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**NORMAL SUBGROUPS OF MONOMIAL GROUPS**

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**1. Introduction.** Let  $U$  be the set consisting of  $x_1, x_2, x_3, \dots, x_n$ . Let  $H$  be a fixed group. A *monomial substitution* of  $U$  over  $H$  is a transformation of the form,

$$y = \left( \begin{array}{cccccc} x_1 & x_2 & x_3 & \cdots & x_n \\ h_1 x_{j_1} & h_2 x_{j_2} & h_3 x_{j_3} & \cdots & h_n x_{j_n} \end{array} \right) \begin{array}{l} x_j \in U \\ h_i \in H \end{array}$$

where the mapping of the  $x$ 's is one-to-one. The  $h_j$  are called the factors of  $y$ . If

$$y_1 = \left( \begin{array}{cccccc} x_1 & x_2 & x_3 & \cdots & x_n \\ k_1 x_{j_1} & k_2 x_{j_2} & k_3 x_{j_3} & \cdots & k_n x_{j_n} \end{array} \right)$$

then

$$yy_1 = \left( \begin{array}{cccccc} x_1 & x_2 & x_3 & \cdots & x_n \\ h_1 k_{i_1} x_{j_{i_1}} & h_2 k_{i_2} x_{j_{i_2}} & h_3 k_{i_3} x_{j_{i_3}} & \cdots & h_n k_{i_n} x_{j_{i_n}} \end{array} \right).$$

By this definition of multiplication the set of all substitutions form a group  $\Sigma_n(H)$ . Denote by  $V$  the set of all substitutions of the form

$$y = \left( \begin{array}{cccccc} x_1 & x_2 & x_3 & \cdots & x_n \\ h_1 x_1 & h_2 x_2 & h_3 x_3 & \cdots & h_n x_n \end{array} \right) = [h_1, h_2, h_3, \dots, h_n].$$

Then  $V$ , called the basis group, is a normal subgroup of  $\Sigma_n(H)$ . A permutation is an element of the form

$$\left( \begin{array}{cccccc} x_1 & x_2 & \cdots & x_n \\ ex_{i_1} & ex_{i_2} & \cdots & ex_{i_n} \end{array} \right) = \left( \begin{array}{cccccc} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{array} \right).$$

where  $e$  is the identity of  $H$ . Cyclic representation will also be used for elements of this type. The set  $S_n$  of all such elements is a subgroup of  $\Sigma_n(H)$ . Furthermore  $\Sigma_n(H) = V \cup S$ ,  $V \cap S = E$  where  $E$  is the identity of  $\Sigma_n(H)$ . Any element  $y$  of  $\Sigma_n(H)$  can be written as  $y = vs$  where  $v \in V$  and  $s \in S$ . Ore [1] has studied this group for finite  $U$  and some of his results have been extended in [2] and [3].

The normal subgroups of  $\Sigma_n(H) = \Sigma_n$  for  $U$  a finite set have been determined in [1]. The normal subgroups for  $o(U) = B = \mathfrak{S}_u$ ,  $u \geq 0$ , where  $o(U)$  means the number of elements of  $U$ , have been determined for rather general cases in [2] and [3]. The subset  $\Sigma_{A,n}(H) = \Sigma_{A,n}$  of elements of the form  $y = vs$  with  $s$  in the alternating group  $A_n$  is a

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subgroup of  $\Sigma_n$ . The normal subgroups of  $\Sigma_{A,n}$  are known for all  $n$  except 3 and 4 [2]. This paper determines the normal subgroups of  $\Sigma_{A,n}$  for  $n = 3, 4$  that are not contained in the basis group, thus filling a gap in the theory.

**2. The normal subgroups of  $\Sigma_{A,3}$  not contained in the basis group  $V$ .** We shall consider first the normal subgroups  $M$  that contain pure permutations.

**THEOREM 1.** *Let  $M$  be normal in  $\Sigma_{A,3}$ ,  $A_3 \subset M$ . Then  $N = M \cap V$  is a normal subgroup of  $\Sigma_{A,3}$ . The subgroup  $M = N \cup A_3$ . There exists a normal subgroup  $S_1$  of  $H$  such that  $H/S_1$  is Abelian and such that  $N$  consists of all elements  $v = [h_1, h_2, h_3]$  for which  $h_1 h_2 h_3 \in S_1$ .*

*Proof.* The intersection of two normal subgroups is again normal so  $N$  is normal in  $\Sigma_{A,3}$ .

Clearly  $M \supset (N \cup A_3)$ . Let  $y = vs$  be arbitrary in  $M$ . Then  $ys^{-1} = v$  belongs to  $M \cap V = N$  so  $M \subset (N \cup A_3)$ .

Let  $v = [h_1, h_2, h_3]$  be arbitrary in  $N$ . Form  $y = v(1, 2, 3)$ , which is in  $M$ . All of the elements  $y_1 = v_1 y v_1^{-1}$ , where  $v_1$  is arbitrary in  $V$  are in  $M$  by  $M$  normal in  $\Sigma_{A,3}$ . For a proper choice of  $v_1$ ,  $y_1 = [h_1 h_2 h_3, e, e]$  (1, 2, 3). Therefore  $N$  contains  $[h_1 h_2 h_3, e, e]$ . Now consider the set  $N_1 \cup N$  of all elements of the form  $[h, e, e]$ . This is a normal subgroup of  $N$ . The elements of  $H$  that occur as the first factors of multiplications of  $N_1$  form a normal subgroup  $S_1$  of  $H$ . We have established that if  $v \in N$  the product of the factors is in  $S_1$ . If  $k_1, k_2, k_3$  are any elements of  $H$  satisfying  $k_1 k_2 k_3 = k$  where  $k$  is in  $S_1$  then  $[k, e, e]$  is in  $N$ . Furthermore  $[k, e, e]$  (1, 2, 3) is in  $M$  and by a proper conjugation with a multiplication  $[k_1, k_2, k_3]$  (1, 2, 3) is in  $M$ . Hence  $[k_1, k_2, k_3]$  is in  $N$ .

Since  $[r_1, r_2, r_2^{-1} r_1^{-1}]$  is in  $N$  for arbitrary  $r_1, r_2$  of  $H$ , its inverse  $[r_1^{-1}, r_2^{-1}, r_1 r_2]$  is also in  $N$ . Therefore  $r_1^{-1} r_2^{-1} r_1 r_2$  is in  $S_1$ . This shows  $r_1 r_2 \equiv r_2 r_1 \pmod{S_1}$  and  $H/S_1$  is Abelian.

**THEOREM 2.** *Let  $N$  be as described in the last sentence of Theorem 1. Then  $N \cup A_3 = M$  is normal in  $\Sigma_{A,3}$ .*

*Proof.* Ore [1, p. 37] has shown  $M$  is normal in  $\Sigma_3$  so it is normal in  $\Sigma_{A,3}$ .

We shall now describe those normal subgroups which do not contain a pure permutation.

**THEOREM 3.** *Let  $S_1 \subset S_2$  be normal subgroups of  $H$  satisfying the conditions  $H/S_1$  is Abelian and  $S_2/S_1$  is isomorphic, by  $\theta$  say, to  $A_3$ .*

Let  $M$  consist of the sets  $T_i = \{vs/s = (1, 2, 3)^i\}$ ,  $i = 0$  or  $1$  or  $2$ , where the factors of substitutions of  $T_i$  run through  $H$  subject to the conditions that their product,  $k$  say, is in  $S_2$  and the coset  $kS_1$  maps onto  $(1, 2, 3)^i$ . Then  $M$  is a normal subgroup of  $\Sigma_{A,3}$ . Conversely if  $M \notin V$  and  $A_3 \notin M$ , then  $M$  has the above form.

*Proof.* We shall establish first that  $M$  is a group. Let  $y_1 = [h_1, h_2, h_3]s_1$  and  $y_2 = [k_1, k_2, k_3]s_2$  be arbitrary elements in  $M$ . We know then that  $h_1h_2h_3S_1\theta = s_1$  and  $k_1k_2k_3S_1\theta = s_2$ . Consider the product  $y_1y_2 = [h_1k_{i_1}, h_2k_{i_2}, h_3k_{i_3}]s_1s_2$ . Since  $H/S_2$  is Abelian and  $\theta$  is an isomorphism  $h_1k_{i_1}h_2k_{i_2}h_3k_{i_3}S_1\theta = h_1h_2h_3k_1k_2k_3S_1\theta = h_1h_2h_3\theta k_1k_2k_3\theta = s_1s_2$ . This shows that if  $y_1y_2$  belongs to  $T_i$  then the coset of the product of the factors maps onto  $(1, 2, 3)^i$ . We show now that when  $y_1$  as above is in  $M$  that its inverse is also in  $M$ . The inverse of  $y_1$  is  $y_1^{-1} = [h_{i_1}^{-1}, h_{i_2}^{-1}, h_{i_3}^{-1}]s_1^{-1}$ . We must show  $h_{i_1}^{-1}h_{i_2}^{-1}h_{i_3}^{-1}$  belongs to  $S_2$  and  $h_{i_1}^{-1}h_{i_2}^{-1}h_{i_3}^{-1}S_1\theta = s_1^{-1}$ . The first of these follows from  $h_1h_2h_3$  in  $S_2$  and  $H/S_2$  Abelian. The second follows from the observation that  $h_3^{-1}h_2^{-1}h_1^{-1}S_1\theta = s_1^{-1}$  and  $H/S_1$  is Abelian.

It remains to show that  $M$  is normal in  $\Sigma_{A,3}$ . Let  $y_1 = [h_1, h_2, h_3]s_1$  and  $y_3 = [g_1, g_2, g_3]s$  be arbitrary elements of  $M$  and  $\Sigma_{A,3}$  respectively. We must show that the product

$$y_3y_1y_3^{-1} = [g_1h_{i_1}g_{j_1}^{-1}, g_2h_{i_2}g_{j_2}^{-1}, g_3h_{i_3}g_{j_3}^{-1}]ss_1s^{-1} = vs_1$$

is in  $M$ . The product of the factors is in  $S_2$  since  $H/S_2$  is Abelian and  $h_1h_2h_3$  is in  $S_2$ . Finally

$$g_1h_{i_1}g_{j_1}^{-1}g_2h_{i_2}g_{j_2}^{-1}g_3h_{i_3}g_{j_3}^{-1}S_1\theta = h_1h_2h_3S_1\theta = s_1.$$

We now give the proof of the converse. Two elements  $vs$  and  $v_1s_1$  of  $M$  are defined to be equivalent if  $s = s_1$ . This is an equivalence relation and induces the partition  $T_0 = \{vs/s = E\}$ ,  $T_1 = \{vs/s = (1, 2, 3)\}$ ,  $T_2 = \{vs/s = (1, 3, 2)\}$  on  $M$ . We note that one of the sets  $T_1$  or  $T_2$  is nonempty since  $M \notin V$ . In fact, since at least one of them is not empty, they are each nonempty.

If an arbitrary element  $y = vs = [h_1, h_2, h_3](1, 2, 3)$  of  $T_1$  is conjugated by  $[h_3, h_2^{-1}, e]$  the resulting elements  $[h_3h_1h_2, e, e](1, 2, 3)$  is also in  $T_1$ . Since  $s_1ys_1^{-1} = s_1vs_1^{-1}s_1ss_1^{-1} = v_1s_1$  is in  $M$  for all  $s_1$  of  $A_3$  we can show that  $[h_1h_2h_3, e, e](1, 2, 3)$  and  $[h_2h_3h_1, e, e](1, 2, 3)$  also belong to  $T_1$ . When  $y_1 = [a, e, e](1, 2, 3)$  is in  $T_1$  then  $(1, 2, 3)y_1(1, 3, 2) = [e, e, a](1, 2, 3)$  and  $(1, 3, 2)y_1(1, 2, 3) = [e, a, e](1, 2, 3)$  are also in  $T_1$ .

Similarly it can be shown that  $T_2$  contains elements of the form  $[b, e, e](1, 3, 2)$  and with every such element  $[e, b, e](1, 3, 2)$ ,  $[e, e, b](1, 3, 2)$ . In particular  $[h_2h_1h_3, e, e](1, 3, 2)$  is in  $T_2$  where  $[h_1, h_1, h_3](1, 3, 2)$  is arbitrary in  $T_2$ . When  $[a, e, e]$  is in  $T_0$ , then  $[e, a, e]$  and  $[e, e, a]$  are also in  $T_0$ .

Now denote by  $R$  the set of elements of the form  $[a, e, e]s$ . Let  $S_2$  be the set of elements of  $H$  that occur as first factors of elements of  $R$ . We shall show that  $S_2$  is a normal subgroup of  $H$ . Choose arbitrary elements  $m_1 = [a_1, e, e]s_1$  and  $m_2 = [a_2, e, e]s_2$  of  $R$ . If  $s_1 = E$  then  $m_1m_2 = [a_1a_2, e, e]s_2$  is again in  $R$  and  $a_1a_2$  belongs to  $S_2$ . If  $s_1 = (1, 2, 3)$  we work with  $m_3 = [e, a_2, e]s_2$  and form  $m_1m_3 = [a_1a_2, e, e](1, 2, 3)s_2$ . Again we have shown  $a_1a_2 \in S_2$ . Finally if  $s_1 = (1, 3, 2)$  we let  $m_4 = [e, e, a_2]s_2$  and consider  $m_1m_4 = [a_1a_2, e, e](1, 3, 2)s_2$ . In any case we see that  $S_2$  is closed. When  $m_1 \in R$  then  $m_1^{-1}$  which is  $[a_1^{-1}, e, e]$ ,  $[e, a_1^{-1}, e]s_1^{-1}$ , or  $[e, e, a_1^{-1}]s_1^{-1}$  also belongs to  $M$ . By the earlier argument we see that  $R$  must contain  $[a_1^{-1}, e, e]s_1^{-1}$ . This shows  $a_1^{-1} \in S_2$ . Let  $a \in S_2$  and  $h \in H$ . Then, by the definition of  $\Sigma_{A,3}$  and  $S_2$ ,  $[a, e, e]s \in M$  and  $[h, h, h] \in \Sigma_{A,3}$ . Now since  $M$  is normal in  $\Sigma_{A,3}$ ,  $[h, h, h][a, e, e]s[h^{-1}, h^{-1}, h^{-1}] = [hah^{-1}, e, e]s \in M$ . Therefore,  $hah^{-1}$  is in  $S_2$ . We have just shown  $S_2$  is normal in  $H$ .

Substitutions in  $R \cap V = N_1$  are of the form  $[a, e, e]$ . The first factors form a subgroup,  $S_1$ , of  $H$ . That  $S_1$  is normal in  $H$  follows from  $M$  normal in  $\Sigma_{A,3}$ . By the definition of the two groups  $S_1$  is a subgroup of  $S_2$ .

To show that  $H/S_1$  is Abelian we let  $h_1, h_2$  be arbitrary elements of  $H$  and show  $h_1h_2h_1^{-1}h_2^{-1}$  is in  $S_1$ . Choose an element  $[b_1, b_2, b_3](1, 2, 3)$  from  $T_1$  and conjugate it by each of the three elements  $[e, h_2h_1b_1; h_1b_1b_2]$ ,  $[e, b_1, b_1b_2]$ , and  $[e, h_2^{-1}h_1^{-1}b_1, h_1^{-1}b_1b_2]$ . The resulting elements, which must be in  $M$ , are  $y_1 = [h_1^{-1}h_2^{-1}, h_2, h_1b_1b_2b_3](1, 2, 3)$ ,  $y_2 = [e, e, b_1b_2b_3](1, 2, 3)$ , and  $y_3 = [h_1h_2, h_2^{-1}, h_1^{-1}b_1b_2b_3](1, 2, 3)$ . The product  $y_4 = y_2y_1^{-1} = [h_2h_1, h_2^{-1}, h_1^{-1}]$  is also in  $M$ . Now form  $y_5 = y_3y_3^{-1} = [h_2^{-1}h_1^{-1}, h_2, h_1]$ . Finally consider  $y_4y_5 = [h_2h_1h_2^{-1}h_1^{-1}, e, e]$  which is in  $M$ . Therefore,  $h_2h_1h_2^{-1}h_1^{-1}$  is in  $S_1$ . In addition this also establishes that  $H/S_2$  is Abelian. Earlier we had  $[h_2h_1h_3, e, e](1, 3, 2)$  in  $T_2$ . By  $H/S_2$  Abelian  $h_1h_2h_3 \in S_2$  also.

We now define a mapping from  $S_2$  onto  $A_3$  as follows. For an element  $a$  of  $S_2$  which occurs as a first factor of a substitution  $y = [a, e, e]s$  we let  $a\theta = s$ . Certainly by this definition every element of  $S_2$  will be mapped. If any element of  $S_2$  is assumed to be mapped onto two different elements of  $A_3$  a computation, using the properties already stated for  $R$  and  $M$ , will show that  $M$  contains a pure permutation contrary to the case we are currently investigating. For example, suppose  $a\theta = (1, 2, 3)$  and  $a\theta = (1, 3, 2)$ . Then  $y_1 = [a, e, e](1, 3, 2)$ ,  $y_2 = [a, e, e](1, 2, 3)$ ,  $y_1^{-1} = [e, e, a^{-1}](1, 2, 3)$ , and  $y_3 = [e, a^{-1}, e](1, 2, 3)$  all belong to  $M$ . So  $[a, e, e](1, 2, 3)[e, a^{-1}, e](1, 2, 3) = (1, 3, 2)$  belongs to  $M$ . This mapping also preserves multiplication. For let  $a_1\theta = s_1, a_2\theta = s_2$ . This means that  $R$  contains the elements  $[a_1, e, e]s_1, [a_2, e, e]s_2$ . But  $M$  also contains  $vs_2$  where  $v$  has two factors of  $e$  and  $a_2$  a factor in the position that  $s_1$  sends  $x_1$  into. Therefore,  $[a_1a_2, e, e]s_1s_2$  belongs to  $R$  and  $a_1a_2\theta = s_1s_2 = a_1\theta a_2\theta$ . The definition of the mapping makes it clear that the kernel

of the homomorphism is precisely  $S_1$ . Therefore,  $S_2/S_1 \cong A_3$ .

It has already been pointed out that if  $y = vs$  is an element of  $T_1$  or  $T_2$  then the product of the factors  $h_1h_2h_3$  of  $v$  is in  $S_2$ . If  $[a_1, a_2, a_3]$  is in  $M \cap V$  then since  $y_5 = [h_2^{-1}h_1^{-1}, h_2, h_1]$  is also in  $M$  for arbitrary  $h_1, h_2$  of  $H$  it follows that  $[a_1, a_2, a_3][a_2a_3, a_2^{-1}, a_3^{-1}] = [a_1a_2a_3, e, e]$  is in  $M$ . This shows that the product of factors of elements in  $T_0$  is in  $S_1$ . Now let us assume that  $b_1, b_2, b_3$  are elements of  $H$  whose product is in  $S_2$ . Then  $(b_1b_2b_3)\theta = (1, 2, 3)^i$  for  $i = 0$ , or 1, or 2. We will show that there is an element  $y = vs$  of  $T_1$  whose factors are  $b_1, b_2$ , and  $b_3$ . In the case where  $i = 0$  we know that  $M$  contains an element  $[b_1b_2b_3, e, e]$ . The element  $y_4 = [h_2h_1, h_2^{-1}, h_1^{-1}]$  and its inverse  $y_4^{-1} = [h_1^{-1}h_2^{-1}, h_2, h_1]$  are also in  $M$  for all  $h_1, h_2$  of  $H$  so choose  $h_2 = b_2, h_1 = b_3$ . Then the product  $[b_1b_2b_3, e, e][b_3^{-1}b_2^{-1}, b_2, b_3] = [b_1, b_2, b_3]$  is in  $M$ . When  $i = 1$  we have  $[b_1b_2b_3, e, e](1, 2, 3)$  in  $M$  and by choosing  $h_2 = b_3^{-1}b_2^{-1}, h_1 = b_2$  and computing  $[b_1b_2b_3, e, e](1, 2, 3)[b_3, b_3^{-1}b_2^{-1}, b_2] = [b_1, b_2, b_3](1, 2, 3)$ . Finally if  $i = 2$  we have  $[b_1b_2b_3, e, e](1, 3, 2)$  in  $M$  and by choosing  $h_2 = b_3, h_1 = b_3^{-1}b_2^{-1}$  and computing we have  $[b_1, b_2, b_3](1, 3, 2)$  in  $T_2$ .

**3. The normal subgroups of  $\Sigma_{A,4}$  not contained in the basis group  $V$ .**  
All proofs in this section except for the proof of Lemma 1 are similar to the corresponding proofs for  $\Sigma_{A,3}$  so will be omitted.

**LEMMA 1.** *Let  $M$  be normal in  $\Sigma_{A,4}$ ,  $M \not\subset V$ . Then the Klein group is contained in  $M$ .*

*Proof.* We will first show that  $M$  contains elements of the form  $y = vs$  where  $s \neq E$  is in the Klein group. Hereafter  $K$  will mean the Klein group.

There is at least one element in  $M$  of the form  $y = vs$   $s \neq E, s \in A_4$ . If  $s$  is not in  $K$  then  $s$  is a three cycle, and we assume without loss of generality that  $s = (1, 3, 4)$ . If  $y$  is conjugated by  $(1, 4)(2, 3)$  the resulting element  $y_1 = v_1(1, 4, 2)$  and its inverse are also in  $M$ . Therefore,  $yy_1^{-1} = v_2(1, 3)(2, 4)$  is in  $M$ . We have just shown that  $M$  has an element of the form  $y = vs$  where  $s$  is in  $K$  and  $s \neq E$ . We assume without loss of generality that  $s = (1, 2)(3, 4)$  and  $v = [a_1, a_2, a_3, a_4]$ . Form the elements

$$\begin{aligned} y_1 &= v_1yv_1^{-1} = [e, a_2^{-1}, e, a_3][a_1, a_2, a_3, a_4](1, 2)(3, 4)[e, a_2, e, a_3^{-1}] \\ &= [a_1a_2, e, e, a_3a_4](1, 2)(3, 4) \text{ and } y_2 = y_3yy_3^{-1} \\ &= [e, a_2^{-1}, a_4^{-1}, e](1, 3, 4)[a_1, a_2, a_3, a_4](1, 2)(3, 4)[e, a_2, e, a_4](1, 4, 3) \\ &= [a_3a_4, e, e, a_1, a_2](1, 3)(2, 4). \end{aligned}$$

Since  $M$  is normal in  $\Sigma_{A,4}$ ,  $y_1$  and  $y_2$  are in  $M$ . Therefore  $y_1y_2^{-1} =$

$(1, 4)(2, 3)$  is in  $M$ . This shows  $S = M \cap A_4 \neq E$ . But  $M$  is normal in  $\Sigma_{A_4}$  so  $S$  is normal in  $A_4$ . This means  $S$  is  $K$  or  $A_4$ .

We shall now describe the normal subgroups  $N$  which is the intersection of  $M$  and the basis group  $V$ .

**THEOREM 1.** *Let  $M$  be normal in  $\Sigma_{A_4}$ ,  $M \not\subset V$ ,  $A_4 \subset M$ . Then  $N = M \cap V$  is a normal subgroup of  $\Sigma_{A_4}$ ,  $M = N \cup A_4$ . There exists a normal subgroup  $S_1$  of  $H$  such that  $H/S_1$  is Abelian and such that  $N$  consists of all elements  $v = [h_1, h_2, h_3, h_4]$  for which  $h_1 h_2 h_3 h_4 \in S_1$ .*

**THEOREM 2.** *Let  $N$  be as described in the last sentence of Theorem 1. Then  $N \cup A_4 = M$  is normal in  $\Sigma_{A_4}$ .*

We shall now describe those normal subgroups which contain no elements of the form  $y = vs$  where  $s$  is a three cycle.

**THEOREM 3.** *Let  $M$  be normal in  $\Sigma_{A_4}$ ,  $M \not\subset V$ ,  $M$  contains no elements of the form  $y = vs$  where  $s$  is a three cycle,  $M \cap V = N$ . Then  $M = N \cup K$ . Furthermore if  $N_1$  is as described in the last sentence of Theorem 1 then  $N_1 \cup K$  is normal in  $\Sigma_{A_4}$ .*

We shall now describe those normal subgroups which contain elements of the form  $y = vs$ , where  $s$  is a three cycle, but which do not contain a pure three cycle.

**THEOREM 4.** *Let  $S_1 \subset S_2$  be normal subgroups of  $H$  satisfying the conditions  $H/S_1$  is Abelian and  $S_2/S_1$  is isomorphic to  $A_3$ . Let  $M$  consist of the sets*

$$T_i = \{vs/s = (1, 2, 3)_i \text{ mod } K\}, \quad i = 0, 1, 2,$$

where the factors of substitutions of  $T_i$  run through  $H$  subject to the condition that their product,  $k$  say, is in  $S_2$  and  $kS_1$  maps onto  $(1, 2, 3)^i$ . Then  $M$  is a normal subgroup of  $\Sigma_{A_4}$ . Conversely, if  $M$  is normal subgroup of  $\Sigma_{A_4}$  such that  $M \not\subset V$  and  $A_4 \not\subset M$ ,  $M$  contains elements of the form  $y = vs$  where  $s$  is a three cycle, then  $M$  has the above form.

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# Pacific Journal of Mathematics

Vol. 12, No. 2

February, 1962

William George Bade and Robert S. Freeman, <i>Closed extensions of the Laplace operator determined by a general class of boundary conditions</i> . . . . .	395
William Browder and Edwin Spanier, <i>H-spaces and duality</i> . . . . .	411
Stewart S. Cairns, <i>On permutations induced by linear value functions</i> . . . . .	415
Frank Sydney Cater, <i>On Hilbert space operators and operator roots of polynomials</i> . . . . .	429
Stephen Urban Chase, <i>Torsion-free modules over <math>K[x, y]</math></i> . . . . .	437
Heron S. Collins, <i>Remarks on affine semigroups</i> . . . . .	449
Peter Crawley, <i>Direct decompositions with finite dimensional factors</i> . . . . .	457
Richard Brian Darst, <i>A continuity property for vector valued measurable functions</i> . . . . .	469
R. P. Dilworth, <i>Abstract commutative ideal theory</i> . . . . .	481
P. H. Doyle, III and John Gilbert Hocking, <i>Continuously invertible spaces</i> . . . . .	499
Shaul Foguel, <i>Markov processes with stationary measure</i> . . . . .	505
Andrew Mattei Gleason, <i>The abstract theorem of Cauchy-Weil</i> . . . . .	511
Allan Brasted Gray, Jr., <i>Normal subgroups of monomial groups</i> . . . . .	527
Melvin Henriksen and John Rolfe Isbell, <i>Lattice-ordered rings and function rings</i> . . . . .	533
Amnon Jakimovski, <i>Tauberian constants for the <math>[J, f(x)]</math> transformations</i> . . . . .	567
Hubert Collings Kennedy, <i>Group membership in semigroups</i> . . . . .	577
Eleanor Killam, <i>The spectrum and the radical in locally <math>m</math>-convex algebras</i> . . . . .	581
Arthur H. Kruse, <i>Completion of mathematical systems</i> . . . . .	589
Magnus Lindberg, <i>On two Tauberian remainder theorems</i> . . . . .	607
Lionello A. Lombardi, <i>A general solution of Tonelli's problem of the calculus of variations</i> . . . . .	617
Marvin David Marcus and Morris Newman, <i>The sum of the elements of the powers of a matrix</i> . . . . .	627
Michael Bahir Maschler, <i>Derivatives of the harmonic measures in multiply-connected domains</i> . . . . .	637
Deane Montgomery and Hans Samelson, <i>On the action of <math>SO(3)</math> on <math>S^n</math></i> . . . . .	649
J. Barros-Neto, <i>Analytic composition kernels on Lie groups</i> . . . . .	661
Mario Petrich, <i>Semicharacters of the Cartesian product of two semigroups</i> . . . . .	679
John Sydney Pym, <i>Idempotent measures on semigroups</i> . . . . .	685
K. Rogers and Ernst Gabor Straus, <i>A special class of matrices</i> . . . . .	699
U. Shukla, <i>On the projective cover of a module and related results</i> . . . . .	709
Don Harrell Tucker, <i>An existence theorem for a Goursat problem</i> . . . . .	719
George Gustave Weill, <i>Reproducing kernels and orthogonal kernels for analytic differentials on Riemann surfaces</i> . . . . .	729
George Gustave Weill, <i>Capacity differentials on open Riemann surfaces</i> . . . . .	769
G. K. White, <i>Iterations of generalized Euler functions</i> . . . . .	777
Adil Mohamed Yaqub, <i>On certain finite rings and ring-logics</i> . . . . .	785