

Pacific Journal of Mathematics

**TAUBERIAN CONSTANTS FOR THE $[J, f(x)]$
TRANSFORMATIONS**

AMNON JAKIMOVSKI

TAUBERIAN CONSTANTS FOR THE $[J, f(x)]$ TRANSFORMATIONS

AMNON JAKIMOVSKI

1. Introduction. Let $\{s_n\}$ ($n \geq 0$) ($s_n = a_0 + \dots + a_n$) be a sequence of real or complex numbers. Denote by $t(x)$ a linear transform T

$$t(x) = \sum_{n=0}^{\infty} c_n(x) s_n$$

of $\{s_n\}$ supposed convergent for all sufficiently large values of x . In addition to classical Abelian and Tauberian theorems which give information about one of $\lim_{x \rightarrow \infty} t(x)$ and $\lim_{n \rightarrow \infty} s_n$ when the other exists, it is possible to find estimates of

$$\limsup_{n \rightarrow \infty, k_n \rightarrow \infty} |t(x_n) - s_n|$$

when neither $\lim t(x)$ nor $\lim s_n$ is supposed to exist but $\{a_n\}$ is assumed to satisfy the condition

$$(1.1) \quad \limsup_{n \rightarrow \infty} |na_n| < +\infty.$$

Such estimates were obtained first by H. Hadwiger [4] for the Abel transform $t(x)$. Delange [3] developed a general theory for such estimates when $x_n = qn$, where q is some fixed positive number. Usually the estimates proved have the form

$$\limsup_{n \rightarrow \infty, x_n \rightarrow \infty} |t(x_n) - s_n| \leq C \limsup_{n \rightarrow \infty} |na_n|.$$

We call the constant C a Tauberian constant associated with the transformation T .

In this paper we shall prove some Hadwiger-type inequalities for a class of $[J, f(x)]$ transformations (see § 2). In these results the connection between n and x_n will be more general than the relation $x_n = qn$.

As a consequence of the main result of this paper we shall obtain the interesting result that for any sequence $\{s_n\}$ satisfying (1.1) the set of limit points of $\{s_n\}$ and the set of limit points of the Borel transform of $\{s_n\}$ are the same set

2. The $[J, f(x)]$ transformations. The class of $[J, f(x)]$ transformations was defined in [5], where it was shown that the $[J, f(x)]$ transfor-

Received April 13, 1961. The research reported in this paper was sponsored in part by the Air Force Office of Scientific Research, Air Research and Development Command, U.S.A. Air Force, through its European Office.

mations are the sequence-to-function analogues to the Hausdorff transformations. The definition of the $[J, f(x)]$ transformations is the following.

DEFINITION. Suppose $f(x)$ is a real or complex function defined for all $x > x_0 \geq 0$. Let $f^{(n)}(x)$ exists for all $x > x_0$ and $n = 1, 2, \dots$. For a given sequence $\{s_n\}$ ($n \geq 0$) we define the $[J, f(x)]$ transform $t(x)$ of $\{s_n\}$ by

$$t(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} f^{(n)}(x) s_n$$

supposed convergent for all sufficiently large x . We say that $\{s_n\}$ is summable $[J, f(x)]$ to s if $\lim_{x \rightarrow \infty} t(x) = s$.

We shall denote in the rest of this paper by $d_n(x)$, $n = 0, 1, \dots$, $x > 0$, the function $(-1)^n (x^n/n!) f^{(n)}(x)$.

EXAMPLES. The $[J, f(x) \equiv (x+1)^{-1}]$ transformation is the Abel transformation. The $[J, f(x) \equiv e^{-x}]$ transformation is the Borel transformation. The $[J, f(x) \equiv (x+1)^{-\alpha}]$ ($\alpha > -1$) transformation is known as the $A^{(\alpha)}$ transformation.

The following necessary and sufficient conditions for the regularity (that is that for each convergent sequence the T transform of the sequence is also convergent to the same limit) of the $[J, f(x)]$ transformations were obtained in [5].

THEOREM (2.A). *The $[J, f(x)]$ transformation is regular if, and only if, there exists a function*

$$(2.1) \quad \beta(t) \text{ of bounded variation in } 0 \leq t \leq 1$$

such that

$$(2.2) \quad \beta(0) = \beta(0+) = 0, \beta(1) = \beta(1-0) = 1$$

and

$$(2.3) \quad f(x) = \int_0^1 t^x d\beta(t) \quad \text{for } x > x_0.$$

3. Tauberian constants. One of the main results of this section is finding the best (in a certain sense) Tauberian constant associated with a certain class of $[J, f(x)]$ transformations.

THEOREM (3.1). *Suppose the $[J, f(x)]$ transformation is regular, that is (by Theorem (2.A)) there exists a function $\beta(t)$ satisfying (2.1), (2.2) and (2.3). Let $q > 0$ be a constant. If $\beta(t)$ is a nondecreasing function and the integrals*

$$(3.1) \quad \int_0^x \frac{\beta(t)}{t \log 1/t} dt, \int_0^{\infty-} \frac{1 - (t + 1)f(t)}{t(t + 1)} dt = \lim_{x \uparrow \infty} \int_0^x \frac{1 - (t + 1)f(t)}{t(t + 1)} dt$$

exist, then for any sequence $\{s_n\}$ satisfying (1.1) we have

$$(3.2) \quad \limsup_{n \rightarrow \infty, x \rightarrow \infty, nx^{-1} \rightarrow q} \left| s_n - \sum_{m=0}^{\infty} d_m(x)s_m \right| \leq A_q \cdot \limsup_{n \rightarrow \infty} |na_n|$$

where

$$(3.3) \quad A_q = \gamma(\text{Euler's constant}) + \log q - \int_0^{\infty-} \frac{1 - (t + 1)f(t)}{t(t + 1)} dt + 2 \int_0^{e^{-q}} \frac{\beta(t)}{t \log 1/t} dt.$$

Moreover, the constant A_q is the best in the following sense. There is a real sequence $\{s_n\}$ such that $0 < \limsup_{n \rightarrow \infty} |na_n| < +\infty$ and the members of the inequality (3.2) are equal.

In the proof of Theorem (3.1) we shall use the following two theorems. The first is proved by R. P. Agnew in [1], and the second is proved in [6], Theorem 7d, page 295.

THEOREM (3.A). Suppose $\{s_n\}$ is any bounded (real or complex) sequence. Let $\{c_n(x)\}$ be a sequence of functions defined for $0 < x < \infty$ and satisfying

$$(3.4) \quad \lim_{x \rightarrow \infty} c_n(x) = 0 \quad \text{for } n = 0, 1, \dots.$$

$$(3.5) \quad \limsup_{x \rightarrow \infty} \sum_{n=0}^{\infty} |c_n(x)| = M < +\infty.$$

Then we have

$$(3.6) \quad \limsup_{x \rightarrow \infty} \left| \sum_{n=0}^{\infty} c_n(x)s_n \right| \leq M \cdot \limsup_{n \rightarrow \infty} |s_n|.$$

Moreover, M is the best constant in the following sense. There exists a bounded sequence $\{s_n\}$, $\limsup |s_n| > 0$, such that the members of inequality (3.6) are equal.

THEOREM (3.B). If $\beta(t)$ is a normalized function of bounded variation in $0 \leq t \leq R$ for every $R > 0$ (that is $\beta(0) = 0$, $\beta(t) = \frac{1}{2}\{\beta(t-0) + \beta(t+0)\}$) and if the integral

$$(3.7) \quad f(x) = \int_0^{\infty-} e^{-xt} d\beta(t)$$

converges for some x , then

$$(3.8) \quad \lim_{x \rightarrow \infty} \sum_{n=0}^{\lceil xt \rceil} d_n(x) = \beta(t), \quad \text{for } 0 < t < \infty.$$

Proof of Theorem (3.1). We have formally, for $x > 0$,

$$(3.9) \quad \begin{aligned} s_n - \sum_{m=0}^{\infty} d_m(x) s_m &= \sum_{k=0}^n a_k - \sum_{m=0}^{\infty} d_m(x) \sum_{k=0}^m a_k \\ &= \sum_{k=0}^n a_k - \sum_{k=0}^{\infty} a_k \sum_{m=k}^{\infty} d_m(x). \end{aligned}$$

We have (see [5])

$$(3.10) \quad \sum_{m=0}^{\infty} d_m(x) = 1, \quad 0 < x < \infty.$$

Therefore, at least formally, for $x > 0$, $n = 1, 2, \dots$,

$$(3.11) \quad \begin{aligned} s_n - \sum_{m=0}^{\infty} d_m(x) s_m &= \sum_{k=1}^n (ka_k) k^{-1} \left\{ 1 - \sum_{m=k}^{\infty} d_m(x) \right\} \\ &\quad - \sum_{k=n+1}^{\infty} (ka_k) k^{-1} \sum_{m=k}^{\infty} d_m(x). \end{aligned}$$

From the fact that $\beta(t)$ is nondecreasing it follows that

$$(3.12) \quad d_k(x) \geq 0 \quad \text{for } x > 0 \quad \text{and } k = 0, 1, \dots.$$

In order to justify our computations for sequences $\{s_n\}$ satisfying (1.1) it is enough, by (3.12), to show that the two expressions

$$\begin{aligned} \sum_{k=1}^n k^{-1} \left\{ 1 - \sum_{m=k}^{\infty} d_m(x) \right\} \\ \sum_{k=n+1}^{\infty} k^{-1} \sum_{m=k}^{\infty} d_m(x) \end{aligned}$$

converge for $x > 0$. The convergence of the last two expressions will follow from (3.15), (3.20) and (3.22). Suppose the convergence of the last two expressions was proved then in order to complete the proof it is enough to show, by (3.11) and Theorem (3.A), that we have

$$(3.13) \quad \lim_{x \rightarrow \infty} k^{-1} \left\{ 1 - \sum_{m=k}^{\infty} d_m(x) \right\} = 0 \quad \text{for } k = 1, 2, \dots.$$

$$(3.14) \quad \lim_{n \rightarrow \infty, x \rightarrow \infty, nx^{-1} \rightarrow q} \left\{ \sum_{k=1}^n k^{-1} \left[1 - \sum_{m=k}^{\infty} d_m(x) \right] + \sum_{k=n+1}^{\infty} k^{-1} \sum_{m=k}^{\infty} d_m(x) \right\} = A_q.$$

Now, by (3.10),

$$1 - \sum_{m=k}^{\infty} d_m(x) = \sum_{m=0}^{k-1} d_m(x).$$

From

$$d_m(x) = \frac{x^m}{m!} \int_0^1 t^x \left(\log \frac{1}{t}\right)^m d\beta(t), \quad \text{for } m = 0, 1, \dots$$

we get $\lim_{x \rightarrow \infty} d_m(x) = 0$ for $m = 0, 1, \dots$, which proves (3.13). The expression in brackets on the left-hand side of (3.14) might be written in the form

$$\begin{aligned} (3.15) \quad & \sum_{k=1}^n k^{-1} \left[1 - \sum_{m=k}^{\infty} d_m(x) \right] + \sum_{k=n+1}^{\infty} k^{-1} \sum_{m=k}^{\infty} d_m(x) \\ &= \sum_{k=1}^n k^{-1} - \sum_{k=1}^{\infty} k^{-1} \sum_{m=k}^{\infty} d_m(x) + 2 \sum_{k=n+1}^{\infty} k^{-1} \sum_{m=k}^{\infty} d_m(x) \\ &\equiv \sum_{k=1}^n k^{-1} - S_1(x) + 2S_{2,n}(x). \end{aligned}$$

Now, by (3.10), for $k = 1, 2, \dots$,

$$\begin{aligned} (3.16) \quad & \frac{d}{dx} \sum_{m=k}^{\infty} d_m(x) = \frac{d}{dx} \left\{ 1 - \sum_{m=0}^{k-1} d_m(x) \right\} \\ &= -\frac{d}{dx} \sum_{m=0}^{k-1} d_m(x) \\ &= (-1)^k \frac{x^{k-1}}{(k-1)!} f^{(k)}(x). \end{aligned}$$

We have also

$$\begin{aligned} (3.17) \quad & \sum_{m=k}^{\infty} d_m(x) = \int_0^{\infty} e^{-xt} \sum_{m=k}^{\infty} \frac{(xt)^m}{m!} d\{1 - \beta(e^{-t})\} \\ &= \int_0^1 \left\{ t \sum_{m=k}^{\infty} (m!)^{-1} \left(\log \frac{1}{t}\right)^m \right\} d\beta(t^{1/x}). \end{aligned}$$

By the Helly-Bray theorem (see [6], Theorem 16.4, page 31) we have from (3.17)

$$(3.18) \quad \lim_{x \downarrow 0} \sum_{m=k}^{\infty} d_m(x) = 0, \quad \text{for } k = 1, 2, \dots$$

From (3.16) and (3.18) we have

$$\sum_{m=k}^{\infty} d_m(x) = \int_0^x (-1)^k \frac{t^{k-1}}{(k-1)!} f^{(k)}(t) dt, \quad k = 1, 2, \dots$$

Therefore

$$(3.19) \quad S_1(x) = \sum_{k=1}^{\infty} \int_0^x (-1)^k \frac{t^{k-1}}{k!} f^{(k)}(t) dt$$

(and, by Fubini's theorem and (3.1), if the last integral exists)

$$= \int_0^x t^{-1} \sum_{k=1}^{\infty} d_k(t) dt$$

(and by (3.10))

$$= \int_0^x \frac{1 - f(t)}{t} dt .$$

The last integral exists by Fubini's theorem, because

$$\begin{aligned} \int_0^x \frac{1 - f(t)}{t} dt &= \int_0^x t^{-1} \left\{ 1 - \int_0^1 u^t d\beta(u) \right\} dt \\ &= \int_0^x dt \int_0^1 \frac{1 - u^t}{t} d\beta(u) \\ &= \int_0^1 d\beta(u) \int_0^x \frac{1 - u^t}{t} dt \end{aligned}$$

and the last integral exists, as is easy to see. Therefore

$$(3.20) \quad S_1(x) = \log(x + 1) + \int_0^x \frac{1 - (t + 1)f(t)}{t(t + 1)} dt .$$

Hence

$$(3.21) \quad \lim_{n \rightarrow \infty, x \rightarrow \infty, nx^{-1} \rightarrow q} \left\{ \sum_{k=1}^n k^{-1} - S_1(x) \right\} = \gamma + \log q - \int_0^{\infty-} \frac{1 - (t + 1)f(t)}{t(t + 1)} dt .$$

In the same way that we obtained (3.20) we get

$$(3.22) \quad \begin{aligned} S_{2,n}(x) &= \int_0^x t^{-1} \sum_{m=n+1}^{\infty} d_m(t) dt \\ &= \int_0^x t^{-1} \left\{ 1 - \sum_{m=0}^n d_m(t) \right\} dt \\ &= \int_0^{x/n} u^{-1} \left\{ 1 - \sum_{m=0}^n d_m(nu) \right\} du . \end{aligned}$$

Now we get from Theorem (3.B) (with $x = nu$, $t = u^{-1}$, $0 < u \leq 1$) and Lebesgue's theorem on the integration of boundedly convergent series, since

$$f(x) = \int_0^{\infty} e^{-xt} d\{1 - \beta(e^{-t})\}$$

that

$$(3.23) \quad \begin{aligned} \lim_{n \rightarrow \infty, x \rightarrow \infty, nx^{-1} \rightarrow q} S_{2,n}(x) &= \int_0^{q^{-1}} u^{-1} \beta(e^{-u^{-1}}) du \\ &= \int_0^{e^{-q}} \frac{\beta(v)}{v \log 1/v} dv . \end{aligned}$$

(3.21) and (3.23) show that (3.14) is true. This completes the proof. Q.E.D.
Using the following expression for Euler's constant

$$\gamma = \int_0^\infty \frac{1 - (t + 1)e^{-t}}{t(t + 1)} dt$$

(see Bromwich [2], page 507) we obtain for A_q of Theorem (3.1) the expression

$$(3.24) \quad A_q = \log q + \int_0^{\infty-} \frac{f(t) - e^{-t}}{t} dt + 2 \cdot \int_0^{e^{-q}} \frac{\beta(t)}{t \log 1/t} dt$$

and we may state Theorem (3.1) in the equivalent form

THEOREM (3.2). *If the suppositions of Theorem (3.1) are satisfied then the number A_q of Theorem (3.1) has the representation (3.24).*

4. Some consequences. We shall indicate here some consequences of the results of § 3.

EXAMPLE (4.1). If we choose in Theorem (3.2) $\beta(t) = 0$, for $0 \leq t < e^{-1}$, $\beta(t) = 1$, for $e^{-1} \leq t \leq 1$, and $q > 0$ is a fixed number than for each sequence $\{s_n\}$ satisfying (1.1) we have

$$\limsup_{n \rightarrow \infty, x \rightarrow \infty, nx^{-1} \rightarrow q} \left| s_n - e^{-x} \sum_{m=0}^{\infty} s_m \frac{x^m}{m!} \right| \leq |\log q| \cdot \limsup_{n \rightarrow \infty} |na_n|.$$

Moreover the constant $|\log q|$ is the best possible in the sense of Theorem (3.1).

Example (4.1) is an immediate consequence of Theorem (3.2).

If we choose in Example (4.1) $q = 1$ we get

THEOREM (4.1). *For a sequence $\{s_n\}$ satisfying (1.1) we have*

$$\lim_{n \rightarrow \infty, x \rightarrow \infty, nx^{-1} \rightarrow 1} \left| s_n - e^{-x} \sum_{m=0}^{\infty} s_m \frac{x^m}{m!} \right| = 0;$$

and therefore the set of limit points of $\{s_n\}$ and the set of limit points of the Borel transform of $\{s_n\}$ are the same set.

Theorem (4.1) raises the following problem: It is known that Borel summability of a sequence and the condition $\sqrt{n}a_n = O(1)$ imply the convergence of the sequence; now by Theorem (4.1) the stronger condition $na_n = O(1)$ implies that the set of limit points of the Borel transform and the set of limit points of the sequence are the same set. We may ask therefore if it is true in general, or for which transformations it is true, that if a Tauberian condition stronger than the appropriate Tauberian condition for the transformation is satisfied then the set of limit points of the transform and the set of limit points of the sequence are the same set.

EXAMPLE (4.2). If we choose in Theorem (3.1) $\beta(t) = t$ we obtain for the Abel transformation $A_q = \gamma + \log q + 2 \int_q^\infty v^{-1} e^{-v} dv$.

The last consequence was obtained by R. P. Agnew in [1].

EXAMPLE (4.3). If we choose in Theorem (3.1)

$$\beta(t) = \frac{1}{\Gamma(\alpha + 1)} \int_0^t \left(\log \frac{1}{u}\right)^\alpha du, \quad \alpha > -1,$$

then

$$A_q = \gamma + \log q - \int_0^1 \frac{1 - (1-t)^\alpha}{t} dt + \frac{2}{\Gamma(\alpha + 1)} \int_q^\infty v^\alpha e^{-v} \log \frac{v}{q} dv.$$

(3.2) assumes, in this particular case, the form, for $y = x/(x + 1)$,

$$(4.1) \quad \limsup_{n \rightarrow \infty, y \rightarrow 1, n(1-y) \rightarrow q} \left| s_n - (1-y)^{\alpha+1} \sum_{m=0}^{\infty} \binom{m+\alpha}{m} s_m y^m \right| \leq A_q \cdot \limsup |n a_n|.$$

Example (4.2) is the particular case $\alpha = 0$ of Example (4.2).

$c_n^{(\alpha)}$ ($\alpha > -1$) the Cesàro transform of order α of $\{s_n\}$ (in short the (C, α) transform) is defined by

$$c_n^{(\alpha)} = \left\{ \binom{n+\alpha}{n} \right\}^{-1} \sum_{m=0}^n \binom{n-m+\alpha-1}{n-m} s_m, \quad n = 0, 1, \dots$$

If in Example (4.3) we replace $\{s_n\}$ by the sequence $\{c_n^{(\alpha)}\}$, the (C, α) transform of a sequence $\{c_n\}$, we obtain from (4.1) the following result.

EXAMPLE (4.4). Suppose $\alpha > -1, q > 0$. For a sequence $\{s_n\}$ denote by $\{c_n^{(\alpha)}\}$ and $\{a_n^{(\alpha)}\}$, respectively, the sequences of the (C, α) transform of $\{s_n\}$ and $\{n a_n\}$. Then for each sequence $\{s_n\}$ with a bounded sequence $\{a_n^{(\alpha)}\}$ we have

$$(4.3) \quad \limsup_{n \rightarrow \infty, y \rightarrow 1, n(1-y) \rightarrow q} \left| c_n^{(\alpha)} - \sum_{m=0}^{\infty} a_m y^m \right| \leq B_q \cdot \limsup_{m \rightarrow \infty} |a_m^{(\alpha)}|$$

where B_q is the constant A_q of Example (4.3). Moreover the constant B_q is the best in the following sense. There is a real sequence $\{s_n\}$ such that $0 < \limsup |a_n^{(\alpha)}| < \infty$ and the members of inequality (4.3) are equal.

5. The minimum of the function A_q . Now we investigate the behaviour of A_q of Theorem (3.1) as a function of $q > 0$.

THEOREM (5.1). Suppose $\beta(t)$ is a function satisfying (2.1) and (2.2). Define $f(x)$ by (2.3). If $\beta(t)$ is nondecreasing and the integrals (3.1) exist then for $A_q (q > 0)$ defined by (3.3), as a function of q we

have

(5.1) $A_q \geq 0$ for $q > 0$.

(5.2) A_q is a continuous function for $q > 0$.

(5.3) $\lim_{q \uparrow \infty} A_q = +\infty$.

(5.4) $\lim_{q \downarrow 0} A_q = +\infty$.

(5.5) A_q has an absolute minimum for $q > 0$.

(5.6) If $\beta(t)$ is a continuous function the value of the absolute minimum of A_q is obtained at the point (or points) q_0 satisfying $\beta(e^{-q_0}) = 1/2$.

Proof. (5.1) follows from the inequality (3.2). (5.2) follows from the fact that $\log q$ is a continuous function and that an integral is a continuous function of its limits. (5.3) follows from the fact that

$$\lim_{q \rightarrow \infty} \log q = +\infty; \lim_{q \rightarrow \infty} \int_0^{e^{-q}} \frac{\beta(t)}{t \log 1/t} dt = 0.$$

We prove (5.4) in the following way: By (2.2) $\beta(1) = \beta(1 - 0) = 1$ and $\beta(t) \geq 0$, therefore for sufficiently small $\delta, 0 < \delta < 1$, if $1 - \delta \leq t \leq 1$ then $\beta(t) > 2/3$. Therefore for all sufficiently small $q > 0$ we have

$$\begin{aligned} \log q + 2 \int_0^{e^{-q}} \frac{\beta(t)}{t \log 1/t} dt &\geq \log q + \frac{4}{3} \int_{1-\delta}^{e^{-}} \frac{dt}{t \log 1/t} \\ &= \frac{4}{3} \log \frac{1}{1-\delta} + \frac{1}{3} \log \frac{1}{q} \\ &\rightarrow \infty \text{ as } q \downarrow 0. \end{aligned}$$

This proves (5.4). (5.5) follows immediately from (5.1)–(5.4). If $\beta(t)$ is continuous then A_q has a continuous derivative

$$\frac{d}{dq} A_q = q^{-1} \{1 - 2\beta(e^{-q})\}$$

and the absolute minimum of A_q , for $q > 0$, is given by $(d/dq)A_q = 0$ or by $\beta(e^{-q}) = 1/2$. This proves (5.6).

Thus we see that for the transformation of Example (5.2)

$$\min_{q>0} A_q = A_{\log 2} = \gamma + \log \log 2 + 2 \int_{\log 2}^{\infty} v^{-1} e^{-v} dv.$$

This result was obtained by R. P. Agnew in [1].

For the transformation of Example (4.3) and for the B_q of Example (4.4) we see that

$$\min_{q>0} A_q = A_{q_0} = \min_{q>0} B_q$$

where q_0 is given by equation

$$\{\Gamma(\alpha + 1)\}^{-1} \int_0^{a_0} t^\alpha e^{-t} dt = \frac{1}{2}.$$

6. Conclusion. We saw in § 5 that the function A_q (of q) obtained in Theorem (3.1) has an absolute minimum for some $q_0 > 0$. We shall denote this minimum by $B = B(f(x))$. That is $B = A_{q_0}$.

Denote by z' a limit point of a sequence $\{s_n\}$. We denote by z'' a limit point of a linear transform of $\{s_n\}$. Then we obtain from Theorem (3.1) the following result concerning limit points z' and z'' .

THEOREM (6.1). *If the suppositions of Theorem (3.1) are satisfied then for any sequence $\{s_n\}$ satisfying (1.1) and its $[J, f(x)]$ transform we have: (i) To each z' corresponds at least one z'' such that*

$$(6.1) \quad |z' - z''| \leq B \cdot \limsup_{n \rightarrow \infty} |na_n|.$$

(ii) *To each z'' corresponds at least one z' such that*

$$(6.2) \quad |z'' - z'| \leq B \cdot \limsup_{n \rightarrow \infty} |na_n|.$$

We do not know if the constant B in (6.1) and in (6.2) is the best possible (the smallest).

BIBLIOGRAPHY

1. R. P. Agnew, *Abel transforms and partial sums of Tauberian series*, Ann. of Math., (2), **50** (1949), 110-117.
2. T. J. I'a Bromwich, *An introduction to the theory of infinite series*, Macmillan and Co. (1942).
3. H. Delange, *Sur les théorèmes inverses des procédés de sommation, I*, Ann. Écoles Norm. (3), **67** (1950), 99-160.
4. H. Hadwiger, *Über ein Distanztheorem bei der A-Limitierung*, Comment. Math Helv. **16** (1944), 209-214.
5. A. Jakimovski, *The sequence-to-function analogues to Hausdorff transformations*, Bulletin of the Research Council of Israel, **8F** (1960), 135-154.
6. D. V. Widder, *The Laplace Transform*, Princeton University Press, 1946.

THE HEBREW UNIVERSITY, JERUSALEM

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RALPH S. PHILLIPS

Stanford University
Stanford, California

M. G. ARSOVE

University of Washington
Seattle 5, Washington

A. L. WHITEMAN

University of Southern California
Los Angeles 7, California

LOWELL J. PAIGE

University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH

T. M. CHERRY

D. DERRY

M. OHTSUKA

H. L. ROYDEN

E. SPANIER

E. G. STRAUS

F. WOLF

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

* * *

AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insa'susha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 12, No. 2

February, 1962

William George Bade and Robert S. Freeman, <i>Closed extensions of the Laplace operator determined by a general class of boundary conditions</i>	395
William Browder and Edwin Spanier, <i>H-spaces and duality</i>	411
Stewart S. Cairns, <i>On permutations induced by linear value functions</i>	415
Frank Sydney Cater, <i>On Hilbert space operators and operator roots of polynomials</i>	429
Stephen Urban Chase, <i>Torsion-free modules over $K[x, y]$</i>	437
Heron S. Collins, <i>Remarks on affine semigroups</i>	449
Peter Crawley, <i>Direct decompositions with finite dimensional factors</i>	457
Richard Brian Darst, <i>A continuity property for vector valued measurable functions</i>	469
R. P. Dilworth, <i>Abstract commutative ideal theory</i>	481
P. H. Doyle, III and John Gilbert Hocking, <i>Continuously invertible spaces</i>	499
Shaul Foguel, <i>Markov processes with stationary measure</i>	505
Andrew Mattei Gleason, <i>The abstract theorem of Cauchy-Weil</i>	511
Allan Brasted Gray, Jr., <i>Normal subgroups of monomial groups</i>	527
Melvin Henriksen and John Rolfe Isbell, <i>Lattice-ordered rings and function rings</i>	533
Amnon Jakimovski, <i>Tauberian constants for the $[J, f(x)]$ transformations</i>	567
Hubert Collings Kennedy, <i>Group membership in semigroups</i>	577
Eleanor Killam, <i>The spectrum and the radical in locally m-convex algebras</i>	581
Arthur H. Kruse, <i>Completion of mathematical systems</i>	589
Magnus Lindberg, <i>On two Tauberian remainder theorems</i>	607
Lionello A. Lombardi, <i>A general solution of Tonelli's problem of the calculus of variations</i>	617
Marvin David Marcus and Morris Newman, <i>The sum of the elements of the powers of a matrix</i>	627
Michael Bahir Maschler, <i>Derivatives of the harmonic measures in multiply-connected domains</i>	637
Deane Montgomery and Hans Samelson, <i>On the action of $SO(3)$ on S^n</i>	649
J. Barros-Neto, <i>Analytic composition kernels on Lie groups</i>	661
Mario Petrich, <i>Semicharacters of the Cartesian product of two semigroups</i>	679
John Sydney Pym, <i>Idempotent measures on semigroups</i>	685
K. Rogers and Ernst Gabor Straus, <i>A special class of matrices</i>	699
U. Shukla, <i>On the projective cover of a module and related results</i>	709
Don Harrell Tucker, <i>An existence theorem for a Goursat problem</i>	719
George Gustave Weill, <i>Reproducing kernels and orthogonal kernels for analytic differentials on Riemann surfaces</i>	729
George Gustave Weill, <i>Capacity differentials on open Riemann surfaces</i>	769
G. K. White, <i>Iterations of generalized Euler functions</i>	777
Adil Mohamed Yaqub, <i>On certain finite rings and ring-logics</i>	785