Pacific Journal of Mathematics

NORMAL SUBGROUPS OF MONOMIAL GROUPS

ALLAN BRASTED GRAY, JR.

Vol. 12, No. 2 February 1962

NORMAL SUBGROUPS OF MONOMIAL GROUPS

ALLAN B. GRAY, JR.

1. Introduction. Let U be the set consisting of $x_1, x_2, x_3, \cdots x_n$. Let H be a fixed group. A *monomial substitution* of U over H is a transformation of the form,

$$y = \begin{pmatrix} x_1 , & x_2 , & x_3 , \cdots, & x_n \\ h_1 x_{j_1}, & h_2 x_{i_2}, & h_3 x_{i_3}, \cdots, & h_n x_{i_n} \end{pmatrix} \begin{matrix} x_j \in U \\ h_i \in H \end{matrix}$$

where the mapping of the x's is one-to-one. The h_j are called the factors of y. If.

$$y_1 = \begin{pmatrix} x_1 , & x_2 , & x_3 , \cdots, & x_n \\ k_1 x_{j_1}, & k_2 x_{j_2}, & k_3 x_{j_3}, \cdots, & k_n x_{j_n} \end{pmatrix}$$

then

$$yy_1 = \begin{pmatrix} x_1 , & x_2 , & x_3 , & \cdots, & x_n \\ h_1 k_{i_1} x_{j_{i_1}}, & h_2 k_{i_2} x_{j_{i_2}}, & h_3 k_{i_3} x_{j_{i_3}}, & \cdots, & h_n k_{i_n} x_{j_{i_n}} \end{pmatrix} .$$

By this definition of multiplication the set of all substitutions form a group $\Sigma_n(H)$. Denote by V the set of all substitutions of the form

$$y = \begin{pmatrix} x_1 , & x_2 , & x_3 , \cdots, & x_n \\ h_1 x_1, & h_2 x_2, & h_3 x_3, \cdots, & h_n x_n \end{pmatrix} = [h_1, h_2, h_3, \cdots, h_n] .$$

Then V, called the basis group, is a normal subgroup of $\Sigma_n(H)$. A permutation is an element of the form

$$\begin{pmatrix} x_1 , & x_2 , \cdots, & x_n \\ e x_{i_1}, & e x_{i_2}, \cdots, & e x_{i_n} \end{pmatrix} = \begin{pmatrix} 1, & 2, \cdots, & n \\ i_1, & i_2, \cdots, & i_n \end{pmatrix} .$$

where e is the identity of H. Cyclic representation will also be used for elements of this type. The set S_n of all such elements is a subgroup of $\Sigma_n(H)$. Furthermore $\Sigma_n(H) = V \cup S$, $V \cap S = E$ where E is the identity of $\Sigma_n(H)$. Any element y of $\Sigma_n(H)$ can be written as $y = vs$ where $v \in V$ and $s \in S$. Ore [1] has studied this group for finite U and some of his results have been extended in [2] and [3].

The normal subgroups of $\Sigma_n(H) = \Sigma_n$ for U a finite set have been determined in [1]. The normal subgroups for $o(U) = B = \aleph_u$, $u \geqq 0$, where $o(U)$ means the number of elements of U, have been determined for rather general cases in [2] and [3]. The subset $\Sigma_{A,n}(H) = \Sigma_{A,n}$ of elements of the form $y = vs$ with s in the alternating group A_n is a

Received May 22, 1961.

subgroup of Σ_n. The normal subgroups of $\Sigma_{A,n}$ are known for all n except 3 and 4 [2]. This paper determines the normal subgroups of $\Sigma_{A,n}$ for $n = 3, 4$ that are not contained in the basis group, thus filling a gap in the theory.

2. The normal subgroups of $\Sigma_{A,3}$ not contained in the basis group V. We shall consider first the normal subgroups M that contain pure permutations.

THEOREM 1. *Let M be normal in $\Sigma_{A,3}$, $A_3 \subset M$. Then $N = M \cap V$ is a normal subgroup of $\Sigma_{A,3}$. The subgroup $M = N \cup A_3$. There exists a normal subgroup S_1 of H such that H/S_1 is Abelian and such that N consists of all elements $v = [h_1, h_2, h_3]$ for which $h_1h_2h_3 \in S_1$.*

Proof. The intersection of two normal subgroups is again normal so N is normal in $\Sigma_{A,3}$.

Clearly $M \supset (N \cup A_3)$. Let $y = vs$ be arbitrary in M. Then $ys^{-1} = v$ belongs to $M \cap V = N$ so $M \subset (N \cup A_3)$.

Let $v = [h_1, h_2, h_3]$ be arbitrary in N. Form $y = v(1, 2, 3)$, which is in M. All of the elements $y_1 = v_1yv_1^{-1}$, where v_1 is arbitrary in V are in M by M normal in $\Sigma_{A,3}$. For a proper choice of v_1, $y_1 = [h_1h_2h_3, e, e]$ $(1, 2, 3)$. Therefore N contains $[h_1h_2h_3, e, e]$. Now consider the set $N_1 \cup N$ of all elements of the form $[h, e, e]$. This is a normal subgroup of N. The elements of H that occur as the first factors of multiplications of N_1 form a normal subgroup S_1 of H. We have established that if $v \in N$ the product of the factors is in S_1. If k_1, k_2, k_3 are any elements of H satisfying $k_1k_2k_3 = k$ where k is in S_1 then $[k, e, e]$ is in N. Furthermore $[k, e, e]$ $(1, 2, 3)$ is in M and by a proper conjugation with a multiplication $[k_1, k_2, k_3]$ $(1, 2, 3)$ is in M. Hence $[k_1, k_2, k_3]$ is in N.

Since $[r_1, r_2, r_2^{-1}r_1^{-1}]$ is in N for arbitrary r_1, r_2 of H, its inverse $[r_1^{-1}, r_2^{-1}, r_1r_2]$ is also in N. Therefore $r_1^{-1}r_2^{-1}r_1r_2$ is in S_1. This shows $r_1r_2 \equiv r_2r_1 \bmod S_1$ and H/S_1 is Abelian.

THEOREM 2. *Let N be as described in the last sentence of Theorem 1. Then $N \cup A_3 = M$ is normal in $\Sigma_{A,3}$.*

Proof. Ore [1, p. 37] has shown M is normal in Σ_3 so it is normal in $\Sigma_{A,3}$.

We shall now describe those normal subgrous which do not contain a pure permutation.

THEOREM 3. *Let $S_1 \subset S_2$ be normal subgroups of H satisfying the conditions H/S_1 is Abelian and S_2/S_1 is isomorphic, by θ say, to A_3.*

Let M consist of the sets $T_i = \{vs/s = (1, 2, 3)^i\}$, $i = 0$ or 1 or 2, where the factors of substitutions of T_i run through H subject to the conditions that their product, k say, is in S_2 and the coset kS_1 maps onto $(1, 2, 3)^i$. Then M is a normal subgroup of $\Sigma_{A,3}$. Conversely if $M \notin V$ and $A_3 \notin M$, then M has the above form.

Proof. We shall establish first that M is a group. Let $y_1 = [h_1, h_2, h_3]s_1$ and $y_2 = [k_1, k_2, k_3]s_2$ be arbitrary elements in M. We know then that $h_1h_2h_3S_1\theta = s_1$ and $k_1k_2k_3S_1\theta = s_2$. Consider the product $y_1y_2 = [h_1k_{i_1}, h_2k_{i_2}, h_3k_{i_3}]s_1s_2$. Since H/S_2 is Abelian and θ is an isomorphism $h_1k_{i_1}h_2k_{i_2}h_3k_{i_3}S_1\theta = h_1h_2h_3k_1k_2k_3S_1\theta = h_1h_2h_3\theta k_1k_2k_3\theta = s_1s_2$. This shows that if y_1y_2 belongs to T_i then the coset of the product of the factors maps onto $(1, 2, 3)^i$. We show now that when y_1 as above is in M that its inverse is also in M. The inverse of y_1 is $y_1^{-1} = [h_{i_1}^{-1}, h_{i_2}^{-1}, h_{i_3}^{-1}]s_1^{-1}$. We must show $h_{i_1}^{-1}h_{i_2}^{-1}h_{i_3}^{-1}$ belongs to S_2 and $h_{i_1}^{-1}h_{i_2}^{-1}h_{i_3}^{-1}S_1\theta = s_1^{-1}$. The first of these follows from $h_1h_2h_3$ in S_2 and H/S_2 Abelian. The second follows from the observation that $h_3^{-1}h_2^{-1}h_1^{-1}S_1\theta = s_1^{-1}$ and H/S_1 is Abelian.

It remains to show that M is normal in $\Sigma_{A,3}$. Let $y_1 = [h_1, h_2, h_3]s_1$ and $y_3 = [g_1, g_2, g_3]s$ be arbitrary elements of M and $\Sigma_{A,3}$ respectively. We must show that the product

$$y_3y_1y_3^{-1} = [g_1h_{i_1}g_{j_1}^{-1}, g_2h_{i_2}g_{j_2}^{-1}, g_3h_{i_3}g_{j_3}^{-1}]ss_1s^{-1} = vs_1$$

is in M. The product of the factors is in S_2 since H/S_2 is Abelian and $h_1h_2h_3$ is in S_2. Finally

$$g_1h_{i_1}g_{j_1}^{-1}g_2h_{i_2}g_{j_2}^{-1}g_3h_{i_3}g_{j_3}^{-1}S_1\theta = h_1h_2h_3S_1\theta = s_1 \, .$$

We now give the proof of the converse. Two elements vs and v_1s_1 of M are defined to be equivalent if $s = s_1$. This is an equivalence relation and induces the partition $T_0 = \{vs/s = E\}$, $T_1 = \{vs/s = (1, 2, 3)\}$, $T_2 = \{vs/s = (1, 3, 2)\}$ on M. We note that one of the sets T_1 or T_2 is nonempty since $M \notin V$. In fact, since at least one of them is not empty, they are each nonempty.

If an arbitrary element $y = vs = [h_1, h_2, h_3]$ $(1, 2, 3)$ of T_1 is conjugated by $[h_3, h_2^{-1}, e]$ the resulting elements $[h_3h_1h_2, e, e]$ $(1, 2, 3)$ is also in T_1. Since $s_1ys_1^{-1} = s_1vs_1^{-1}s_1ss_1^{-1} = v_1s$ is in M for all s_1 of A_3 we can show that $[h_1h_2h_3, e, e](1, 2, 3)$ and $[h_2h_3h_1, e, e](1, 2, 3)$ also belong to T_1. When $y_1 = [a, e, e](1, 2, 3)$ is in T_1 then $(1, 2, 3)y_1(1, 3, 2) = [e, e, a](1, 2, 3)$ and $(1, 3, 2)y_1(1, 2, 3) = [e, a, e](1, 2, 3)$ are also in T_1.

Similarly it can be shown that T_2 contains elements of the form $[b, e, e,](1, 3, 2)$ and with every such element $[e, b, e](1, 3, 2)$, $[e, e, b](1, 3, 2)$. In particular $[h_2h_1h_3, e, e](1, 3, 2)$ is in T_2 where $[h_1, h_1, h_3](1, 3, 2)$ is arbitrary in T_2. When $[a, e, e]$ is in T_0, then $[e, a, e]$ and $[e, e, a]$ are also in T_0.

Now denote by R the set of elements of the form $[a, e, e]s$. Let S_2 be the set of elements of H that occur as first factors of elements of R. We shall show that S_2 is a normal subgroup of H. Choose arbitrary elements $m_1 = [a_1, e, e]s_1$ and $m_2 = [a_2, e, e]s_2$ of R. If $s_1 = E$ then $m_1m_2 = [a_1a_2, e, e]s_2$ is again in R and a_1a_2 belongs to S_2. If $s_1 = (1, 2, 3)$ we work with $m_3 = [e, a_2, e]s_2$ and form $m_1m_3 = [a_1a_2, e, e](1, 2, 3)s_2$. Again we have shown $a_1a_2 \in S_2$. Finally if $s_1 = (1, 3, 2)$ we let $m_4 = [e, e, a_2]s_2$ and consider $m_1m_4 = [a_1a_2, e, e](1, 3, 2)s_2$. In any case we see that S_2 is closed. When $m_1 \in R$ then m_1^{-1} which is $[a_1^{-1}, e, e]$, $[e, a_1^{-1}, e]s_1^{-1}$, or $[e, e, a_1^{-1}]s_1^{-1}$ also belongs to M. By the earlier argument we see that R must contain $[a_1^{-1}, e, e]s_1^{-1}$. This shows $a_1^{-1} \in S_2$. Let $a \in S_2$ and $h \in H$. Then, by the definition of $\Sigma_{A,3}$ and S_2, $[a, e, e]s \in M$ and $[h, h, h] \in \Sigma_{A,3}$. Now since M is normal in $\Sigma_{A,3}$, $[h, h, h][a, e, e]s[h^{-1}, h^{-1}, h^{-1}] = [hah^{-1}, e, e]s \in M$. Therefore, hah^{-1} is in S_2. We have just shown S_2 is normal in H.

Substitutions in $R \cap V = N_1$ are of the form $[a, e, e]$. The first factors form a subgroup, S_1, of H. That S_1 is normal in H follows from M normal in $\Sigma_{A,3}$. By the definition of the two groups S_1 is a subgroup of S_2.

To show that H/S_1 is Abelian we let h_1, h_2 be arbitrary elements of H and show $h_1h_2h_1^{-1}h_2^{-1}$ is in S_1. Choose an element $[b_1, b_2, b_3](1, 2, 3)$ from T_1 and conjugate it by each of the three elements $[e, h_2h_1b_1; h_1b_1b_2]$, $[e, b_1, b_1b_2]$, and $[e, h_2^{-1}h_1^{-1}b_1, h_1^{-1}b_1b_2]$. The resulting elements, which must be in M, are $y_1 = [h_1^{-1}h_2^{-1}, h_2, h_1b_1b_2b_3](1, 2, 3)$, $y_2 = [e, e, b_1b_2b_3](1, 2, 3)$, and $y_3 = [h_1h_2, h_2^{-1}, h_1^{-1}b_1b_2b_3](1, 2, 3)$. The product $y_4 = y_2y_1^{-1} = [h_2h_1, h_2^{-1}, h_1^{-1}]$ is also in M. Now form $y_5 = y_2y_3^{-1} = [h_2^{-1}h_1^{-1}, h_2, h_1]$. Finally consider $y_4y_5 = [h_2h_1h_2^{-1}h_1^{-1}, e, e]$ which is in M. Therefore, $h_2h_1h_2^{-1}h_1^{-1}$ is in S_1. In addition this also establishes that H/S_2 is Abelian. Earlier we had $[h_2h_1h_3, e, e]$ $(1, 3, 2)$ in T_2. By H/S_2 Abelian $h_1h_2h_3 \in S_2$ also.

We now define a mapping from S_2 onto A_3 as follows. For an element a of S_2 which occurs as a first factor of a substitution $y = [a, e, e]s$ we let $a\theta = s$. Certainly by this definition every element of S_2 will be mapped. If any element of S_2 is assumed to be mapped onto two different elements of A_3 a computation, using the properties already stated for R and M, will show that M contains a pure permutation contrary to the case we are currently investigating. For example, suppose $a\theta = (1, 2, 3)$ and $a\theta = (1, 3, 2)$. Then $y_1 = [a, e, e](1, 3, 2)$, $y_2 = [a, e, e](1, 2, 3)$, $y_1^{-1} = [e, e, a^{-1}](1, 2, 3)$, and $y_3 = [e, a^{-1}, e](1, 2, 3)$ all belong to M. So $[a, e, e](1, 2, 3)[e, a^{-1}e](1, 2, 3) = (1, 3, 2)$ belongs to M. This mapping also preserves multiplication. For let $a_1\theta = s_1$, $a_2\theta = s_2$. This means that R contains the elements $[a_1, e, e]s_1$, $[a_2, e, e]s_2$. But M also contains vs_2 where v has two factors of e and a_2 a factor in the position that s_1 sends x_1 into. Therefore, $[a_1a_2, e, e]s_1s_2$ belongs to R and $a_1a_2\theta = s_1s_2 = a_1\theta a_2\theta$. The definition of the mapping makes it clear that the kernel of the homomorphism is precisely S_1. Therefore, $S_2/S_1 \cong A_3$.

It has already been pointed out that if $y = vs$ is an element of T_1 or T_2 then the product of the factors $h_1h_2h_3$ of v is in S_2. If $[a_1, a_2, a_3]$ is in $M \cap V$ then since $y_5 = [h_2^{-1}h_1^{-1}, h_2, h_1]$ is also in M for arbitrary h_1, h_2 of H it follows that $[a_1, a_2, a_3][a_2a_3, a_2^{-1}, a_3^{-1}] = [a_1a_2a_3, e, e]$ is in M. This shows that the product of factors of elements in T_0 is in S_1. Now let us assume that b_1, b_2, b_3 are elements of H whose product is in S_2. Then $(b_1b_2b_3)\theta = (1, 2, 3)^i$ for $i = 0$, or 1, or 2. We will show that there is an element $y = vs$ of T_1 whose factors are b_1, b_2, and b_3. In the case where $i = 0$ we know that M contains an element $[b_1b_2b_3, e, e]$. The element $y_4 = [h_2h_1, h_2^{-1}, h_1^{-1}]$ and its inverse $y_4^{-1} = [h_1^{-1}h_2^{-1}, h_2, h_1]$ are also in M for all h_1, h_2 of H so choose $h_2 = b_2, h_1 = b_3$. Then the product $[b_1b_2b_3, e, e][b_3^{-1}b_2^{-1}, b_2, b_3] = [b_1, b_2, b_3]$ is in M. When $i = 1$ we have $[b_1b_2b_3, e, e](1, 2, 3)$ in M and by choosing $h_2 = b_3^{-1}b_2^{-1}, h_1 = b_2$ and computing $[b_1b_2b_3, e, e](1, 2, 3)[b_3, b_b^{-1}b_2^{-1}, b_2] = [b_1, b_2, b_3](1, 2, 3)$. Finally if $i = 2$ we have $[b_1b_2b_3, e, e](1, 3, 2)$ in M and by choosing $h_2 = b_3, h_1 = b_3^{-1}b_2^{-1}$ and computing we have $[b_1, b_2, b_3](1, 3, 2)$ in T_2.

3. The normal subgroups of $\Sigma_{A,4}$ not contained in the basis group V. All proofs in this section except for the proof of Lemma 1 are similar to the corresponding proofs for $\Sigma_{A,3}$ so will be omitted.

LEMMA 1. *Let M be normal in $\Sigma_{A,4}$, $M \not\subset V$. Then the Klein group is contained in M.*

Proof. We will first show that M contains elements of the form $y = vs$ where $s \neq E$ is in the Klein group. Hereafter K will mean the Klein group.

There is at least one element in M of the form $y = vs$ $s \neq E, s \in A_4$. If s is not in K then s is a three cycle, and we assume without loss of generality that $s = (1, 3, 4)$. If y is conjugated by $(1, 4)(2, 3)$ the resulting element $y_1 = v_1(1, 4, 2)$ and its inverse are also in M. Therefore, $yy_1^{-1} = v_2(1, 3)(2, 4)$ is in M. We have just shown that M has an element of the form $y = vs$ where s is in K and $s \neq E$. We assume without loss of generality that $s = (1, 2)(3, 4)$ and $v = [a_1, a_2, a_3, a_4]$. Form the elements

$$\begin{aligned} y_1 = v_1yv_1^{-1} &= [e, a_2^{-1}, e, a_3][a_1, a_2, a_3, a_4](1, 2)(3, 4)[e, a_2\, e, a_3^{-1}] \\ &= [a_1a_2, e, e, a_3a_4](1, 2)(3, 4) \text{ and } y_2 = y_3yy_3^{-1} \\ &= [e, a_2^{-1}, a_4^{-1}, e](1, 3, 4)[a_1, a_2, a_3, a_4](1, 2)(3, 4)[e, a_2, e, a_4](1, 4, 3) \\ &= [a_3a_4, e, e, a_1, a_2](1, 3)(2, 4). \end{aligned}$$

Since M is normal in $\Sigma_{A,4}$, y_1 and y_2 are in M. Therefore $y_1y_2^{-1} =$

(1, 4)(2, 3) is in M. This shows $S = M \cap A_4 \neq E$. But M is normal in $\Sigma_{A,4}$ so S is normal in A_4. This means S is K or A_4.

We shall now describe the normal subgroups N which is the intersection of M and the basis group V.

THEOREM 1. *Let M be normal in $\Sigma_{A,4}$, $M \not\subset V$, $A_4 \subset M$. Then $N = M \cap V$ is a normal subgroup of $\Sigma_{A,4}$, $M = N \cup A_4$. There exists a normal subgroup S_1 of H such that H/S_1 is Abelian and such that N consists of all elements $v = [h_1, h_2, h_3, h_4]$ for which $h_1h_2h_3h_4 \in S_1$.*

THEOREM 2. *Let N be as described in the last sentence of Theorem* 1. *Then $N \cup A_4 = M$ is normal in $\Sigma_{A,4}$.*

We shall now describe those normal subgroups which contain no elements of the form $y = vs$ where s is a three cycle.

THEOREM 3. *Let M be normal in $\Sigma_{A,4}$, $M \not\subset V$, M contains no elements of the form $y = vs$ where s is a three cycle, $M \cap V = N$. Then $M = N \cup K$. Furthermore if N_1 is as described in the last sentence of Theorem 1 then $N_1 \cup K$ is normal in $\Sigma_{A,4}$.*

We shall now describe those normal subgroups which contain elements of the form $y = vs$, where s is a three cycle, but which do not contain a pure three cycle.

THEOREM 4. *Let $S_1 \subset S_2$ be normal subgroups of H satisfying the conditions H/S_1 is Abelian and S_2/S_1 is isomorphic to A_3. Let M consist of the sets*

$$T_i = \{vs/s = (1, 2, 3)^{\prime}_i \bmod K\}\ , \qquad i = 0, 1, 2,$$

where the factors of substitutions of T_i run through H subject to the condition that their product, k say, is in S_2 and kS_1 maps onto $(1, 2, 3)^i$. Then M is a normal subgroup of $\Sigma_{A,4}$. Conversely, if M is normal subgroup of $\Sigma_{A,4}$ such that $M \not\subset V$ and $A_4 \not\subset M$, M contains elements of the form $y = vs$ where s is a three cycle, then M has the above form.

BIBLIOGRAPHY

1. R. Baer, *Die kompositionsreihe der Gruppe aller eineindeutigen abbildungen einer unendlichen Reihe auf sich*, Studia Mathematica, **5** (1934), 15-17.

2. R. Crouch, *Monomial groups*, Trans. Amer. Math. Soc., **80** (1955), 187-215.

3. R. Crouch and W. R. Scott, *Normal subgroups of monomial groups*, Proc. Amer. Math. Soc., **8** (1957), 931-936.

4. O. Ore. *Theory of monomial groups*, Trans. Amer. Math. Soc., **51** (1942), 15-64.

NEW MEXICO STATE UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Pacific Journal of Mathematics

Vol. 12, No. 2 February, 1962