

# Pacific Journal of Mathematics

**A NOTE ON WEAK SEQUENTIAL CONVERGENCE**

RALPH DAVID MCWILLIAMS

# A NOTE ON WEAK SEQUENTIAL CONVERGENCE

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1. Let  $X$  be a real Banach space,  $J_X$  the canonical mapping from  $X$  into  $X^{**}$ , and  $K(X)$  the set of all elements  $F$  in  $X^{**}$  which are  $X^*$ -limits of sequences in  $J_X X$ . Thus  $F \in K(X)$  if and only if there exists a sequence  $\{x_n\}$  in  $X$  such that

$$(1.1) \quad F(f) = \lim_n f(x_n)$$

for all  $f \in X^*$ . While the closure of  $J_X X$  in the  $X^*$ -topology is  $X^{**}$  [4, p. 229], it is not true in general that  $K(X) = X^{**}$ . By using properties of the space of continuous real functions defined on a real interval, we shall prove that the subspace  $K(X)$  is norm-closed in  $X^{**}$ .

2. If  $x$  is a bounded real function defined on a closed interval  $[a, b]$ , let  $\|x\| = \sup \{|x(s)| : a \leq s \leq b\}$ . If  $x$  is a bounded Baire function of the first class, then there exists a sequence  $\{x_n\} \subset \mathcal{C}[a, b]$  such that  $x(s) = \lim_n x_n(s)$  for all  $s \in [a, b]$  and  $\|x_n\| = \|x\|$  for all  $n$  [2, p. 138]. However, if a bounded function  $x$  is the pointwise limit of an unbounded sequence of elements of a subspace  $X$  of  $\mathcal{C}$ , then it is not necessarily true that  $x$  is the pointwise limit of a bounded sequence in  $X$ .

LEMMA 1. *Let  $X$  be a subspace of  $\mathcal{C}$ , and let  $x$  be a real function which is the pointwise limit of a bounded sequence in  $X$ . Then there exists a sequence  $\{x_n\}$  in  $X$  such that  $x$  is the pointwise limit of  $\{x_n\}$  and  $\|x_n\| = \|x\|$  for all  $n$ .*

*Proof.* If  $\{y_n\}$  is a sequence in  $X$  which converges pointwise to  $x$ , with  $\sup_n \|y_n\| = M < \infty$ , let continuous functions  $\varphi, \varphi_1, \varphi_2, \dots$  be defined by

$$(2.1) \quad \begin{aligned} \varphi(s) &\equiv \|x\| \\ \varphi_n(s) &= \max(y_n(s), \|x\|) \end{aligned}$$

for all  $s \in [a, b]$ . Then  $\{\varphi_n\}$  converges to  $\varphi$  in the  $\mathcal{C}^*$ -topology of  $\mathcal{C}$  [1, p. 224], and hence [3, p. 36] for each positive integer  $n$  there exist nonnegative numbers  $a_{n1}, \dots, a_{nk_n}$  such that

$$(2.2) \quad \sum_{k=1}^{k_n} a_{nk} = 1, \quad \left| \sum_{k=1}^{k_n} a_{nk} \varphi_{n+k} - \varphi \right| < n^{-1}.$$

Define  $\{z_n\} \subset X$  by

$$(2.3) \quad z_n = \sum_{k=1}^{k_n} a_{nk} y_{n+k}.$$

Then  $\{z_n\}$  converges pointwise to  $x$ , and  $-M \leq z_n(s) \leq \|x\| + n^{-1}$  for each  $n$ .

If a sequence  $\{\psi_n\}$  is now defined in  $\mathcal{E}$  by  $\psi_n = \min(z_n, -\varphi)$ , an argument like that used with  $\{\varphi_n\}$  shows that there exist, for each  $n$ , nonnegative numbers  $b_{n1}, \dots, b_{nj_n}$  such that

$$(2.4) \quad \sum_{j=1}^{j_n} b_{nj} = 1, \quad \left| \left| \sum_{j=1}^{j_n} b_{nj} \psi_{n+j} + \varphi \right| \right| < n^{-1}.$$

If  $\{u_n\} \subset X$  is defined by

$$(2.5) \quad u_n = \sum_{j=1}^{j_n} b_{nj} z_{n+j},$$

then  $x$  is the pointwise limit of  $\{u_n\}$ , and  $\|u_n\| \rightarrow \|x\|$  as  $n \rightarrow \infty$ . Since it may be assumed that  $\|u_n\| \neq 0$  for each  $n$ , the desired sequence  $\{x_n\}$  is obtained by defining  $x_n = (\|x\|/\|u_n\|) u_n$ .

3. The conjugate space  $\mathcal{E}^*$  of  $\mathcal{E}$  is equivalent with the space of all finite regular signed Borel measures on  $[a, b]$ , under a mapping  $U$  such that if  $f \in \mathcal{E}^*$  and  $\mu_f = Uf$ , then

$$(3.1) \quad f(x) = \int_a^b x d\mu_f$$

for all  $x \in \mathcal{E}$  [4, p. 397]. It follows that if  $X$  is a closed subspace of  $\mathcal{E}$  and  $F \in X^{**}$ , then  $F \in K(X)$  if and only if there exists a bounded, pointwise-convergent sequence  $\{y_n\}$  in  $X$  with the property that

$$(3.2) \quad F(f|X) = \int_a^b (\lim y_n) d\mu_f$$

for all  $f \in \mathcal{E}^*$ .

**LEMMA 2.** *If  $X$  is a real Banach space and  $F \in K(X)$ , then there exists a sequence  $\{x_n\}$  in  $X$  such that  $F$  is the  $X^*$ -limit of  $\{J_x x_n\}$  and  $\|x_n\| = \|F\|$  for all  $n$ .*

*Proof. Case 1.* If  $X$  is a closed subspace of  $\mathcal{E}$  and  $F \in K(X)$ , there must be a bounded, pointwise-convergent sequence  $\{y_n\} \subset X$  such that (3.2) holds for all  $f \in \mathcal{E}^*$ . If  $x(s) = \lim_n y_n(s)$  for  $a \leq s \leq b$ , then by Lemma 1 there exists a sequence  $\{x_n\}$  in  $X$  such that  $x$  is the pointwise limit of  $\{x_n\}$  and  $\|x_n\| = \|x\|$  for all  $n$ . Thus  $F$  is the  $X^*$ -limit of  $\{J_x x_n\}$  and it is easily verified that  $\|F\| = \|x_n\|$  for each  $n$ .

*Case 2.* If  $X$  is an arbitrary real Banach space and  $F \in K(X)$ , then there is a sequence  $\{y_n\}$  in  $X$  such that  $F$  is the  $X^*$ -limit of  $\{J_x y_n\}$ . If  $Y$  is the closed subspace of  $X$  generated by  $\{y_n\}$ , we can define

$G \in Y^{**}$  by

$$(3.3) \quad G(f|Y) = F(f) \text{ for all } f \in X^*,$$

and this definition is unambiguous since  $F$  is the  $X^*$ -limit of a sequence in  $J_X Y$ . It is easy to verify that  $G \in K(Y)$  and  $\|G\| = \|F\|$ . Since  $Y$  is separable,  $Y$  is equivalent with a closed subspace of  $\mathcal{C}$  [1, p. 185], and hence by Case 1, there is a sequence  $\{x_n\}$  in  $Y$  such that  $G$  is the  $Y^*$ -limit of  $\{J_Y x_n\}$  and  $\|x_n\| = \|G\| = \|F\|$  for all  $n$ . Finally, if  $f \in X^*$ , then

$$(3.4) \quad F(f) = G(f|Y) = \lim_n f(x_n),$$

so  $F$  is the  $X^*$ -limit of  $\{J_X x_n\}$ , and the lemma is proved.

4. THEOREM. *If  $X$  is a real Banach space, then  $K(X)$  is norm-closed in  $X^{**}$ .*

*Proof.* If  $F \in \overline{K(X)}$ , then there is a sequence  $\{F_n\}$  in  $K(X)$  such that  $F_n \rightarrow F$  in norm, and  $\|F_n - F_{n-1}\| < 2^{-n}$  for each  $n > 1$ . If we let  $F_0 = 0$ , then by Lemma 2 there exists, for each  $n \geq 1$ , a sequence  $\{x_{nk}\}$  in  $X$  such that  $\|x_{nk}\| = \|F_n - F_{n-1}\|$  for all  $k$  and such that  $F_n - F_{n-1}$  is the  $X^*$ -limit of  $\{J_X x_{nk}\}$ .

For each  $k$  the series  $\sum_{n=1}^{\infty} x_{nk}$  converges to an element  $x_k \in X$  such that

$$\left\| x_k - \sum_{n=1}^j x_{nk} \right\| < 2^{-j} \text{ for each } j.$$

Given  $0 \neq f \in X^*$  and  $\varepsilon > 0$ , there exist positive integers  $J$  and  $K$  such that  $2^{-j} < \varepsilon/(3\|f\|)$  and  $|F_j(f) - f(\sum_{n=1}^J x_{nk})| < \varepsilon/3$  for all  $k \geq K$ . Hence for  $k \geq K$ ,

$$(4.1) \quad |F(f) - f(x_k)| \leq |(F - F_J)(f)| + \left| F_J(f) - f\left(\sum_{n=1}^J x_{nk}\right) \right| + \left| f\left(\sum_{n=1}^J x_{nk} - x_k\right) \right| < \varepsilon,$$

so that  $F$  is the  $X^*$ -limit of  $\{J_X x_k\}$ .

### REFERENCES

1. S. Banach, *Théorie des opérations linéaires*, Warsaw, 1932
2. C. Goffman, *Real functions*, New York, Rinehart, 1953.
3. E. Hille and R. S. Phillips, *Functional analysis and semigroups*, Amer. Math. Soc. Colloquium Publications, **31**, 1957.
4. A. E. Taylor, *Functional analysis*, New York, Wiley, 1958.



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