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DERIVATIVES OF THE HARMONIC MEASURES IN MULTIPLY-CONNECTED DOMAINS

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1. Introduction. Basic definitions and some known results.

The Geometric Function Theory encounters serious difficulties when dealing with multilply-connected domains, due to the fact that no domain function is known in an explicit form for a domain of connectivity greater than two. It is for this reason that one at least tries to find properties of the domain functions in terms of geometric characteristics of their domains.

In this paper we search for information on domain functions and certain families of functions, defined for p-connected domains, that can be deduced form properties of functions defined in *symmetric* domains of connectivity $2(p-1)$. We also extend the results to infinitely-connected domains.

Let Δ be a domain in the z-plane, and let $L^2(\Delta)$ be the class of functions which are regular and square integrable in Δ. Let $l^2(\Delta)$ be its subclass consisting of those functions which have a single-valued integral in Δ. Both classes form separable Hilbert Spaces $[L^2(\Delta)]$ and $[l^2(\Delta)]$ under the scalar multiplication

$$(1.1) \qquad (f;\bar{g}) = \iint_{\Delta} f(z)\,\overline{g(z)}\,d\omega\,, \qquad d\omega = dx\,dy,\ z = x + iy\,.$$

Let $h^2(\Delta)$ be the class of functions which belong to the orthogonal complement of $[l^2(\Delta)]$ with respect to $[L^2(\Delta)]$. The Hilbert space $[h^2(\Delta)]$ has a finite dimension $p-1$, if Δ is p-connected and none of its boundary components reduces to a point. (See Bergman [3]). If Δ is infinitely-connected, this space has in general an infinite dimension (See Virtanen [9]).

According to Virtanen [9] (See also Nevanlinna [7], one can construct an orthogonal basis for $h^2(\Delta)$ as follows: Let $C_1, C_2, \cdots$ be a homology basis of cycles in Δ, subject therefore to the following conditions:

(1) Any cycle in Δ is homologous to a finite chain of these cycles.

(2) No chain is homologous to zero, unless its coefficients are all zero.

We can also assume that each cycle C_j, $(j = 1, 2, \cdots)$[1] is an oriented analytic Jordan curve. If $K_\Delta(z, \zeta)$ is the Bergman kernel function for the class $L^2(\Delta)$, then the functions

$$(1.2) \qquad F_j'(z;\Delta) = i \oint_{C_j} \overline{K_\Delta(\zeta, z)}\,d\bar{\zeta}\,, \qquad j = 1, 2, \cdots\,.$$

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[1] Throughout this paper the notation $1, 2, \cdots$ will mean a finite or infinite sequence, as the case may be.

span the space $[h^2(\Delta)]$; and if a subsequence $F'_{j_k}(z;\Delta)$ of them is obtained by omitting those functions which depend on the previous ones, we can orthonormalize its elements and thus obtain a basis $\varphi_n(z;\Delta)$ $(n=1,2,\cdots)$ for $[h^2(\Delta)]$. The coeficients used in this process are real:

$$\varphi_n(z;\Delta)=\sum_{k=1}^{n} c_k^{(n)}\, F'_{j_k}(z;\Delta)\;; \qquad c_k \text{ real}, n=1,2,\cdots . \tag{1.3}$$

We recall also that

$$F'_j(z;\Delta)=2\frac{\partial}{\partial z}\,\omega_j(z;\Delta)\;, \quad 2\frac{\partial}{\partial z}=\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\;, \tag{1.4}$$

where $\omega_j(z;\Delta)$ are the harmonic measures of that part of the boundary of Δ, which lies inside the cycle C_j. (See Bergman [3], Virtanen [9]).

Clearly, the classes $L^2(\Delta)$ and $l^2(\Delta)$ decrease monotonically if the domain Δ increases, and this fact plays an important role in the development of the theory of the kernel function. Unfortunately, a monotonic property for the class $h^2(\Delta)$ does not hold in general. It is therefore interesting to notice that if B is a symmetric domain with respect to the real axis, and the extended real axis is contained in B, then

$$h^2(D)\subset h^2(B)\;, \tag{1.5}$$

where D is that part of B which lies in the upper part of the finite plane. (Section 2). We study the connection between the bases for the classes $h^2(B)$ and $h^2(D)$, and deduce various relations for the kernel functions and the l-kernels for the corresponding classes. (Sections 2, 3).

2. Symmetric domains. Let D be a finitely or infinitely-connected domain in the z-plane, the boundary of which consists of the real axis I as well as other boundary components which shall be denoted by Γ. We assume that I is an *isolated* boundary component and we denote by D^* and Γ^* the reflections of D and Γ with respect to I, respectively. Let B be the symmetric domain bounded by Γ and Γ^*.

Let C_j $(j=1,2,\cdots)$ be a basis for the homology group of D, as described in §1. We choose the orientation on C_j to be counter clockwise and therefore I is not in the interior of C_j $(j=1,2,\cdots)$. We also assume that C_1 encloses Γ. Let C_j^* $(j=1,2,\cdots)$ be the reflection of C_j with respect to I, oriented counter clockwise. Clearly, the C_j^*'s form a homology basis for D^*, and $C_1, C_2, \cdots, C_2^*, C_3^*, \cdots$ form a homology basis for B. We also denote by Γ_j, Γ_j^* those parts of Γ and Γ^* which are enclosed by C_j and C_j^*, respectively $(j=1,2,\cdots)$. $(\Gamma\equiv\Gamma_1)$.

The harmonic measures $\omega_j(z;D)$ of Γ_j, with respect to D, vanish on I and therefore can be continued into D^*. They satisfy:

$$\omega_j(z; D) = -\omega_j(\bar{z}; D) , \qquad z \in B, \ j = 1, 2, \cdots . \tag{2.1}$$

Thus, the functions

$$F_j'(z; D) \equiv 2\frac{\partial \omega_j(z; D)}{\partial z} , \qquad j = 1, 2, \cdots , \tag{2.2}$$

are single-valued and regular throughtout B, and satisfy

$$F_j'(z; D) = \overline{-F_j'(\bar{z}; D)} , \qquad j = 1, 2, \cdots, \ z \in B . \tag{2.3}$$

Here and elsewhere we use the notation

$$F_j'(\bar{z}; D) = \frac{dF_j(\zeta; D)}{d\zeta}\bigg|_{\zeta=\bar{z}} . \tag{2.4}$$

By (1.3) it now follows that

$$\varphi_n(z; D) = \overline{-\varphi_n(\bar{z}; D)}, \qquad z \in B , \ n = 1, 2, \cdots . \tag{2.5}$$

THEOREM 2.1. *If B is a symmetric domain with respect to the real axis, and contains the real axis, then*

$$h^2(D) \subset h^2(B) . \tag{2.6}$$

D denotes that part of B which lies in the upper half plane.

Proof. The functions $\varphi_j(z; D)/\sqrt{2}$ $(j = 1, 2, \cdots)$ belong to the class $h^2(B)$ and form an orthonormal system in B. Indeed, if $g(z) \in l^2(B)$ then a fortiori $g(z) \in l^2(D)$ and $\overline{g(\bar{z})} \in l^2(D)$, $z \in D$; hence, by (2.5),

$$\iint_B \varphi_n(z; D)\overline{g(z)}\, d\omega = \iint_D \varphi_n(z; D)\overline{g(z)}\, d\omega - \iint_D \overline{\varphi_n(z; D)g(\bar{z})}\, d\omega = 0 . \tag{2.7}$$

Similarly,

$$\begin{aligned} \iint_B \varphi_n(z; D)\overline{\varphi_m(z; D)}\, d\omega &= \iint_D \varphi_n(z; D)\overline{\varphi_m(z; D)}\, d\omega \\ &+ \iint_D \overline{\varphi_n(z; D)}\varphi_m(z; D)\, d\omega = 2\delta_{nm} . \end{aligned} \tag{2.8}$$

If $f(z) \in h^2(D)$, then $f(z) = \Sigma_n a_n \varphi_n(z; D)$, where $\Sigma_n |a_n|^2 < \infty$. Thus $\Sigma_n |\sqrt{2}\, a_n|^2 < \infty$; hence $f(z) = \Sigma_n \sqrt{2}\, a_n (\varphi_n(z)/\sqrt{2}) \in h^2(B)$.

Consider again the symmetric domain B, and let $\omega_j(z; B)$, $\omega_j^*(z; B)$ be the harmonic measures of Γ_j, Γ_j^*, with respect to B, respectively. We shall prove that

$$\omega_j^*(z; B) = \omega_j(\bar{z}; B) , \qquad z \in B , \ j = 1, 2, \cdots . \tag{2.9}$$

Indeed, we exhaust B by a sequence of finitely-connected subdomains $B_1, B_2, \cdots$,

$$\bar{B}_m \subset B_{m+1}, \bigcup_m B_m = B, \qquad m = 1, 2, \cdots. \tag{2.10}$$

We can assume that each domain is bounded by analytic Jordan curves and is symmetric and contains the extended real axis. Clearly, for each j,

$$\omega_j^*(z; B_m) = \omega_j(\bar{z}; B_m), \tag{2.11}$$

if m is large enough, since both sides are harmonic in B_m and have the same boundary values. By letting m go to infinity, we obtain (2.9). (See Virtanen [9]).

In a similar fashion one obtains:

$$\omega_j(z; B) - \omega_j(\bar{z}; B) = \omega_j(z; D), \ z \in B, \qquad j = 1, 2, \cdots. \tag{2.12}$$

Applying the operation $2(\partial/\partial z)$ to both sides of (2.9) and (2.12), we obtain the important relations:

$$F_j'^*(z; B) = \overline{F_j'(\bar{z}; B)} \tag{2.13}$$

$$F_j'(z; B) - \overline{F_j'(\bar{z}; B)} = F_j'(z; D), \qquad z \in B, \ j = 1, 2, \cdots. \tag{2.14}$$

Since $h^2(D) \subset h^2(B)$, it is of interest to have some information on the orthogonal complement of $[h^2(D)]$ with respect to $[h^2(B)]$.

THEOREM 2.2. *Let B be a symmetric domain with respect to the real axis, which contains the real axis, and let D be that part of B which lies in the upper half plane. Let*[2]

$$\psi_n(z; B) = \varphi_n(z; D)/\sqrt{2} = \sum_{k=1}^{n} c_{nk} F_k'(z; D), \ (c_{nk} \text{ real}), \tag{2.15}$$

$$\chi_n(z; B) = \sum_{k=1}^{n} c_{nk} F_k'(z; B), \tag{2.16}$$

where $\varphi_n(z; D)$ is an orthonormal basis for $[h^2(D)]$, as constructed in §1. *Then, the functions*

$$r_n(z; B) = \chi_n(z; B) - \frac{\psi_n(z; B)}{2} \tag{2.17}$$

are orthogonal to the functions

$$\psi_m(z; B) \qquad n, m = 1, 2, \cdots.$$

Proof. It is well known (see Bergman [3], Virtanen [9]) that the

[2] We assume that $F_k'(z; D)$, $k = 1, 2, \cdots$, are already independent. Note also that $F_1'(z; D) = 2F_1'(z; B)$ (see Section 1).

scalar product (over B) of any two functions from the closed system $\{F_j'(z; D), F_{j+1}'(z; B), j = 1, 2, \cdots\}$ is real. The same holds true for the scalar product of $\psi_m(z; B)$ and $\chi_n(z; B)$. Therefore, in view of (2.5) and (2.14),

$$
\begin{aligned}
(2.18)\qquad & \iint_B \chi_n(z; B)\overline{\psi_m(z; B)}\, d\omega \\
&= \iint_D \chi_n(z; B)\overline{\psi_m(z; B)}\, d\omega - \iint_{D*} \chi_n(z; B)\psi_m(\bar{z}; B) d\omega \\
&= \iint_D (\psi_n(z; B) + \overline{\chi_n(\bar{z}; B)})\,\overline{\psi_m(z; B)}\, d\omega - \iint_D \chi_n(\bar{z}; B)\psi_m(z; B)\, d\omega \\
&= \iint_D \psi_n(z; B)\overline{\psi_m(z; B)}\, d\omega = \frac{1}{2}\delta_{nm} ,
\end{aligned}
$$

where $\delta_{nm} = 0$ if $n \neq m$, $\delta_{nn} = 1$.

On the other hand,

$$
(2.19)\qquad \iint_B \frac{\psi_n(z; B)}{2}\,\overline{\psi_m(z; B)}\, d\omega = \frac{1}{2}\delta_{nm} ;
$$

hence the result follows.

COROLLARY 2.1. *It is easily verified that*

$$
(2.20)\qquad r_n(z; B) = \overline{r_n(\bar{z}; B)} , \qquad r_1(z; B) = 0 ;
$$

hence, by orthonormalizing the functions $r_n(z; B)$, $n = 2, 3, \cdots$ (with real coefficients), a complete orthonormal basis $\psi_n(z; B)$, $\rho_n(z; B)$, $n = 1, 2, \cdots$, is constructed for $[h^2(B)]$, where

$$
(2.21)\qquad \psi_n(z; B) = -\overline{\psi_n(\bar{z}; B)},\ \rho_n(z; B) = \overline{\rho_n(\bar{z}; B)} , \quad n = 1, 2, \cdots .
$$

This result, which later will be of much use, implies:

THEOREM 2.3. *Let B be a symmetric domain with respect to the real axis, and contains the real axis, then, each function $f(z)$ of the class $h^2(B)$ can be represented as a sum of two functions $f_1(z)$ and $f_2(z)$, of the same class, which satisfy*

$$
(2.22)\qquad f_1(z) = -\overline{f_1(\bar{z})} , \quad f_2(z) = \overline{f_2(\bar{z})} .
$$

In fact,

$$
(2.23)\qquad 2f_1(z) = f(z) - \overline{f(\bar{z})} , \quad 2f_2(z) = f(z) + \overline{f(\bar{z})} .
$$

3. **The space $h_r^2(\Delta)$.** Let $h_r^2(\Delta)$ be a subclass of $h^2(\Delta)$, consisting of those functions which have *real* Fourier coefficients with respect to the base $\varphi_n(z; \Delta)$ $(n = 1, 2, \cdots)$. This class forms a real Hilbert space $[h_r^2(\Delta)]$,

with respect to the metric (1.1).

In order to extend some results to the case of a general domain Δ, we exhaust Δ by a sequence of smoothly bounded, finitely-connected subdomains $\{\Delta_m\}$, $m = 1, 2, \cdots$ which satisfy

$$\bar{\Delta}_m \subset \Delta_{m+1}, \quad \bigcup_m \Delta_m = \Delta , \qquad m = 1, 2, \cdots . \tag{3.1}$$

LEMMA 3.1. *If $f(z) \in h_r^2(\Delta)$, there exists a sequence of functions $\{f_m(z)\}$, $f_m(z) \in h_r^2(\Delta_m)$, $m = 1, 2, \cdots$, which converges in the mean to $f(z)$, and therefore converges uniformly to $f(z)$ on each compact set in Δ.*

Proof. Given $f(z)$, it follows that

$$f(z) = \Sigma_n a_n \varrho_n(z; \Delta) , \qquad a_n \text{ real}, \ \Sigma_n a_n^2 < \infty . \tag{3.2}$$

The series converges in the mean. If m is large enough, Δ_m contains a given cycle C_j which encloses a part $\Gamma_i^{(m)}$ of the boundary of Δ_m. It is known (see Virtanen [9]) that $F_j'(z; \Delta_m)$ converges in the mean to $F_j'(z, \Delta)$ if $m \to \infty$ (we define $F_j'(z; \Delta_m)$ to be zero outside Δ_m). One now applies the Gram-Schmidt orthonormalization process to a subsequence $\{F_{j_k}'(z; \Delta_m)\}$, where k is chosen in such a way that each $F_{j_k}'(z; \Delta)$ is independent on the functions $F_{j_l}'(z; \Delta)$, $l = 1, 2, \cdots, k-1$. One obtains for each m an orthonormal system

$$\varphi_n(z, \Delta_m) = \sum_{k=1}^{N_m} c_k^{(m)} F_{j_k}'(z; \Delta_m), \qquad c_k^{(m)} \text{ real}, \ k = 1, 2, \cdots, N_m . \tag{3.3}$$

N_m is the number of cycles C_{j_k} which are in Δ_m. It can be proved by induction that $\varphi_n(z; \Delta_m)$ converges in the mean to $\varphi_n(z; \Delta)$, defined by (1.3) and that $c_k^{(m)} \to c_k$, as $m \to \infty$. Consequently, the functions

$$f_m(z) = \sum_{n=1}^{N_m} a_n \varphi_n(z; \Delta_m) \tag{3.4}$$

converge to $f(z)$ in the mean.

The space $h_r^2(\Delta)$ can be characterized differently. To this effect we apply the l-kernel defined (Bergman Schiffer [4], Schiffer [5]) as

$$l_\Delta(z, \zeta) = \frac{1}{\pi(z-\zeta)^2} + \frac{2}{\pi} \frac{\partial^2 G_\Delta(z, \zeta)}{\partial z \partial \zeta} , \tag{3.5}$$

where $G_\Delta(z, \zeta)$ is the Green function of the domain Δ.

LEMMA 3.2. *The kernels $l_{\Delta_m}(z, \zeta)$ for a fixed ζ, converges in the mean to $l_\Delta(z, \zeta)$. Δ_m is the exhaustion* (3.1).

Proof. We use the known relations (Bergman-Schiffer [4])

$$\iint_{\Delta_m} l_{\Delta_m}(z,\zeta)\overline{l_{\Delta_m}(z,\zeta)}\,d\omega_z = K_{\Delta_m}(\zeta,\zeta) - \Gamma_{\Delta_m}(\zeta,\zeta)\,, \tag{3.6}$$

where

$$\Gamma_{\Delta_m}(\zeta,\zeta) = \frac{1}{\pi^2}\iint_{\tilde{\Delta}_m} \frac{d\omega_z}{|z-\zeta|^4}\,, \qquad \zeta \in \Delta_m\,. \tag{3.7}$$

$\tilde{\Delta}_m$ is the complement of Δ_m with respect to the z-plane.

$$T_{\Delta_m} f \equiv \iint_{\Delta_m} l(z,\zeta)\overline{f(z)}\,d\omega_z = \frac{1}{\pi}\iint_{\Delta_m} \frac{\overline{f(z)}}{(z-\zeta)^2}\,d\omega_z,\ \zeta \in \Delta\,, \tag{3.8}$$

for $f(z) \in L^2(\Delta_m)$. The improper integral on the right hand side of (3.8) is to be understood in the sense of the limit of integrals over the domains $\Delta_{\varepsilon,m}$ which are obtained from Δ_m by removing circles around ζ with radius ε, $\varepsilon \to 0$. Let $p \geqq m$, then

$$\begin{aligned}\iint_{\Delta_m} |l_{\Delta_p}(z,\zeta) - l_{\Delta_m}(z,\zeta)|^2\,d\omega_z &\leqq K_{\Delta_p}(\zeta,\zeta) - \Gamma_{\Delta_p}(\zeta,\zeta) \\ &+ K_{\Delta_m}(\zeta,\zeta) - \Gamma_{\Delta_m}(\zeta,\zeta) - 2\,Re\iint_{\Delta_m} l_{\Delta_p}(z,\zeta)\overline{l_{\Delta_m}(z,\zeta)}\,d\omega_z\,.\end{aligned} \tag{3.9}$$

By (3.8), the last integral can be estimated:

$$\begin{aligned}\iint_{\Delta_m} l_{\Delta_p}(z,\zeta)\overline{l_{\Delta_m}(z,\zeta)}\,d\omega_z &= \frac{1}{\pi}\iint_{\Delta_m} \frac{l_{\Delta_p}(z,\zeta)}{(\bar{z}-\bar{\zeta})^2}\,d\omega_z \\ &= \iint_{\Delta_p} l_{\Delta_p}(z,\zeta)\overline{l_{\Delta_p}(z,\zeta)}\,d\omega_z - \frac{1}{\pi}\iint_{\Delta_p-\Delta_m} \frac{l_{\Delta_p}(z,\zeta)}{(\bar{z}-\bar{\zeta})^2}\,d\omega_z \\ &= K_{\Delta_p}(\zeta,\zeta) - \Gamma_{\Delta_p}(\zeta,\zeta) - \frac{1}{\pi}\iint_{\Delta_p-\Delta_m} \frac{l_{\Delta_p}(z,\zeta)}{(\bar{z}-\bar{\zeta})^2}\,d\omega_z\,.\end{aligned} \tag{3.10}$$

By Schwarz inequality, the modulus of the last integral can be made arbitrarily small, if m is large enough, and also $K_{\Delta_m}(\zeta,\zeta)$ tends to $K_\Delta(\zeta,\zeta)$ (See Virtanen [9]); hence the right hand side of (3.9) tends to zero if $m \to \infty$.

The limit function is $l_\Delta(z,\zeta)$ as defined by 3.5, since $G_{\Delta_m}(z,\zeta)$ converges to $G_\Delta(z,\zeta)$.

THEOREM 3.1. (Bergman-Schiffer) *The function $f(z) \in h^2(\Delta)$ belongs to the class $h_r^2(\Delta)$ if and only if*

$$T_\Delta f \equiv \iint_\Delta l_\Delta(z,\zeta)\,\overline{f(z)}\,d\omega_z = -f(\zeta), \qquad \zeta \in \Delta\,. \tag{3.11}$$

This theorem was proved in [4] for the case of a sufficiently smooth bounded, finitely-connected domain. If $f(z) \in h_r^2(\Delta)$, where Δ is a general

domain, it follows from Lemma 3.1 and Lemma 3.2 that $T_{\Delta_m} f_m(z) = -f_m(\zeta)$ implies $T_\Delta f(z) = -f(\zeta)$. Conversly, if $f(z) \in h^2(\Delta)$ then

$$f(z) = \Sigma \alpha_n \varphi_n(z; \Delta) + i\Sigma \beta_n \varphi_n(z; \Delta), \quad \alpha_n, \beta_n \text{ real} ; \tag{3.12}$$

hence $T_\Delta f(z) = -f(z)$ can only happen if $\beta_n = 0$.

If is known [4] that $l_\Delta(z, \zeta)$ can be represented as a sum of two kernels:

$$l_\Delta(z, \zeta) = l_\Delta^{(1)}(z, \zeta) + l_\Delta^{(h)}(z, \zeta) , \tag{3.13}$$

where[3]

$$l_\Delta^{(h)}(z, \zeta) = -\sum_n \varphi_n(z, \Delta)\varphi_n(\zeta, \Delta) , \tag{3.14}$$

and $l_\Delta^{(1)}(z, \zeta)$ belongs to $l^2(\Delta)$, for each fixed ζ. We therefore obtain:

COROLLARY. *A necessary and sufficient condition that a function $f(z)$ of the class $L^2(\Delta)$ belongs to $h_r^2(\Delta)$ is*

$$T_\Delta^{(r)} \equiv \iint_\Delta l_\Delta^{(h)}(z, \zeta)\overline{f(z)}\, d\omega_z = -f(\zeta) , \qquad \zeta \in \Delta . \tag{3.15}$$

REMARK. Clearly, $K_\Delta^{(h)}(z, \zeta) - l_\Delta^{(h)}(z, \zeta) \in h_r^2(\Delta)$ as a function of z, where

$$K_\Delta^{(h)}(z, \zeta) = \sum_n \varphi_n(z; \Delta)\overline{\varphi_n(\zeta; \Delta)} . \tag{3.16}$$

In the case of a symmetric domain B, Theorem 2.1 can be slightly improved to the effect that $h_r^2(D) \subset h_r^2(B)$. Also, if $f(z) \in h_r^2(D)$, it follows from (2.5) that

$$f(z) = -\overline{f(\bar{z})} . \tag{3.17}$$

Limiting ourselves to the spaces $h_r^2(B)$, $h_r^2(D)$, we can now construct functions in the latter space from functions in the former one:

THEOREM 3.2. *If B is a symmetric domain with respect to the real axis, and contains the real axis, and if $f(z) \in h_r^2(B)$ then $f(z) - \overline{f(\bar{z})} \in h_r^2(D)$, where D is that part of B which lies in the upper half plane.*

Proof. We exhaust B by the symmetric domains $\{B_m\}$ (See (2.10)), and consider the functions $f_m(z)$, defined in (3.4), where $f(z)$ has the representation (3.2). The functions $\varphi_n(z; B_m)$ are finite linear combinations, with real coefficients, of $F'_{j_k}(z; B_m)$ and $F^{*\prime}_{j_k}(z; B_m)$; consequently, by (2.13) and (2.14), $\varphi_n(z; B_m) - \varphi_n(\bar{z}; B_m)$ are finite linear combintations, with real coefficients, of $F'_{j_k}(D_m)$, which, in turn, are finite linear combintations,

[3] The generalization to a general domain is obvious.

with real coefficients, of $\varphi_n(z; D_m)$. Since $[h_r^2(\Delta_m)]$ has finite dimensions, it follows that

$$f_m(z) - \overline{f_m(\bar{z})} \equiv g_m(z) \in h_r^2(D_m) . \tag{3.18}$$

Thus, by Lemma 3.1 if $m \to \infty$, the left hand side converges to the function $f(z) - \overline{f(\bar{z})} = g(z)$. But $g(z) \in L^2(B) \subset L^2(D)$, hence $g_m(z)$ tends to $g(z)$ which is in $h_r^2(D)$.

This theorem could also be proved directly by the use of Corollary 2.1. We can also use this corollary and the series developments (3.14), (3.16) in order to prove by direct calculation the following

THEOREM 3.3. *If B is a symmetric domain with respect to the real axis, and contains the real axis, then the following identities hold for $z, \zeta \in B$.*

$$K_B^{(h)}(z, \zeta) = K_B^{(h)}(\bar{\zeta}, \bar{z}) , \tag{3.19}$$

$$l_B^{(h)}(z, \zeta) = \overline{l_B^{(h)}(\bar{z}, \bar{\zeta})} , \tag{3.20}$$

$$K_B^{(h)}(z, \bar{\zeta}) + l_B^{(h)}(z, \zeta) = K_D^{(h)}(z, \bar{\zeta}) , \tag{3.21}$$

$$K_D^{(h)}(z, \bar{\zeta}) = \overline{K_D^{(h)}(\bar{\zeta}, z)} = l_D^{(h)}(z, \zeta) . \tag{3.22}$$

Here D denotes that part of B, which lies in the upper half plane.

REMARK. From the symmerty of B, it follows that the Green function satisfies the identity

$$G_B(z, \zeta) = G_B(\bar{z}, \bar{\zeta}) , \qquad z, \zeta \in B , \tag{3.23}$$

therefore, by (3.5) and by the well known formula (See [4], [5])

$$K_\Delta(z, \zeta) = -\frac{2}{\pi} \frac{\partial^2 G_\Delta(z, \zeta)}{\partial z \partial \bar{\zeta}} , \qquad z, \zeta \in \Delta , \tag{3.24}$$

it follows also that

$$K_B(z, \zeta) = K_B(\bar{\zeta}, \bar{z}) \tag{3.25}$$

$$l_B(z, \zeta) = \overline{l_B(\bar{z}, \bar{\zeta})} . \tag{3.26}$$

Hence, similar identities hold also for the kernels $K_B^l(z, \zeta)$ and $l_B^l(z, \zeta)$.

Making use of Corollary 2.1 (2.15), (3.14), (3.16), we also find that

$$K_B^{(h)}(z, \zeta) - \tfrac{1}{2} K_D^{(h)}(z, \zeta) = \tfrac{1}{2} l_D^{(h)}(z, \bar{\zeta}) - l_B^{(h)}(z, \bar{\zeta}) = \sum_n \rho_n(z) \overline{\rho_n(\zeta)} ; \tag{3.27}$$

hence,

THEOREM 3.4. *If B is a symmetric domain with respect to the real*

axis I, and contains the real axis, and if D is that part of B which lies in the upper half plane, then

$$K_B^{(h)}(z, z) \geqq \tfrac{1}{2} K_D^{(h)}(z, z) , \qquad z \in B,\ z \notin I . \tag{3.28}$$

$$l_B^{(h)}(z, \bar{z}) \leqq \tfrac{1}{2} l_D^{(h)}(z, \bar{z}) , \qquad z \in B,\ z \notin I . \tag{3.29}$$

4. Concluding remarks. A domain Δ is said to be of zero span (of the class 0_{AD}, in the terminology of the theory of Riemann Surfaces), if the class $l^2(\Delta)$ reduces to the zero function. (See Ahlfors-Sario [2], Nevanlinna [7], Schiffer [8]). This property is invariant under univalent conformal mappings, and the boundary of Δ can be characterized by the fact that it is a removable singularity for all the functions, defined in a neighbourhood of the boundary and belong to the class l^2 in that neighbourhood. (See Painlevé Problem, [7]).

Clearly, for such domains,

$$K_\Delta(z, \zeta) = K_\Delta^{(h)}(z, \zeta) \qquad l(z, \zeta) = l^{(h)}(z, \zeta) . \tag{4.1}$$

Therefore, if B is a symmetric domain of zero span, one can replace the class $h^2(B)$ by $L^2(B)$ in all our previous results. One may also deduce properties of the class $h^2(D)$ from known properties of the class $L^2(B)$.

In Maschler [6], it was conjectured that if there exists a point in a domain Δ, at which all the functions of the class $h^2(\Delta)$ vanish, then $h^2(\Delta)$ consists of the zero function alone. (In the later case, the domain is simply-connected except for punctures which are removable with respect to the function of $h^2(\Delta)$). This conjecture was proved there under the assumption that Δ has two boundary components which are not completely point-like. An analogous statement is correct for the classes $L^2(\Delta)$ (see Nevanlinna [7]) and $l^2(\Delta)$ (see Ahlfors Beurling [1]).[4] The above mentioned conjecture is trivially true if Δ has a zero span.

Using the representation (2.21), one derives easily, by (2.15), that *there exists a point t in D, at which all the functions of the class $h^2(D)$ vanish, if and only if $f(t) = \overline{f(\bar{t})}$ for all the functions $f(z)$ of the class $h_r^2(B)$.* Thus, if Γ contains a boundary component which is not completely point-like, then, for any point t in D, there exists a function of the class $h_r^2(B)$, which does not take a conjugate value at $\bar{t}$.

It would be of interest to study also the question whether it is possible that all the functions $f(z)$ of the class $h_r^2(B)$ satisfy at a particular point t, $t \in D$, the relation $f(t) = -\overline{f(\bar{t})}$. This is equivalent to equality in (3.28) and (3.29), for $z = t$. (See (3.27)).

[4] The situation in the general case of a Riemann surface is studied in Virtanen [**10**].

REFERENCES

1. L. V. Ahlfors and A. Beurling, *Conformal invariants and function-theoretic null-sets.* Acta Math., **83** (1950), 101–129.
2. L. V. Ahlfors and L. Sario, *Riemann Surfaces,* Princeton Univ. Press, Princeton New Jersey (1960).
3. S. Bergman, *The kernel function and conformal mapping.* Math. Surveys V, Amer. Math. Soc., New York (1950).
4. S. Bergman and M. Schiffer, *Kernel functions and conformal mapping,* Composition Math., **8** (1950), 205–249.
5. R. Courant, *Dirichlet's principle, Conformal mapping and minimal surfaces,* Appendix by M. Schiffer. Interscience pub. New York (1950).
6. M. Maschler, *Analytic functions of the classes L^2 and l^2 and their kernel functions,* Rendiconti Circ. Mat. Palermo, Ser. II, **8** (1959), 163–177.
7. R. Nevanlinna, *Uniformisierung,* Springer-Verlag, Berlin (1953).
8. M. Schiffer, *The span of multiply-connected domains,* Duke Math. J., **10** (1943), 209–216.
9. K. I. Virtanen, *Über eine Integraldarstellung von quadratische integrierbaren analytischen Differentialen,* Ann. Acad. Scient. Fenn. Ser. A, I, **69** (1950).
10. K. I. Virtanen, *Über Extremalfunktionen auf offenen Riemannschen Flächen,* Ann. Acad. Scient. Fenn. Ser. A, I, **141** (1952).

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