \lambda(C^*C) .$$

Therefore

$$E_r[\lambda(C^*C)] = E_r[\lambda(D_\sigma^2C^*G^2C)]$$

for all C. It follows that

$$\prod_{i=1}^{r} d_{\sigma(i)}^2g_i^2 = 1$$

for any permutation σ of $1, \cdots, n$. From this we conclude that

$$d_1^2 = \cdots = d_n^2 = d^2$$

and similarly

$$g_1^2 = \cdots = g_n^2 = g^2 .$$

Then $G = gI$, $D = dI$ and $U = gQ$, $V = dP$, i.e. U, V are scalar multiples of unitary matrices. Now,

$$\begin{aligned} E_r[\lambda(A^*A)] &= E_r[\lambda((UAV)^*(UAV))] \\ &= E_r[\lambda(|g|^2 V^*A^*AV)] \\ &= E_r[\lambda(|gd|^2 A^*A)] \\ &= |gd|^{2r} E_r[\lambda(A^*A)] \,. \end{aligned}$$

Hence $|gd|^{2r} = 1$ and we can choose U and V to be gdQ and P which are unitary. This completes the proof.

We remark that in case $r = 1$ T does not necessarily have the form indicated in (2) and (3). For

$$E_1[\lambda(A^*A)] = tr(A^*A) = \sum_{(i,j)=(1,1)}^{(m,n)} |a_{ij}|^2 \,,$$

and if T is merely a unitary operator on $M_{m,n}$

$$E_1[\lambda((T(A))^*T(A))] = E_1[\lambda(A^*A)] \,.$$

For example T can be the operator that interchanges the (1, 2) and (2, 1) elements of every $A \in M_{m,n}$ (assume $m, n > 2$) and this cannot be accomplished by any pre- and post-multipliplication as in (2) or (3).

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U. S. NATIONAL BUREAU OF STANDARDS, WASHINGTON, D. C.

THE UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 12, No. 1 January, 1962