ON TWO TAUBERIAN REMAINDER THEOREMS

MAGNUS LINDBERG
1. Introduction. In this paper we are going to treat two different tauberian problems by a common method. The first of these problems originates from a question raised by P. Erdős [6] and the second was suggested by F. Brownell [5]. Our method gives in both cases slightly better results than those hitherto known.

In connection with his work on the Prime-number-theorem Erdős came to consider the following problem:

If $A$ is a nondecreasing function, such that

\begin{equation}
\int_0^x A(x - t)dA(t) = \frac{x^2}{2} + O(x), \quad x \to \infty,
\end{equation}

what can be said about the order of magnitude of $a(x) = A(x) - x$?

Erdős proved that $a(x) = o(x)$ and constructed a counter example showing that (1.1) couldn’t give $a(x) = o(\sqrt{x})$.

Later on Avakumović [2] gave the result $a(x) = O(x^{\varepsilon + \frac{1}{2}})$ for every $\varepsilon > 0$.

In 1956 Bojanic, Jurkat and Peyerimhoff [4] improved this to $a(x) = O(x^{\varepsilon - \frac{1}{2}})$, this being the best result known till now.

Here we will study the more general case, when the remainder in (1.1) is $O(x^q),\ 0 \leq q < 2$. Assuming this our result is $a(x) = O(x^{q - \frac{1}{2}})$. Thus we are able to remove the logarithm in the last estimate.

The second of our problems was presented as a research problem in the Bulletin of the American Mathematical Society [5] by F. Brownell:

"Let $F(x)$ be a real valued function of real $x \geq 0$ which is of bounded variation over every finite interval $[0, N]$, which is continuous at $x = 0$ with $F(0) = 0$, and which has $\int_0^\infty e^{-tx}|dF(x)| < +\infty$ for real $t > 0$. With $s = t + iv, t$ and $v$ real, define $g$ by the Lebesque-Stieltjes integral $g(s) = \int_0^\infty e^{-sx}dF(x)$, analytic in the region $t > 0$. Let $F$ satisfy the conditions that

\begin{equation}
g(t) = b + 0(e^{-\varepsilon t}), \quad t \to 0+\n\end{equation}

for some real constants $c > 0$ and $b$, and that

\begin{equation}
F(x) + Kx^\nu
\end{equation}

be strictly increasing over $x \geq 1$ for some real constants $K > 0$ and $\nu \geq 1$.

Is it true as conjectured, that

Received April 1961.
as $x \to \infty$ if in addition to (1.2) and (1.3) it is also assumed that $g(iv) = \lim_{t \to 0^+} g(t + iv)$ exists finite for all $v \neq 0$, that the resulting $g(s)$ is continuous in $t \geq 0$ and $s \neq 0$, and that over all such $s$

$$|g(s)| \leq M_1|s|^{-\gamma} + M_2$$

for some finite constants $M_1$ and $M_2$ and $\gamma > 0$.

That

$$F(x) = O(x^{\nu-\gamma/2})$$

can be proved even if assumption (1.4) is excluded has been shown by several authors. In the case $\nu = 1$, this result was obtained by Avakumović [1] and the general case has been treated by Ganelius [9] and Korevaar [11], basing their works on the results of Freud [7, 8] (cf. Ganelius [10]).

The example $F(x) = \int_0^x (\sin \sqrt{t}) \, t^{\nu-1} \, dt$ (Korevaar [11]), for integer $\nu$ satisfies (1.2), (1.3) and (1.6) but not (1.5), thus proving (1.6) to be the best possible under (1.2) and (1.3) only. But the example also violates (1.4).

In this paper we show that (1.5) can be obtained even under the following additional assumption:

$$(1.4)' \quad |g(s)| < M_1|s|^{1+3\gamma+\eta} + M_2, \quad \cap Re \, s > 0$$

and for some finite constants $M_1$, $M_2$ and $\gamma > 0$. It is also sufficient if $\nu > 1/2$. This is evidently a weaker assumption than that suggested by Brownell.

The common treatment of these problems was suggested to me by Professor T. Ganelius. I would like to thank him for his help and valuable advice.

2. A lemma on Laplace-transforms. In order to sum up the common properties of the proofs of the two theorems we state the following lemma:

**Lemma.** Let $F$ be a real valued function on $[0, \infty)$, such that $F(x) + Kx^\alpha$ is nondecreasing over $x > x_0$ for some positive constants $K$ and $\alpha$:

Define the function $f$ by

$$f(s) = \int_0^\infty e^{-sx} F(x) \, dx$$

where the integral is supposed to be absolutely convergent for $Re \, s =
Then if
\[
(2.2) \quad \int_{-\infty}^{+\infty} |f(t + iv)| v^{-\omega^{-1}} \sin^2(\omega v) dv = O(t^{\beta}) ,
\]
t \to 0+ , where \( \omega = t^{(\alpha-1+\beta)/2} = t^\gamma \) and \( \alpha + \beta + 1 > 0 \) it follows that
\[
(2.3) \quad F(x) = O(x^{(\alpha-\beta-1)/2}) , \quad x \to +\infty .
\]

Proof. Let \( G_\omega \) be the function defined by
\[
G_\omega(x) = \begin{cases} 1 & \text{if } |x| < 2\omega \\ 0 & \text{if } |x| \geq 2\omega \end{cases}.
\]
The Fourier transform of \( G_\omega \) is the function \( H_\omega \), defined by
\[
(2\pi)^{-1} \int_{-\infty}^{+\infty} f(t + iv)e^{itx} H_\omega(v) dv
\]
and Parseval’s formula gives
\[
(2.4) \quad \int_{T-2\omega}^{T+2\omega} e^{-tx} F(x) (1 - (2\omega)^{-1} |T - x|) dx .
\]
Taking the absolute value of both sides we get
\[
\left| \int_{T-2\omega}^{T+2\omega} e^{-tx} F(x) (1 - (2\omega)^{-1} |T - x|) dx \right| \leq \pi^{-1} \int_{-\infty}^{+\infty} |f(t + iv)| \omega^{-1} v^{-2} \sin^2(\omega v) dv = o(t^\beta) .
\]

If we in formula (2.4) put \( T = t^{-1} + 2t^\gamma \) we may conclude, \( F(x) + Kx^a \) being nondecreasing, that for \( T - 2t^\gamma \leq \tau \leq T + 2t^\gamma \)
\[
F(\tau) \geq F(t^{-1}) - K((t^{-1} + 4t^\gamma)^a - t^{-a}) = F(t^{-1}) - 4 \alpha K t^\gamma (t^{-1} + 4t^\gamma)^{a-1} \geq F(t^{-1}) - K t^\gamma - t^{-\alpha+1} .
\]

Suppose that \( F(t^{-1}) \geq 0 \). By aid of (2.5) we infer from (2.4) that
\[
K_{t}^{\beta} \geq \int_{t^{-1}}^{t^{-1} + 4t^\gamma} e^{-tx} F(x) (1 - (2t^\gamma)^{-1} |T - x|) dx \geq \int_{t^{-1}}^{t^{-1} + 4t^\gamma} e^{-tx}(F(t^{-1}) - K t^\gamma - t^{-\alpha+1}) (1 - (2t^\gamma)^{-1} |T - x|) dx \geq e^{-5} F(t^{-1}) 2t^\gamma - K_{t} e^{-t\gamma - t^{-\alpha+1}} 2t^\gamma \geq 2e^{-5} F(t^{-1}) t^\gamma - K_{t}^{\beta} .
\]
Consequently \( F(t^{-1}) \leq K_{t}^{\beta - \alpha + 1/2} \).

If \( F(t^{-1}) < 0 \) we can in the same way show that \( F(t^{-1}) \geq -K_{t}^{\beta - \alpha + 1/2} \), choosing \( T = t^{-1} - 2t^\gamma \). Hence our lemma is proved.

3. On a nonlinear Tauberian theorem. We shall now apply our
lemma to problem of Erdős, mentioned in the introduction.

**Theorem 1.** Let $A$ be a nondecreasing function, defined on $[0, \infty)$ and such that

\[
(3.1) \quad h(x) = \int_0^x A(x-t) \, dA(t) = \frac{x^2}{2} + O(x^q), \quad x \to +\infty
\]

where $q$ is some real number, $0 \leq q < 2$.

Then

\[
(3.2) \quad A(x) = x + O(x^{(q+1)/3}), \quad x \to +\infty.
\]

**Proof.** We are going to prove the theorem in the case when $A$ is a normalized function, i.e. $A(0) = 0$ and $A(x) = 2^{-i}(A(x-0) + A(x+0))$ for $x > 0$. Owing to the nondecreasing of $A$ this is not a restriction, since formula (3.2) will be correct for any nondecreasing function $A$, if it is so for the corresponding normalized function. We also assume the function $h$ to be normalized, because normalization is possible at the countable set of points, where it may not be defined (see [12] p. 84).

Our problem is now reduced to getting an estimate of the type (2.2). For this purpose we use some of the results of Bojanic, Jurkat and Peyerimhoff [4].

Since

\[
A^2\left(\frac{x}{2}\right) = A\left(\frac{x}{2}\right) \int_0^{x/2} dA(u) \leq \int_0^{x/2} A(x-u) \, dA(u)
\]

\[
\leq \int_0^x A(x-u) \, dA(u) = O(x^2),
\]

we get $A(x) = O(x)$, which implies that the integral $(s = t + iv)$

\[
(3.3) \quad f(s) = \int_0^\infty e^{-su} dA(u)
\]

is absolutely convergent for $t > 0$, and that

\[
(3.4) \quad f^2(s) = \int_0^\infty e^{-su} h(u) \, du = s \int_0^\infty e^{-su} h(u) \, du \quad \text{for} \quad t > 0 \quad ([12] \text{ p. 91}).
\]

According to the assumptions on $h$, we have for the function $g$, defined by $g(x) = h(x) - x^{q/2}$, the estimate

\[
(3.5) \quad |g(x)| \leq K_1(x^q + 1) \quad \text{for some} \quad K_1 \quad \text{and every} \quad x \geq 0.
\]

Putting the function $g$ into (3.4) we get

\[
(3.6) \quad f^2(s) = s \int_0^\infty e^{-su}\left(\frac{u^2}{2} + g(u)\right) \, du = s^{-3} + s \int_0^\infty e^{-su} g(u) \, du.
\]
We now restrict $t$ to the interval $(0,1)$ and take the absolute of (3.6)\n
\begin{align}
|f^2(s) - s^{-2}| &\leq |s| \int_0^\infty K_t(u^q + 1) e^{-s^2 t} du \\
&= K_t \Gamma(q + 1) |s| t^{-(q+1)} + K_1 |s| t^{-1} \leq K_2 |s| t^{-(q+1)}.
\end{align}

This formula may also be expressed as

\begin{equation}
|f^3(s) = s^{-1}(1 + r_s(s) s^2 t^{-(q+1)}), \quad \text{where } |r_s(s)| \leq K_3.
\end{equation}

If $|s|^3 t^{-(q+1)} < (2K_3)^{-1}$ we conclude from (3.8) that

\begin{equation}
f(s) = s^{-1}(1 + r_s(s) s^2 t^{-(q+1)}), \quad \text{where } |r_s(s)| \leq K_3.
\end{equation}

Hence

\begin{align}
|f(s) - s^{-1}| &\leq K_3 |s|^2 t^{-(q+1)} \leq \sqrt{K_3} |s|^{1/2} t^{-(q+1)/2}.
\end{align}

If $|s|^3 t^{-(q+1)} \geq (2K_3)^{-1}$ we get from (3.8) the estimate $|f(s)| \leq |s|^{-1} (1 + |s|^3 t^{-(q+1)} K_3)^{1/2} \leq \sqrt{3K_3} |s|^{1/2} t^{-(q+1)/2}$ and

\begin{align}
|f(s) - s^{-1}| &\leq |s|^{1/2} t^{-(q+1)/2} (\sqrt{3K_3} + \sqrt{2K_3}) \\
&\leq |s|^2 t^{-(q+1)/2} K_3 (\sqrt{6} + 2).
\end{align}

We now define a new function $a$ by

$$a(u) = \begin{cases} A(u) - u & \text{if } u \geq 0 \\ 0 & \text{if } u < 0 \end{cases}$$

and a function $L$ by

$$L(s) = s^{-2}(f(s) - s^{-1}) = \int_0^\infty e^{-s^2 t - s^2 u} (A(u) - u) du \quad \text{for } t > 0.$$  

The estimates (3.9) and (3.10) give us

\begin{equation}
|L(s)| \leq K_3 |s| t^{-(q+1)} \quad \text{and} \quad |L(s)| \leq K_3 |s|^{1/2} t^{-(q+1)/2}.
\end{equation}

These are the estimates corresponding to those given by Bojanic, Jurkat and Peyerimhoff in [4].

We are now going to use our lemma with $a$ as the function $F$, and $L$ as the function $f$ in formula (2.1). The number $\alpha$ is 1 and we need an estimate of the type (2.2):

\begin{align}
\int_{-\infty}^{+\infty} |L(t + iv)| |w^{-1} v^{2} \sin^2(\omega v)| dv \\
\leq 2K_3 \int_0^{(q+1)/3} |s|^2 t^{-(q+1)} \omega dv + 2K_3 \int_{-(q+1)/3}^{+\infty} \omega^{-1} v^{2} |s|^{-1/2} t^{-(q+1)/2} dv \\
= 2K_3 \omega t^{-q} \int_0^{(q-2)/3} (1 + w)^{1/2} dw.
\end{align}
\[ + 2K_3 \omega^{-1} t^{-2-(q/3)} \int_{t(q-2)/3}^{+\infty} u^{-2}(1 + u^2)^{-1/4} \, du \]

\[ \leq 2K_3(\sqrt{2} t^{1-q} \omega + t^{-(q+1)/3} \omega + (2/3) \omega^{-1} t^{-2-(q/3)} t^{-(3/2)(q-2)/3}) \]

\[ \leq 2K_3(3t^{-(q+1)/3} \omega + \omega^{-1} t^{-(q+1)}) = 8K_3 t^{-(2/3)(q+1)} \]

for

\[ w = t^{\gamma} = t^{(1-(3/2)(q+1)-1)/2} = t^{-(q+1)/3}. \]

With \( \alpha = 1, \beta = -(2/3)(q + 1) \) and \( \alpha + \beta = (1/3)(1 - 2q) > -1 \) we apply the lemma and get (3.2):

\[ a(x) = O(x^{(q+1)/3}). \]

Putting \( q = 1 \) we obtain that \( h(x) = x^2/2 + O(x) \) implies \( A(x) = x + O(x^{2/3}) \).

As an example that the conclusion \( A(x) = x + o(x^{2/3}) \) is false, we give

\[ A(x) = \begin{cases} 10^n & \text{if } 10^n - 4^{-1} 10^{2q/3} < x < 10^n + 4^{-1} 10^{2q/3}, \ n = 1, 2, \ldots \\ x & \text{everywhere else} \end{cases} \]

REMARK. If the assumption (3.1) is formulated

\[ h(x) = x^{2p} \Gamma^p(p + 1) \Gamma^{-1}(2p + 1) + O(x^p) \quad (p > 0, 0 \leq q < 2p) \]

we can with the same method as above prove

\[ A(x) = x^p + O(x^q), \quad \text{where } d = (4p^3 + p + pq + 4q)/(6p + 9), \ x \to \infty. \]

\( p = q \) gives for instance \( d = 5(p^3 + p)/(3(2p + 3)) \sim (5/6)p \) as \( p \to \infty \).

4. On Brownell's conjecture.

THEOREM 2. Let \( F \) be a real valued function on \([0, \infty)\), which is of bounded variation over every finite interval \([0, N]\), which is continuous at \( x = 0 \) with \( F(0) = 0 \), and for which \( \int_0^\infty e^{-t^2} |dF(x)| < +\infty \) for real \( t > 0 \). With \( t \) and \( v \) real and \( s = t + iv \) we define \( g \) by the Lesbesque-Stieltjes integral \( g(s) = \int_0^\infty e^{-t^2} dF(x) \), analytic in the region \( t > 0 \). Let \( F \) satisfy the three conditions that

(4.1) \( g(t) = b + O(e^{-\alpha t}), \ t \to 0+, \) for some real constants \( c > 0 \) and \( b, \)

(4.2) \( F(x) + Kx^\alpha \) is nondecreasing over \( x \geq 1 \) for some real constants \( K > 0 \) and \( \alpha > 1/2, \)

(4.3) \( |g(s)| \leq M_1 |s|^{\alpha-\eta+\eta} + M_2 \) for \( t > 0 \) and some real constants \( M_1, M_2 \) and \( 1 > \eta > 0, \eta < \alpha - 1/2. \)

Then as \( x \to + \infty \)
4.4) \( F(x) = o(x^{a-1/2}) \), and actually is \( O(x^{a-1/2-\varepsilon}) \) for some \( \varepsilon > 0 \).

Proof. Define \( \tilde{g}(s) = g(s) - b \), so \( \tilde{g}(t) = O(e^{-ct}) \) in (4.1). Also \( \tilde{g}(s) = \int_0^\infty e^{-sz}dF(x) - b = s \int_0^\infty F(x)e^{-sz} dx - b = s \int_0^\infty [F(x) - b] e^{-sz} dx \).

From (4.3) we get \( |\tilde{g}(s)| \leq K_1 |s|^{1-2a+\eta} \) if \( |s| \leq 2 \) and \( |\tilde{g}(s)| \leq K_1 \) if \( |s| \geq 1 \) for some real constant \( K_1 \).

We now define a function \( h \) by \( h(s) = K_1^{-1} s^{2a-1-\eta} \tilde{g}(s) \).

This function evidently satisfies the following inequalities:

\[
|h(s)| \leq 1 \text{ if } |s| \leq 2 \text{ and } |h(s)| \leq |s|^{2a-1-\eta} \text{ if } |s| \geq 1.
\]

In order to use our lemma, our problem is now to estimate the integral

\[
\int_{-\infty}^{+\infty} |\tilde{g}(t + iv)| |t + iv|^{-1} \omega^{-1} v^{-2} \sin^2(\omega v) dv
\]

\[
= K_1 \int_{-\infty}^{+\infty} |h(t + iv)| |t + iv|^{-2} \omega^{-1} v^{-2} \sin^2(\omega v) dv.
\]

The following notations will be used

\[
\delta = \eta/(2(2a + 1)), \quad \varepsilon = \delta \eta/3.
\]

The number \( \omega \) will be \( t^{-1/2+\varepsilon} = t^\varepsilon \).

Inserting the estimates for \( h \) on the right side of (4.6) we obtain

\[
\int_{-\infty}^{+\infty} |\tilde{g}(t + iv)| |t + iv|^{-1} \omega^{-1} v^{-2} \sin^2(\omega v) dv
\]

\[
\leq 2K_1 \int_{-1}^{+\infty} v^{-3} \omega^{-1} dv + 2K_1 \int_{1/2+\delta}^{1} v^{-2} \omega^{-1} v^{-2} dv
\]

\[
+ K_1 \int_{-1/2+\delta}^{1/2+\delta} |h(t + iv)| t^{-2a} dv
\]

\[
= O(t^{1/2-\varepsilon}) + O(t^{1/2-\varepsilon} t^{-(2a+1-\eta)(1/2+\delta)}) + K_1 I_t
\]

\[
= O(t^{-2a+\varepsilon}) + K_1 I_t, \quad t \to 0 +,
\]

where

\[
I_t = \int_{-1/2+\delta}^{1/2+\delta} \omega |h(t + iv)| t^{-2a} dv.
\]

To get a suitable bound for \( I_t \) we have to improve the estimate \( |h(s)| \leq 1 \) for \( |s| \leq 2 \) by aid of (4.1). We apply a theorem of Milloux's ([3] p. 134–137) stating that if \( f(\xi) \) is regular in \( |\xi| < 1 \) and \( |f(\xi)| \leq 1 \) there and if \( |f(y)| \leq m < 1 \) for real \( y \), \( 0 \leq y < 1 \), then the estimate

\[
|f(\xi)| < m^{(1/2)(1-|\xi|)}
\]

is valid in the unit circle.

Let us now consider the circle with radius 1 and centre \( z = 1 \). In
this circle \(|h(z)| \leq 1\) and \(|h(u)| = K, |u|^{a-1-\eta} |\hat{g}(u)| \leq K_t t^{a-1-\eta} e^{-\epsilon t}\) for real \(u, 0 < u \leq t\).

We now map this circle onto the unit circle by \(w(z) = (z-t)/(z(t-1)-t)\). Then the line \(\text{Im } z = 0, 0 < \text{Re } z \leq t\) is mapped on \(\text{Im } w = 0, 0 \leq \text{Re } w < 1\) and the line \(\text{Re } z = t\) is mapped on a circle with diameter from \(w = -(1-t)^{-1}\) to \(w = 0\).

In order to use the theorem of Milloux’s mentioned above for \(f(w(z)) = h(z)\) we have to estimate

\[
\min_{|v| < t^{1/2} + \delta} (1 - |w(t + iv)|) = 1 - |w(t + it^{1/2} + \delta)| = 1 - t^{1/2 + \delta} ((t^2 - 2t) + ((t - 1)t^{1/2 + \delta})^{1/2} = (((t^2 - 2t)^2 + (t - 1)t^{1/2} - 1)((t^2 - 2t)^2 + (t - 1)t^{1/2}) = (2t^{1-\delta} + O(t))/(1 + O(t^-2)) = 2t^{1-\delta} + O(t).
\]

Thus we are now able to estimate the integral \(I_t\), as

\[
I_t \leq \int_{-t^{1/2} + \delta}^{t^{1/2} + \delta} (t-1/2+\epsilon) \left[ K_\delta t^{a-1-\gamma} e^{-\epsilon t}(t^{2t^{1-3\delta} + O(1)}) t^{-2a} \right] dv = \leq K_\delta t^{3+3+\gamma-2\epsilon + (2a-1-\eta)(t^{1-3\delta} + O(1)) e^{-\epsilon t^{2-2\delta} + O(1)} = O(1) \text{ as } t \to 0^+.
\]

Introducing this estimate in (4.7) we get, since \(2\epsilon < \alpha\),

\[
\int_{-\infty}^{+\infty} |\hat{g}(t + iv)| |t + iv|^{-\omega - 1} v^{-2} \sin^2(\omega v) dv = O(t^{1/2 + 2\epsilon}).
\]

Returning to the definition of \(\hat{g}(s)\), we finally apply the lemma with \(\alpha, \beta = -\alpha + 2\epsilon, \gamma = (\alpha + \beta - 1)/2 = \epsilon - 1/2\) and conclude that

\[
F(x) = O(x^{a-1/2-\epsilon}) = O(x^{a-1/2})
\]

and the proof is finished.

**References**


UNIVERSITY OF GOTHENBURG, SWEDEN
Mathematical papers intended for publication in the Pacific Journal of Mathematics should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insa-susha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.
Pacific Journal of Mathematics
Vol. 12, No. 2 February, 1962

William George Bade and Robert S. Freeman, Closed extensions of the Laplace operator determined by a general class of boundary conditions .......... 395
William Browder and Edwin Spanier, H-spaces and duality ......................... 411
Stewart S. Cairns, On permutations induced by linear value functions .......... 415
Frank Sydney Cater, On Hilbert space operators and operator roots of polynomials .......................................................... 429
Stephen Urban Chase, Torsion-free modules over K[x, y] .......................... 437
Heron S. Collins, Remarks on affine semigroups ...................................... 449
Peter Crawley, Direct decompositions with finite dimensional factors ............ 457
Richard Brian Darst, A continuity property for vector valued measurable functions .......................................................... 469
R. P. Dilworth, Abstract commutative ideal theory ................................ 481
P. H. Doyle, III and John Gilbert Hocking, Continuously invertible spaces .... 499
Shaul Foguel, Markov processes with stationary measure ......................... 505
Andrew Mattei Gleason, The abstract theorem of Cauchy-Weil .................. 511
Allan Brasted Gray, Jr., Normal subgroups of monomial groups ................... 527
Melvin Henriksen and John Rolfe Isbell, Lattice-ordered rings and function rings .......................................................... 533
Amnon Jakimovski, Tauberian constants for the [J. f(x)] transformations .... 567
Hubert Collings Kennedy, Group membership in semigroups ......................... 577
Eleanor Killam, The spectrum and the radical in locally m-convex algebras ........ 581
Arthur H. Kruse, Completion of mathematical systems ................................ 589
Magnus Lindberg, On two Tauberian remainder theorems .......................... 607
Lionello A. Lombardi, A general solution of Tonelli’s problem of the calculus of variations .......................................................... 617
Marvin David Marcus and Morris Newman, The sum of the elements of the powers of a matrix .......................................................... 627
Michael Bahir Maschler, Derivatives of the harmonic measures in multiply-connected domains .......................................................... 637
Deane Montgomery and Hans Samelson, On the action of SO(3) on Sn .......... 649
J. Barros-Neto, Analytic composition kernels on Lie groups ........................ 661
Mario Petrich, Semicharacters of the Cartesian product of two semigroups ........ 679
John Sydney Pym, Idempotent measures on semigroups ............................... 685
K. Rogers and Ernst Gabor Straus, A special class of matrices ..................... 699
U. Shukla, On the projective cover of a module and related results ............... 709
Don Harrell Tucker, An existence theorem for a Goursat problem ................. 719
George Gustave Weill, Reproducing kernels and orthogonal kernels for analytic differentials on Riemann surfaces ........................................... 729
George Gustave Weill, Capacity differentials on open Riemann surfaces ........... 769
G. K. White, Iterations of generalized Euler functions .................................. 777
Adil Mohamed Yaqub, On certain finite rings and ring-logics ....................... 785