REMARKS ON AFFINE SEMIGROUPS

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H. S. COLLINS

A problem of fundamental importance in the study of measure semigroups is the following: if $S$ is a compact topological semigroup and $\tilde{S}$ is the convolution semigroup (with the weak-* topology) of nonnegative normalized regular Borel measures on $S$, what relationship exists between a measure $\mu$ in $\tilde{S}$ and its carrier? In the paper numbered [9, Lemma 5], Wendel proved that when $S$ is a group and $\mu$ an idempotent in $\tilde{S}$, then carrier $\mu$ is a group and $\mu$ is Haar measure on carrier $\mu$. He proved further that the mapping $\mu \rightarrow \text{carrier } \mu$ is a one-to-one mapping from the set of idempotents of $\tilde{S}$ onto the set of closed subgroups of $S$. Glicksberg in [6] extended these results to the case when $S$ is an abelian semigroup. In addition he showed (when $S$ is a group or an abelian semigroup) the structure of the closed subgroups of $\tilde{S}$ to be quite simple: each closed subgroup of $\tilde{S}$ consists of the $G$-translates of Haar measure on some closed normal subgroup of a suitably chosen closed group $G$ of $S$.

It is our purpose in this paper to prove in § 2 that these properties are equivalent in general, each being equivalent to several other properties of some interest (see Theorem 2). One of these conditions is the geometric requirement that $\tilde{S}$ can contain no ‘parallelogram’ whose vertices are $\mu, \nu, \mu \nu, \text{ and } \nu \mu$, with all four of these measures idempotent and $\mu$ and $\nu$ distinct. A crucial lemma of independent interest is that found in Theorem 1 of § 1, where it is shown that a line segment of an affine semigroup (see [3] for definitions) which contains three distinct idempotents consists entirely of idempotents. Several corollaries are drawn from this theorem, among them the result that a compact affine semigroup consists of idempotents (i.e., is a band in the sense of [2]) if and only if it is rectangular, and that this occurs if and only if it is simple (i.e., contains no proper ideals).

References for terminology and notation used here may be found in [3, 6, 8, 9].

1. General affine semigroups. This section is devoted primarily to several results about general affine semigroups. However, we list first without proof two lemmas ([3, Theorem 3] and [4, Theorem 2]) needed in the sequel.

**Lemma 1.** Let $T$ be a compact affine topological semigroup and
let \( K \) be its kernel (=minimal ideal). Then

(a) Each minimal left or right ideal is convex.

(b) \( x \in K \) if and only if \( xTx = \{x\} \); in particular, each point of \( K \) is idempotent.

**Lemma 2.** Let \( S \) be a compact topological semigroup and \( \mu \) be an idempotent in \( \tilde{S} \). Then \( H = \text{carrier } \mu \) is a compact simple semigroup, and for each continuous complex function \( f \) on \( S \) the mapping \( y \rightarrow \int f(xy)d\mu(x) \) is constant on each minimal left ideal of \( H \).

**Theorem 1.** Suppose \( T \) is an affine semigroup and \( L \) is a line segment in \( T \). If there exist three distinct idempotents on \( L \), then \( L \) consists entirely of idempotents, and \( xLx = \{x\} \) for all \( x \in L \).

**Proof.** Let \( e, f, \) and \( g \) be distinct idempotents on \( L \), with \( e \) between \( f \) and \( g \). Then there exists \( 0 < a < 1 \) such that \( e = af + (1 - a)g \), so \( af + (1 - a)g = e = e^2 = a^2f + a(1 - a)fg + (1 - a)g \). Multiplication on the left by \( f \) yields \( af + (1 - a)fg = a^2f + a(1 - a)fg + a(1 - a)fgf + (1 - a)^2g \), or \( af = a^2f + a(1 - a)fgf \). Rewriting this as \( a(1 - a)f = a(1 - a)fgf \) and using the fact that \( a \) is neither zero nor one, we obtain \( f = fgf \). By similar arguments one can show \( gfg = g \), and it follows that both \( fg \) and \( gf \) are idempotents. Again using the fact that \( e \) is an idempotent, \( af + (1 - a)g = e = e^2 = a^2f + a(1 - a)fg + a(1 - a)fgf + (1 - a)^2g \). This can be rewritten as \( a(1 - a)f + a(1 - a)g = a(1 - a)fg + a(1 - a)fgf \), so \( f + g = fg + gf \). If now \( x \) is any point on \( L \), say \( x = bf + (1 - b)g \), then \( x^2 = b^2f + b(1 - b)fg + b(1 - b)gf + (1 - b)^2g = b^2f + b(1 - b)[f + g - gf] + (1 - b)gf + (1 - b)^2g = bf + (1 - b)g = x \), so each \( x \in L \) is an idempotent. By direct computation it then follows readily that \( xLx = \{x\} \), all \( x \in L \).

**Corollary 1.** Every element of an affine semigroup \( T \) is idempotent if and only if \( xTx = \{x\} \), all \( x \in T \). In addition, if \( T \) is a compact affine topological semigroup, the requirement that \( T \) be simple (i.e., \( T \) is its own kernel) is equivalent to each of the above conditions.

**Proof.** If \( xTx = \{x\} \) for all \( x \) in \( T \), then \( x^2 = x \), so \( x^3 = x^2x = xx^2x = x \). Conversely, if \( T \) consists of idempotents, fix \( x \) in \( T \). Then \( y \) in \( T \) implies the line segment \( L \) joining \( x \) and \( y \) contains more than two idempotents, so by the theorem \( x ye \in xLx = \{x\} \); i.e., \( xTx = \{x\} \).

When \( T \) is compact, Lemma 1 shows that \( T \) is simple if and only if \( xTx = \{x\} \), all \( x \in T \).

**Corollary 2.** When \( T \) is the convolution semigroup \( \tilde{S} \) of measures
on a compact semigroup $S$, then each of the conditions of the preceding corollary is equivalent to each of (1) the multiplication in $S$ is either left or right singular; i.e., $xy = x$ for all $x, y \in S$ or $xy = y$ for all $x, y \in S$, (2) the multiplication in $\tilde{S}$ is either left or right singular.

Proof. It was shown in [5, Corollary 3] that $\tilde{S}$ is simple if and only if (1) holds. Now it is clear that here (2) implies (1) since $S$ is a semigroup of $\tilde{S}$. To show the converse, let $C$ be the convex hull of $S$ in $\tilde{S}$. It is known [2, Lemmas 3.1 and 3.2] that $C$ is dense in $\tilde{S}$. From this fact and the requirements on the multiplication in $\tilde{S}$ it follows readily that (1) implies (2).

Corollary 3. If $T$ is an affine semigroup, the following are mutually equivalent:

1. there exist three distinct collinear idempotents in $T$.
2. there exist distinct idempotents $f$ and $g$ in $T$ such that $fg$ and $gf$ are also idempotents and $f + g = fg + gf$.
3. $T$ contains an affine semigroup affinely equivalent to either the closed unit square of the Euclidean plane under the multiplication $(x, y)(a, b) = (x, b)$ or the closed unit interval of reals under left or right singular multiplication.

Proof. It was seen in the proof of Theorem 1 that (1) implies (2). To prove (2) implies (3), let $f$ and $g$ be distinct idempotents, with $fg$ and $gf$ idempotent and $f + g = fg + gf$. Denote by $M$ the manifold generated by $\{f, g, gf\}$ (i.e., $M$ is composed of all sums of the form $af + bg + cgf$, with $a + b + c = 1$). Since $fg = f + g - gf$, it follows that $M \cap T$ contains the convex hull $C$ of $\{f, g, fg, gf\}$. If $gf$ is on the line through $f$ and $g$, say $gf = af + (1 - a)g$, then $gf = gff = af + (1 - a)gf$, so $af = agf$. If $a = 0$, then $gf = g$ and $fg = f + g - gf = f + g - g = f$. It is then easy to see that the closed line segment $L$ from $f$ to $g$ is a semigroup, with left singular multiplication. If $a \neq 0$, then $gf = f$ and $fg = f + g - gf = f + g - f = g$. In this case $L$ is a semigroup, with right singular multiplication.

In the alternate case, $gf$ is not on the line through $f$ and $g$. We use here the identities $gfg = g$ and $fgf = f$ (easily deducible from the equation $fg + gf = f + g$) to show that if $x$ and $y$ are any points of $C$ (say $x = af + bg + [1 - (a + b)]gf$ and $y = cf + dg + [1 - (c + d)]gf$, where $a, b, c, d \geq 0$), then $xy = af + dg + [1 - (a + d)]gf$. The mapping $x \rightarrow (a, b)$ can now be easily verified to be an affine equivalence between $C$ and the unit square, where the latter is given the multiplication $(a, b)(c, d) = (a, d)$. Thus (3) holds.

The final implication (3) implies (1) is obvious, for each of the three affine semigroup mentioned in (3) clearly contain entire line segments.
of idempotents.

**Corollary 4.** The kernel \( K \) of a compact affine topological semigroup \( T \) is non-convex if and only if there exist distinct points \( x \) and \( y \) of \( K \) such that the open line segment between \( x \) and \( y \) misses \( K \).

**Proof.** If such a pair of points exists it is obvious that \( K \) is non-convex. Conversely, if \( K \) is non-convex one can find distinct points \( x \) and \( y \) of \( K \) and a point of \( T \) outside \( K \) on the open line segment \( L \) joining \( x \) to \( y \). It is then clear (since by Lemma 1 every point of \( K \) is an idempotent) that \( L \) misses \( K \), for if \( L \) and \( K \) meet Theorem 1 implies \( \{z\} = zLz = zxz \subset zKz \subset K \), for all \( z \in L \). This concludes the proof.

The preceding corollary shows that the examples of nonconvex kernels given in [3, pp. 111–112] were the only possible kind, for in both of these the non-convexity was shown by exhibiting points \( x \) and \( y \) such that \( L \) missed \( K \). It seems likely that the only way in which a kernel can fail to be convex is for there to be in \( T \) a usual real interval semigroup whose two idempotents are in \( K \).

2. Measure semigroups. Preliminary to our main Theorem 2, several lemmas will be stated and proved. Throughout this section \( \mathcal{S} \) will be (as before) the convolution semigroup of measures on a compact semigroup \( S \). Recall that the carrier of a measure \( \mu \) in \( \mathcal{S} \) is the complement of the largest open set of \( S \) whose \( \mu \) measure is zero. A result needed repeatedly is the fact that the carrier of a product of two measures is the product of the carriers [6, Lemma 2.1]. We say, following Wallace, that a semigroup of \( S \) is *simple* if it contains no proper (two-sided) ideals. The proof of the following lemma is obvious, and is omitted. In Lemma 4, the *carrier* of a subset \( \Gamma \) of \( \mathcal{S} \) is the closure of the set \( \bigcup \{ \text{carrier } \mu : \mu \in \Gamma \} \).

**Lemma 3.** Let \( H \) be a compact semigroup of \( S \), let \( \mathcal{H} \) denote the semigroup of measures on \( H \), and let \( H' \) be the set of measures \( \mu \in \mathcal{S} \) such that carrier \( \mu \subset H \). Then \( \mathcal{H} \) and \( H' \) (the latter with the multiplication and topology inherited from \( \mathcal{S} \)) are affinely equivalent (both topologically and algebraically).

**Lemma 4.** Let \( \Gamma \) be a compact group in \( \mathcal{S} \) with \( \eta \) its identity element. Let \( H \) be the carrier of \( \eta \), and denote by \( G \) the carrier of \( \Gamma \). Then both \( H \) and \( G \) are compact simple semigroups of \( S \) and \( G \) and have the same idempotents. In particular, \( G \) is a group if and only if \( H \) is; in this case, \( H \) is a normal subgroup of \( G \) and \( \eta \) is Haar measure on \( H \).
Remark 4. Let $H$ be a compact semigroup such that $\hat{H}$ contains at most two distinct collinear idempotents. Then the kernel of $H$ is a group.

Proof. Let $\mu$ be in the kernel of $\hat{H}$. By Lemma 1, $\mu$ is idempotent; and $\mu\hat{H}$ and $\hat{H}\mu$ are convex. Since here $\hat{H}$ has at most two collinear idempotents, it is clear that $\mu\hat{H} = \{\mu\} = \hat{H}\mu$; i.e., $\mu$ is the zero of $\hat{H}$. But then (since $\mu$ is both right and left invariant) Rosen's result [7, Corollary 1] implies the kernel of $H$ is a group.

Theorem 2. The following conditions are mutually equivalent:
The carrier of each idempotent measure in $\tilde{S}$ is a group.

(2) No three idempotents of $\tilde{S}$ are collinear.

(3) $\tilde{S}$ contains no affine image of any of the three semigroups mentioned in Corollary 3.

(4) Every compact simple semigroup of $S$ is a group.

(5) The mapping $\mu \rightarrow \text{carrier } \mu$ is one-to-one onto between the set $\tilde{E}$ of idempotents of $\tilde{S}$ and the set of compact simple semigroups of $S$.

(6) The mapping $\mu \rightarrow \text{carrier } \mu$ is one-to-one on $E$.

(7) Each compact group of $\tilde{S}$ consists of the $G$-translates of Haar measure on a compact normal subgroup of some compact group $G$ of $S$.

Proof. (1) implies (2). Let $\mu, \nu \in \tilde{E}$, $0 < a, 0 < b$, and $a + b = 1$ be such that $\phi = a\mu + b\nu \in \tilde{E}$. Let $A = \text{carrier } \mu$ and $B = \text{carrier } \nu$. By (1), $A, B$ and $A \cup B = \text{carrier } \phi$ are groups. It follows then from Lemma 2 that $\mu, \nu, \phi$ are Haar measure on $A, B$, and $A \cup B$ respectively. Let $e, f, g$ be the identities of $A, B$, and $A \cup B$ respectively. It is then clear (since $A, B$ are subgroups of the group $A \cup B$) that $e = f = g$. Suppose there is $t$ in $B/A$ and let $x \in A$. Then $xt \in AB \subset (A \cup B)(A \cup B) \subset A \cup B$, so $xt \in A$ or $xt \in B$. If $xt \in A$, then (inverse of $x$ in $A$) $xt \in A$. This implies $t = ft = et \in A$, a contradiction. Thus $xt \in B$, so $x = xe = xf = xt \cdot (\text{inverse of } t \text{ in } B) \in BB \subset B$; thus $A \subset B$. But then $A \cup B = B$, so carrier $\phi = B = \text{carrier } \nu$. Since normalized Haar measure on the compact group $B$ is unique, it follows that $\phi = \nu$, so (2) is proved.

The equivalence of (2) and (3) follows immediately from Corollary 3 of § 1.

(2) implies (4). Let $H$ be a compact simple semigroup of $S$. It is clear (assuming (2)) that the $H'$ of Lemma 3 cannot contain three distinct collinear idempotents, so the same is true (by Lemma 3) of $\tilde{H}$. Lemma 5 then implies that $H$ (being its own kernel) is a group.

(4) implies (5). If $H = \text{carrier } \mu = \text{carrier } \nu$, with $\mu, \nu \in \tilde{E}$, then by (4) and Lemma 2, $\mu$ and $\nu$ are both normalized Haar measure on the group $H$. Thus $\mu = \nu$ and the mapping $\mu \rightarrow \text{carrier } \mu$ is one-to-one on $\tilde{E}$. To complete the proof of (4) implies (5), let $H$ be a compact simple semigroup of $S$. By (4), $H$ is a group, and then Haar measure $\mu$ on $H$ (extended to $S$, of course) is idempotent and carrier $\mu = H$; i.e., the mapping is onto.

(5) implies (6) is clear. To show (6) implies (2), suppose there exist three distinct collinear idempotents in $\tilde{S}$. There is then by Theorem 1 a nondegenerate line segment $L$ of idempotent measures. In particular then, there exist distinct measures $\mu$ and $\nu$ on $L$ such that carrier $\mu = \text{carrier } \nu$, contradicting (6).

(4) implies (7). Let $\Gamma$ be a compact group in $\tilde{S}$ with identity element $\eta$, let $G$ be the carrier of $\Gamma$, and let $H = \text{carrier } \eta$. By (4) and
Lemmas 4 and 2, $G$ and $H$ are groups with $H$ normal in $G$ and $\eta$ is Haar measure on $H$. Then the proof given by Glicksberg (starting on page 57 of [6] with the phrase “Now suppose $S$ is a (non-abelian) compact group—”) applies to our situation to prove (7) holds, for an examination of his proof reveals that all he needs there is that $H$ be a normal subgroup of the group $G$, with $\eta$ being Haar measure on $H$ (or one could apply Glicksberg's result to $\tilde{G}$).

To conclude, we show (7) implies (1). Let $\mu^G = \mu \in S$ and let $\Gamma$ be the maximal group containing $\mu$ [8, Theorem 2.1]. Then $\Gamma$ is a compact group of $\tilde{S}$ so by (7) there are compact groups $G$ and $H$ of $S$, with $H$ a normal subgroup of $G$, such that $\Gamma = \eta G$, where $\eta$ is Haar measure on $H$. The measure $\eta$ is then invariant on $H$ ($\eta x = x \eta = \eta$, all $x \in H$), so $\{\eta\} = \eta H \subset \eta G = \Gamma$ implies ($\Gamma$ being a group)$\eta = \mu$. Thus carrier $\mu = \text{carrier } \eta = H$, a group. This completes the theorem.

It has already been remarked that condition (1) of Theorem 2 holds in case $S$ is either a group or an abelian semigroup. More generally, this is true if the idempotents of $S$ commute. In fact, if $H$ is a compact simple semigroup of $S$ and $e$ and $f$ are idempotents of $H$, then $ef \in Hf \cap eH$ and $fe \in He \cap fH$. Since here $fe = ef$, this says that the maximal groups $eHe = eH \cap He$ and $fHf = fH \cap Hf$ of $H$ meet. However, two maximal groups which meet coincide [8, Theorem 2.1], so $eHe = fHf$ and $e = f$. But then $H$ has exactly one idempotent and so is a group [8, Theorem 4.3].

**References**

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