

Pacific Journal of Mathematics

A NOTE ON WEAK SEQUENTIAL CONVERGENCE

RALPH DAVID MCWILLIAMS

A NOTE ON WEAK SEQUENTIAL CONVERGENCE

R. D. MCWILLIAMS

1. Let X be a real Banach space, J_x the canonical mapping from X into X^{**} , and $K(X)$ the set of all elements F in X^{**} which are X^* -limits of sequences in $J_x X$. Thus $F \in K(X)$ if and only if there exists a sequence $\{x_n\}$ in X such that

$$(1.1) \quad F(f) = \lim_n f(x_n)$$

for all $f \in X^*$. While the closure of $J_x X$ in the X^* -topology is X^{**} [4, p. 229], it is not true in general that $K(X) = X^{**}$. By using properties of the space of continuous real functions defined on a real interval, we shall prove that the subspace $K(X)$ is norm-closed in X^{**} .

2. If x is a bounded real function defined on a closed interval $[a, b]$, let $\|x\| = \sup \{|x(s)| : a \leq s \leq b\}$. If x is a bounded Baire function of the first class, then there exists a sequence $\{x_n\} \subset \mathcal{C}[a, b]$ such that $x(s) = \lim_n x_n(s)$ for all $s \in [a, b]$ and $\|x_n\| = \|x\|$ for all n [2, p. 138]. However, if a bounded function x is the pointwise limit of an unbounded sequence of elements of a subspace X of \mathcal{C} , then it is not necessarily true that x is the pointwise limit of a bounded sequence in X .

LEMMA 1. *Let X be a subspace of \mathcal{C} , and let x be a real function which is the pointwise limit of a bounded sequence in X . Then there exists a sequence $\{x_n\}$ in X such that x is the pointwise limit of $\{x_n\}$ and $\|x_n\| = \|x\|$ for all n .*

Proof. If $\{y_n\}$ is a sequence in X which converges pointwise to x , with $\sup_n \|y_n\| = M < \infty$, let continuous functions $\varphi, \varphi_1, \varphi_2, \dots$ be defined by

$$(2.1) \quad \begin{cases} \varphi(s) \equiv \|x\| \\ \varphi_n(s) = \max(y_n(s), \|x\|) \end{cases}$$

for all $s \in [a, b]$. Then $\{\varphi_n\}$ converges to φ in the \mathcal{C}^* -topology of \mathcal{C} [1, p. 224], and hence [3, p. 36] for each positive integer n there exist nonnegative numbers a_{n1}, \dots, a_{nk_n} such that

$$(2.2) \quad \sum_{k=1}^{k_n} a_{nk} = 1, \quad \left| \left| \sum_{k=1}^{k_n} a_{nk} \varphi_{n+k} - \varphi \right| \right| < n^{-1}.$$

Define $\{z_n\} \subset X$ by

$$(2.3) \quad z_n = \sum_{k=1}^{k_n} a_{nk} y_{n+k}.$$

Then $\{z_n\}$ converges pointwise to x , and $-M \leq z_n(s) \leq \|x\| + n^{-1}$ for each n .

If a sequence $\{\psi_n\}$ is now defined in \mathcal{E} by $\psi_n = \min(z_n, -\varphi)$, an argument like that used with $\{\varphi_n\}$ shows that there exist, for each n , nonnegative numbers b_{n1}, \dots, b_{nj_n} such that

$$(2.4) \quad \sum_{j=1}^{j_n} b_{nj} = 1, \quad \left| \sum_{j=1}^{j_n} b_{nj} \psi_{n+j} + \varphi \right| < n^{-1}.$$

If $\{u_n\} \subset X$ is defined by

$$(2.5) \quad u_n = \sum_{j=1}^{j_n} b_{nj} z_{n+j},$$

then x is the pointwise limit of $\{u_n\}$, and $\|u_n\| \rightarrow \|x\|$ as $n \rightarrow \infty$. Since it may be assumed that $\|u_n\| \neq 0$ for each n , the desired sequence $\{x_n\}$ is obtained by defining $x_n = (\|x\|/\|u_n\|) u_n$.

3. The conjugate space \mathcal{E}^* of \mathcal{E} is equivalent with the space of all finite regular signed Borel measures on $[a, b]$, under a mapping U such that if $f \in \mathcal{E}^*$ and $\mu_f = Uf$, then

$$(3.1) \quad f(x) = \int_a^b x d\mu_f$$

for all $x \in \mathcal{E}$ [4, p. 397]. It follows that if X is a closed subspace of \mathcal{E} and $F \in X^{**}$, then $F \in K(X)$ if and only if there exists a bounded, pointwise-convergent sequence $\{y_n\}$ in X with the property that

$$(3.2) \quad F(f|X) = \int_a^b (\lim y_n) d\mu_f$$

for all $f \in \mathcal{E}^*$.

LEMMA 2. *If X is a real Banach space and $F \in K(X)$, then there exists a sequence $\{x_n\}$ in X such that F is the X^* -limit of $\{J_x x_n\}$ and $\|x_n\| = \|F\|$ for all n .*

Proof. Case 1. If X is a closed subspace of \mathcal{E} and $F \in K(X)$, there must be a bounded, pointwise-convergent sequence $\{y_n\} \subset X$ such that (3.2) holds for all $f \in \mathcal{E}^*$. If $x(s) = \lim_n y_n(s)$ for $a \leq s \leq b$, then by Lemma 1 there exists a sequence $\{x_n\}$ in X such that x is the pointwise limit of $\{x_n\}$ and $\|x_n\| = \|x\|$ for all n . Thus F is the X^* -limit of $\{J_x x_n\}$ and it is easily verified that $\|F\| = \|x_n\|$ for each n .

Case 2. If X is an arbitrary real Banach space and $F \in K(X)$, then there is a sequence $\{y_n\}$ in X such that F is the X^* -limit of $\{J_x y_n\}$. If Y is the closed subspace of X generated by $\{y_n\}$, we can define

$G \in Y^{**}$ by

$$(3.3) \quad G(f|Y) = F(f) \text{ for all } f \in X^*,$$

and this definition is unambiguous since F is the X^* -limit of a sequence in $J_X Y$. It is easy to verify that $G \in K(Y)$ and $\|G\| = \|F\|$. Since Y is separable, Y is equivalent with a closed subspace of \mathcal{C} [1, p. 185], and hence by Case 1, there is a sequence $\{x_n\}$ in Y such that G is the Y^* -limit of $\{J_Y x_n\}$ and $\|x_n\| = \|G\| = \|F\|$ for all n . Finally, if $f \in X^*$, then

$$(3.4) \quad F(f) = G(f|Y) = \lim_n f(x_n),$$

so F is the X^* -limit of $\{J_X x_n\}$, and the lemma is proved.

4. THEOREM. *If X is a real Banach space, then $K(X)$ is norm-closed in X^{**} .*

Proof. If $F \in \overline{K(X)}$, then there is a sequence $\{F_n\}$ in $K(X)$ such that $F_n \rightarrow F$ in norm, and $\|F_n - F_{n-1}\| < 2^{-n}$ for each $n > 1$. If we let $F_0 = 0$, then by Lemma 2 there exists, for each $n \geq 1$, a sequence $\{x_{nk}\}$ in X such that $\|x_{nk}\| = \|F_n - F_{n-1}\|$ for all k and such that $F_n - F_{n-1}$ is the X^* -limit of $\{J_X x_{nk}\}$.

For each k the series $\sum_{n=1}^\infty x_{nk}$ converges to an element $x_k \in X$ such that

$$\left\| x_k - \sum_{n=1}^j x_{nk} \right\| < 2^{-j} \text{ for each } j.$$

Given $0 \neq f \in X^*$ and $\varepsilon > 0$, there exist positive integers J and K such that $2^{-J} < \varepsilon/(3\|f\|)$ and $|F_J(f) - f(\sum_{n=1}^J x_{nk})| < \varepsilon/3$ for all $k \geq K$. Hence for $k \geq K$,

$$(4.1) \quad |F(f) - f(x_k)| \leq |(F - F_J)(f)| + \left| F_J(f) - f\left(\sum_{n=1}^J x_{nk}\right) \right| + \left| f\left(\sum_{n=1}^J x_{nk}\right) - f(x_k) \right| < \varepsilon,$$

so that F is the X^* -limit of $\{J_X x_k\}$.

REFERENCES

1. S. Banach, *Théorie des opérations linéaires*, Warsaw, 1932
2. C. Goffman, *Real functions*, New York, Rinehart, 1953.
3. E. Hille and R. S. Phillips, *Functional analysis and semigroups*, Amer. Math. Soc. Colloquium Publications, **31**, 1957.
4. A. E. Taylor, *Functional analysis*, New York, Wiley, 1958.

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RALPH S. PHILLIPS

Stanford University
Stanford, California

M. G. ARSOVE

University of Washington
Seattle 5, Washington

A. L. WHITEMAN

University of Southern California
Los Angeles 7, California

LOWELL J. PAIGE

University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH

T. M. CHERRY

D. DERRY

M. OHTSUKA

H. L. ROYDEN

E. SPANIER

E. G. STRAUS

F. WOLF

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Pacific Journal of Mathematics

Vol. 12, No. 1

January, 1962

Jonathan L. Alperin, <i>Groups with finitely many automorphisms</i>	1
Martin Arthur Arkowitz, <i>The generalized Whitehead product</i>	7
John D. Baum, <i>Instability and asymptoticity in topological dynamics</i>	25
William Aaron Beyer, <i>Hausdorff dimension of level sets of some Rademacher series</i>	35
Frank Herbert Brownell, III, <i>A note on Cook's wave-matrix theorem</i>	47
Gulbank D. Chakerian, <i>An inequality for closed space curves</i>	53
Inge Futtrup Christensen, <i>Some further extensions of a theorem of Marcinkiewicz</i>	59
Charles Vernon Coffman, <i>Linear differential equations on cones in Banach spaces</i>	69
Eckford Cohen, <i>Arithmetical notes. III. Certain equally distributed sets of integers</i>	77
John Irving Derr and Angus E. Taylor, <i>Operators of meromorphic type with multiple poles of the resolvent</i>	85
Jacob Feldman, <i>On measurability of stochastic processes in products space</i>	113
Robert S. Freeman, <i>Closed extensions of the Laplace operator determined by a general class of boundary conditions, for unbounded regions</i>	121
Robert E. Fullerton, <i>Geometric structure of absolute basis systems in a linear topological space</i>	137
Dieter Gaier, <i>On conformal mapping of nearly circular regions</i>	149
Andrew Mattei Gleason and Hassler Whitney, <i>The extension of linear functionals defined on H^∞</i>	163
Seymour Goldberg, <i>Closed linear operators and associated continuous linear operators</i>	183
Basil Gordon, Aviezri Siegmund Fraenkel and Ernst Gabor Straus, <i>On the determination of sets by the sets of sums of a certain order</i>	187
Branko Grünbaum, <i>The dimension of intersections of convex sets</i>	197
Paul Daniel Hill, <i>On the number of pure subgroups</i>	203
Robert Peter Holten, <i>Generalized Goursat problem</i>	207
Alfred Horn, <i>Eigenvalues of sums of Hermitian matrices</i>	225
Henry C. Howard, <i>Oscillation and nonoscillation criteria for $y''(x) + f(y(x))p(x) = 0$</i>	243
Taqdir Husain, <i>S-spaces and the open mapping theorem</i>	253
Richard Eugene Isaac, <i>Markov processes and unique stationary probability measures</i>	273
John Rolfe Isbell, <i>Supercomplete spaces</i>	287
John Rolfe Isbell, <i>On finite-dimensional uniform spaces. II</i>	291
N. Jacobson, <i>A note on automorphisms of Lie algebras</i>	303
Antoni A. Kosinski, <i>A theorem on families of acyclic sets and its applications</i>	317
Marvin David Marcus and H. Minc, <i>The invariance of symmetric functions of singular values</i>	327
Ralph David McWilliams, <i>A note on weak sequential convergence</i>	333
John W. Milnor, <i>On axiomatic homology theory</i>	337
Victor Julius Mizel and Malempati Madhusudana Rao, <i>Nonsymmetric projections in Hilbert space</i>	343
Calvin Cooper Moore, <i>On the Frobenius reciprocity theorem for locally compact groups</i>	359
Donald J. Newman, <i>The Gibbs phenomenon for Hausdorff means</i>	367
Jack Segal, <i>Convergence of inverse systems</i>	371
Józef Siciak, <i>On function families with boundary</i>	375
Hyman Joseph Zimmerberg, <i>Two-point boundary conditions linear in a parameter</i>	385