CONTINUOUSLY INVERTIBLE SPACES

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In [4] we introduced the concept of an invertible space. (A topological space $S$ is invertible if, for each open set $U$ in $S$, there is an inverting homeomorphism for $U$, $h : S \rightarrow S$, of $S$ onto itself such that $h(S - U)$ lies in $U$.) This paper is a continuation of the investigation of invertibility in which we concentrate upon that aspect contained in the following definition: A topological space $S$ is continuously invertible if it is invertible and if, for each open set in $S$, there is an inverting homeomorphism that is isotopic to the identity mapping.

It is easy to see that the $n$-sphere $S^n$ enjoys this property for, given an open set $U$ in $S^n$, there is an inverting homeomorphism for $U$ in each component of the space of homeomorphisms of $S^n$ onto itself. It was this observation which suggested our present investigation. We acknowledge a debt of gratitude to the unknown referee of our paper [4] who brought to our attention the papers by Dancer [2], L. Whyburn [6] and Wilder [7]. While they were of little assistance, these papers do contain results allied to our own and hence provide us with a link to the past.

Let $S$ be a continuously invertible space and let $\mathcal{G}(S)$ denote the group of homeomorphisms of $S$ onto itself. (Note that each element of $\mathcal{G}(S)$ is an inverting homeomorphism for some open set, the identity mapping included.) Let $\mathcal{G}(S)$ be the subgroup of $\mathcal{G}(S)$ consisting of all homeomorphisms that are isotopic to the identity mapping. If $x$ is a point of $S$, we let

$$O_x = \{ y \mid y = g(x), \ g \in \mathcal{G}(S) \}$$

denote the total orbit of $x$ and we let

$$P_x = \{ y \mid y = h(x), \ h \in \mathcal{G}(S) \}$$

denote the continuous orbit of $x$.

By Theorem 8 of [4], each total orbit $O_x$ is dense in $S$ and an obvious modification of the same argument proves that each continuous orbit $P_x$ is also dense in $S$. We note that each continuous orbit is connected. For if $y$ is any point of the continuous orbit $P_x$, then the isotopy path of $x$ during an isotopy carrying $x$ onto $y$ is a continuum. Therefore $P_x$ is a union of continua having the point $x$ in common.

**Theorem 1.** If $S$ is a continuously invertible space, then every continuous orbit (and every total orbit) is connected and dense in $S$.

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Also, $S$ itself is connected.

**Theorem 2.** If $S$ is a continuously invertible $T_1$ space and if $c$ is a cut point of $S$, then the continuous orbit $P_c$ of $c$ is $S$ itself.

*Proof.* Suppose there were a point $x$ in $S - P_c$. Since $S - c = A \cup B$ set, we may assume that $x$ lies in $A$. The set $B$ is open under the hypotheses and hence there is an inverting homeomorphism $h$, isotopic to the identity, which carries $x$ into $B$. But then the isotopy path of $x$ must pass through the cut point $c$ and hence the continuous orbits $P_x$ and $P_c$ intersect. This is impossible.

**Corollary.** No continuously invertible Hausdorff continuum has a cut point.

*Proof.* If such a continuum is degenerate, then it has no cut points. If it is nondegenerate, then it has at least two noncut points and the existence of a cut point would contradict Theorem 2.

**Theorem 3.** In a (nondegenerate) continuously invertible Hausdorff space, each continuous orbit is arcwise connected and each point in a total orbit lies on an arc in the orbit.

*Proof.* The isotopy path of a point is a continuous image of the unit interval in a Hausdorff space and hence is a Peano continuum. Then by Theorem 10 of [4], each orbit is homogeneous.

**Corollary.** No orbit in a (nondegenerate) continuously invertible Hausdorff space is degenerate.

**Corollary.** Every continuously invertible (nondegenerate) Hausdorff space is a union of nondegenerate, dense, disjoint, homogeneous, arcwise connected, continuously invertible subspaces.

*Proof.* Each such space is the union of its continuous orbits.

If a continuously invertible Hausdorff space $S$ contains no simple closed curve, then every Peano continuum in $S$ is a dendrite and if $S$ itself is a Peano continuum, then $S$ is a dendrite. But by virtue of the corollary to Theorem 2, a dendrite is not continuously invertible.

**Theorem 4.** Each nondegenerate continuously invertible Peano continuum contains a simple closed curve.
THEOREM 5. If the invertible space $S$ contains a separating proper subcontinuum $C$, then each open set in $S$ contains a separating continuum imbedded in $S$ as is $C$.

Proof. This is an application of Theorem 6 of [4].

THEOREM 6. If an invertible space $S$ is separated by a proper closed set $C$ which is irreducible with respect to separating $S$, then $C$ contains no open set of $S$.

Proof. This is an application of Theorem 5 above.

Next we have an extrinsic characterization of the $n$-sphere which may be compared with the intrinsic characterization given in [3].

THEOREM 7. Let $M$ be a set in $E^{n+1}$ which is continuously invertible and which contains an $n$-sphere, $S$. Then $M$ is the $n$-sphere $S$.

Proof. Suppose that $M - S$ is not empty. There is no loss of generality in assuming that there are points of $M$ in the bounded component $A$ of $E^{n+1} - S = A \cup B$. By Theorem 5, there is an $n$-sphere $S'$ in $A \cap M$ and, in particular, there is an isotopy $H_t$ which carries $S$ onto $S'$ where $H_t(S)$ lies in $M$ for each $t$, $0 \leq t \leq 1$.

Now $M$ cannot contain an $(n+1)$-cell for if it were locally Euclidean of dimension $n + 1$ at any point, then Theorem 1 of [3] would imply that $M$ is an $(n + 1)$-sphere imbedded in $E^{n+1}$ which is impossible. Thus there must be a point $p$ lying in the annular region between $S$ and $S'$ such that $p$ is not in $M$. Similarly, there is a point $q$ in the unbounded domain $B$ such that $q$ is not in $M$. Clearly, the continuous cycles $S$ and $p \cup q$ are linked, whence the the isotopic cycle $S'$ must be linked with $p \cup q$. This is contradictory.

THEOREM 8. The only continuously invertible Peano continua in the plane are the simple closed curves.

Proof. By Theorem 4, each such continuum contains a simple closed curve and then Theorem 7 applies.

We conclude this report with a few results on continuously invertible plane continua which serve only to indicate a direction in which further study may be fruitful.

THEOREM 9. Let $C$ be a continuously invertible plane continuum that is not a simple closed curve. If $x$ and $y$ are two points in the same continuous orbit in $C$, then there is a unique arc in $C$ having
x and y as endpoints.

Proof. By Theorem 3 there is at least one arc joining x and y. If there were another, then C would contain (and hence be) a simple closed curve.

Theorem 10. Let C be a continuously invertible plane continuum that is not a simple closed curve. Then every Peano continuum in C is a simple arc.

Proof. From a previous remark and Theorem 7 we know that every Peano continuum in C is a dendrite. If there is a dendrite in C other than a simple arc, then there would be a simple triod T in C. Now C — T is not empty and hence there is an isotopy carrying T into its complement. The isotopy path of T is a Peano continuum in C and hence is a dendrite. But then the isotopy path of the branch point b of T contains uncountably many branch points in the isotopy path of T. This is impossible because no dendrite contains uncountably many branch points.

Theorem 11. No proper subcontinuum of a continuously invertible plane continuum separates the plane.

Proof. If some proper subcontinuum separates the plane, then by Theorem 5 there is a separating subcontinuum in every open set of the continuum. A construction as in the proof of Theorem 7 using Theorem 7, Chap. 1 of [5] will then yield a contradiction.

Theorem 12. Let C be a continuously invertible plane continuum that is not a simple closed curve. Then every proper subcontinuum of C is arcwise connected.

Proof. Let C' be a proper subcontinuum of C and suppose there are points x, y in C' which lie on no arc in C'. Let U be an open set of C such that $\bar{U}$ lies in $C - C'$. Then there is an isotopy of C with terminal mapping h such that $h(C') = C''$ lies in U. Letting $h(x) = x'$ and $h(y) = y'$, we note that there are arcs $xx'$, $yy'$ from $C'$ to $C''$ and that these arcs must pass into $C - C'$, as we go from x to $x'$ and y to $y'$. It follows that the continuum $D = C' \cup xx' \cup yy' \cup C''$ separates the plane. Thus by Theorem 11, we must have $D = C$. But this, too, is impossible since D obviously contains an open arc as an open set and hence Theorem 1 of [3] concludes that D is a simple closed curve, contrary to hypothesis.
Theorem 13. Let \( C \) be a continuously invertible plane continuum that is not a simple closed curve. Then every proper subcontinuum of \( C \) is an arc in some continuous orbit of \( C \).

Proof. Suppose there were a continuum \( C' \) in \( C \) which did not lie entirely in a continuous orbit. Let \( x \) and \( y \) be points of \( C' \) such that \( y \) is not in \( P_x \). Then the arc from \( x \) to \( y \) given by Theorem 12 together with an arc in \( P_x \) having \( x \) as an interior point will contain a triod \( T \). Then the same argument as in Theorem 10 yields a contradiction. The fact that a proper subcontinuum is an arc easily follows from Theorem 9.

Theorem 14. The only decomposable, continuously invertible plane continua are the simple closed curves.

Proof. Suppose there is a decomposable continuously invertible plane continuum \( C \) that is not a simple closed curve. By definition, \( C = C_1 \cup C_2 \) where \( C_1 \) and \( C_2 \) are proper subcontinua. Then by Theorem 13, both \( C_1 \) and \( C_2 \) are simple arcs. Hence the open set \( C_1 - C_2 \) contains an open arc, whence Theorem 1 of [3] proves that \( C \) is a simple closed curve after all.

In [1], R. H. Bing has shown that a homogeneous plane continuum that contains an arc is necessarily a simple closed curve. Since a continuously invertible continuum contains an arc, we have the following result.

Corollary. The only homogeneous, continuously invertible plane continua are the simple closed curves.

The existence of an indecomposable, continuously invertible plane continuum remains an open question. In §8 of [1], Bing gives an example which may enjoy these properties but this has not been established.

References


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