A NOTE ON WEAK SEQUENTIAL CONVERGENCE

RALPH DAVID MCWILLIAMS
A NOTE ON WEAK SEQUENTIAL CONVERGENCE

R. D. McWilliams

Let $X$ be a real Banach space, $J_X$ the canonical mapping from $X$ into $X^{**}$, and $K(X)$ the set of all elements $F$ in $X^{**}$ which are $X^*$-limits of sequences in $J_X X$. Thus $F \in K(X)$ if and only if there exists a sequence $\{x_n\}$ in $X$ such that

\[(1.1) \quad F(f) = \lim_{n} f(x_n)\]

for all $f \in X^*$. While the closure of $J_X X$ in the $X^*$-topology is $X^{**}$ [4, p. 229], it is not true in general that $K(X) = X^{**}$. By using properties of the space of continuous real functions defined on a real interval, we shall prove that the subspace $K(X)$ is norm-closed in $X^{**}$.

2. If $x$ is a bounded real function defined on a closed interval $[a, b]$, let $\|x\| = \sup \{|x(s)| : a \leq s \leq b\}$. If $x$ is a bounded Baire function of the first class, then there exists a sequence $\{x_n\} \subset C[a, b]$ such that $x(s) = \lim_{n} x_n(s)$ for all $s \in [a, b]$ and $\|x_n\| = \|x\|$ for all $n$ [2, p. 138]. However, if a bounded function $x$ is the pointwise limit of an unbounded sequence of elements of a subspace $X$ of $C$, then it is not necessarily true that $x$ is the pointwise limit of a bounded sequence in $X$.

**Lemma 1.** Let $X$ be a subspace of $C$, and let $x$ be a real function which is the pointwise limit of a bounded sequence in $X$. Then there exists a sequence $\{x_n\}$ in $X$ such that $x$ is the pointwise limit of $\{x_n\}$ and $\|x_n\| = \|x\|$ for all $n$.

**Proof.** If $\{y_n\}$ is a sequence in $X$ which converges pointwise to $x$, with $\sup_n \|y_n\| = M < \infty$, let continuous functions $\varphi, \varphi_1, \varphi_2, \cdots$ be defined by

\[\varphi(s) = \|x\|, \quad \varphi_n(s) = \max(y_n(s), \|x\|)\]

for all $s \in [a, b]$. Then $\{\varphi_n\}$ converges to $\varphi$ in the $C^*$-topology of $C$ [1, p. 224], and hence [3, p. 36] for each positive integer $n$ there exist nonnegative numbers $a_{n1}, \cdots, a_{nk_n}$ such that

\[(2.2) \quad \sum_{k=1}^{k_n} a_{nk} = 1, \quad \left\| \sum_{k=1}^{k_n} a_{nk} \varphi_{n+k} - \varphi \right\| < n^{-1}.\]

Define $\{z_n\} \subset X$ by

\[(2.3) \quad z_n = \sum_{k=1}^{k_n} a_{nk} y_{n+k}.\]

Received February 2, 1961.
Then \( \{z_n\} \) converges pointwise to \( x \), and 
\[-M \leq z_n(s) \leq \|x\| + n^{-1} \]
for each \( n \).

If a sequence \( \{\psi_n\} \) is now defined in \( \mathcal{C} \) by \( \psi_n = \min(z_n, -\varphi) \), an argument like that used with \( \{\varphi_n\} \) shows that there exist, for each \( n \), nonnegative numbers \( b_{n_1}, \ldots, b_{n_{j_n}} \) such that
\[
\sum_{j=1}^{j_n} b_{n_j} = 1, \quad \left\| \sum_{j=1}^{j_n} b_{n_j} \psi_{n + j} + \varphi \right\| < n^{-1}.
\]

If \( \{u_n\} \subset X \) is defined by
\[
u_n = \sum_{j=1}^{j_n} b_{n_j} z_{n+j},
\]
then \( x \) is the pointwise limit of \( \{u_n\} \), and \( \|u_n\| \to \|x\| \) as \( n \to \infty \). Since it may be assumed that \( \|u_n\| \neq 0 \) for each \( n \), the desired sequence \( \{x_n\} \) is obtained by defining \( x_n = (\|x\|/\|u_n\|) \) \( u_n \).

3. The conjugate space \( \mathcal{C}^* \) of \( \mathcal{C} \) is equivalent with the space of all finite regular signed Borel measures on \( [a, b] \), under a mapping \( U \) such that if \( f \in \mathcal{C}^* \) and \( \mu_f = Uf \), then
\[
f(x) = \int_a^b xd\mu_f
\]
for all \( x \in \mathcal{C} \) [4, p. 397]. It follows that if \( X \) is a closed subspace of \( \mathcal{C} \) and \( F \in X^{**} \), then \( F \in K(X) \) if and only if there exists a bounded, pointwise-convergent sequence \( \{y_n\} \) in \( X \) with the property that
\[
F(f \mid X) = \int_a^b (\lim y_n) d\mu_f
\]
for all \( f \in \mathcal{C}^* \).

**Lemma 2.** If \( X \) is a real Banach space and \( F \in K(X) \), then there exists a sequence \( \{x_n\} \) in \( X \) such that \( F \) is the \( X^* \)-limit of \( \{J_x x_n\} \) and \( \|x_n\| = \|F\| \) for all \( n \).

**Proof.** Case 1. If \( X \) is a closed subspace of \( \mathcal{C} \) and \( F \in K(X) \), there must be a bounded, pointwise-convergent sequence \( \{y_n\} \subset X \) such that (3.2) holds for all \( f \in \mathcal{C}^* \). If \( x(s) = \lim y_n(s) \) for \( a \leq s \leq b \), then by Lemma 1 there exists a sequence \( \{x_n\} \) in \( X \) such that \( x \) is the pointwise limit of \( \{x_n\} \) and \( \|x_n\| = \|x\| \) for all \( n \). Thus \( F \) is the \( X^* \)-limit of \( \{J_x x_n\} \) and it is easily verified that \( \|F\| = \|x_n\| \) for each \( n \).

Case 2. If \( X \) is an arbitrary real Banach space and \( F \in K(X) \), then there is a sequence \( \{y_n\} \) in \( X \) such that \( F \) is the \( X^* \)-limit of \( \{J_x y_n\} \). If \( Y \) is the closed subspace of \( X \) generated by \( \{y_n\} \), we can define
\( G \in Y^{**} \) by

\[
(3.3) \quad G(f|Y) = F(f) \text{ for all } f \in X^*,
\]

and this definition is unambiguous since \( F \) is the \( X^* \)-limit of a sequence in \( J_X Y \). It is easy to verify that \( G \in K(Y) \) and \( ||G|| = ||F|| \). Since \( Y \) is separable, \( Y \) is equivalent with a closed subspace of \( \mathcal{C} \) [1, p. 185], and hence by Case 1, there is a sequence \( \{x_n\} \) in \( Y \) such that \( G \) is the \( Y^* \)-limit of \( \{J_Y x_n\} \) and \( ||x_n|| = ||G|| = ||F|| \) for all \( n \). Finally, if \( f \in X^* \), then

\[
(3.4) \quad F(f) = G(f|Y) = \lim_n f(x_n)
\]

so \( F \) is the \( X^* \)-limit of \( \{J_X x_n\} \), and the lemma is proved.

4. Theorem. If \( X \) is a real Banach space, then \( K(X) \) is norm-closed in \( X^{**} \).

Proof. If \( F \in \overline{K(X)} \), then there is a sequence \( \{F_n\} \) in \( K(X) \) such that \( F_n \to F \) in norm, and \( ||F_n - F_{n-1}|| < 2^{-n} \) for each \( n > 1 \). If we let \( F_0 = 0 \), then by Lemma 2 there exists, for each \( n \geq 1 \), a sequence \( \{x_{nk}\} \) in \( X \) such that \( ||x_{nk}|| = ||F_n - F_{n-1}|| \) for all \( k \) and such that \( F_n - F_{n-1} \) is the \( X^* \)-limit of \( \{J_X x_{nk}\} \).

For each \( k \) the series \( \sum_{n=1}^{\infty} x_{nk} \) converges to an element \( x_k \in X \) such that

\[
\left| x_k - \sum_{n=1}^{j} x_{nk} \right| < 2^{-j} \text{ for each } j.
\]

Given \( 0 \neq f \in X^* \) and \( \varepsilon > 0 \), there exist positive integers \( J \) and \( K \) such that \( 2^{-j} < \varepsilon/(3||f||) \) and \( |F_J(f) - f(\sum_{n=1}^{J} x_{nk})| < \varepsilon/3 \) for all \( k \geq K \). Hence for \( k \geq K \),

\[
(4.1) \quad |F(f) - f(x_k)| \leq |(F - F_J)(f)| + |F_J(f) - f(\sum_{n=1}^{J} x_{nk})| + |f(\sum_{n=1}^{J} x_{nk} - x_k)| < \varepsilon,
\]

so that \( F \) is the \( X^* \)-limit of \( \{J_X x_k\} \).

REFERENCES

1. S. Banach, Théorie des opérations linéaires, Warsaw, 1932
2. C. Goffman, Real functions, New York, Rinehart, 1953.