

# Pacific Journal of Mathematics

**SEMICHARACTERS OF THE CARTESIAN PRODUCT OF TWO  
SEMIGROUPS**

MARIO PETRICH

# SEMICHARACTERS OF THE CARTESIAN PRODUCT OF TWO SEMIGROUPS

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1. If  $S$  and  $T$  are semigroups, then by  $S \times T$  we mean the semigroup consisting of the Cartesian product  $S \times T$  of the sets  $S$  and  $T$  with coordinatewise multiplication. The semigroup  $S \times T$  is called the *Cartesian product of the semigroups*  $S$  and  $T$ . A complex-valued multiplicative function on a semigroup  $S$  is called a *semicharacter* of  $S$  if it is different from 0 at some point and is bounded (1.3, [1]). The set of all semicharacters of  $S$  is denoted by  $\hat{S}$ .

We show that  $\widehat{S \times T} = \{\chi | \chi(x, u) = \phi(x)\psi(u) \text{ for some } \phi \in \hat{S}, \psi \in \hat{T}\}$  (2.4). We obtain a similar result for continuous semicharacters of topological semigroups (3.3). One of the most interesting consequences of the above results is a theorem on prime ideals (2.6). A subset  $I$  of a semigroup  $S$  is called a *prime ideal* of  $S$  if  $I$  is a proper (i.e.,  $\neq S$ ) two-sided ideal of  $S$  whose complement in  $S$  is a semigroup. For convenience we also call the empty set a prime ideal (cf. Definitions 2, 2a, [2]). We also prove a theorem concerning continuity of the semicharacters of the Cartesian product  $S \times T$  of two topological semigroups (3.4).

If  $A$  and  $B$  are sets, then  $A - B$  will denote the set of all elements of  $A$  which are not contained in  $B$ . A semigroup will always be nonempty. A nonempty subset  $I$  of  $S$  is said to be an (two-sided) *ideal* of  $S$  if  $xy, yx \in I$  for all  $x \in S, y \in I$ .

All results in this paper are stated for the Cartesian product of two semigroups. However, a simple inductive argument shows that all of them generalize to the Cartesian product of any finite number of semigroups.

This paper is an excerpt from the author's doctoral dissertation. The writer wishes to express his sincere gratitude to Professor Herbert S. Zuckerman for his very kind help in the preparation of this research. The writer also is indebted to Dr. K. A. Ross and Professor Edwin Hewitt for useful suggestions. This research was supported by the Office of Naval Research.

2. If  $S$  and  $T$  are semigroups with two-sided identities, then semicharacters of  $S \times T$  are obtained easily from the semicharacters of  $S$  and  $T$ . (If  $e$  and  $f$  are identities of  $S$  and  $T$ , respectively, then each element  $(x, u)$  of  $S \times T$  can be written as  $(x, f)(e, u)$ .) In 5, [3], Št. Schwarz considers this case for commutative semigroups. We first introduce two definitions.

2.1. DEFINITION. Let  $f$  and  $g$  be arbitrary complex-valued functions defined on sets  $S$  and  $T$ , respectively. We define the function  $(f, g)$  on  $S \times T$  by  $(f, g)(x, u) = f(x)g(u)$  for all  $x \in S, u \in T$ .

2.2. DEFINITION. Let  $S$  and  $T$  be semigroups. We define  $\widehat{S \circ T} = \{\chi | \chi = (\phi, \psi) \text{ for some } \phi \in \widehat{S}, \psi \in \widehat{T}\}$ .

2.3. THEOREM. Let  $S$  and  $T$  be semigroups and let  $\chi \in \widehat{S \times T}$ . Then  $\chi$  can be written uniquely as  $(\phi, \psi)$ , where  $\phi \in \widehat{S}$  and  $\psi \in \widehat{T}$ . If  $(a, b)$  is any element of  $S \times T$  such that  $\chi(a, b) \neq 0$ , then

$$\begin{aligned}\phi(x) &= \frac{\chi(ax, b)}{\chi(a, b)} \text{ for all } x \in S \text{ and} \\ \psi(u) &= \frac{\chi(a, bu)}{\chi(a, b)} \text{ for all } u \in T.\end{aligned}$$

*Proof.* Let  $(a, b)$  be any element of  $S \times T$  such that  $\chi(a, b) \neq 0$  and let  $x$  and  $y$  be elements of  $S$ . Then  $\chi(ax, b)\chi(a, b) = \chi(axa, b^2) = \chi(a, b)\chi(xa, b)$  and after dividing this identity by  $\chi(a, b)$ , we obtain

$$(1) \quad \chi(ax, b) = \chi(xa, b) \text{ for all } x \in S.$$

Let

$$\phi(x) = \frac{\chi(ax, b)}{\chi(a, b)} \text{ for all } x \in S.$$

From (1) we obtain

$$\chi(ax, b)\chi(ay, b) = \chi(ax, b)\chi(ya, b) = \chi(axya, b^2) = \chi(axy, b)\chi(a, b)$$

and consequently

$$\phi(x)\phi(y) = \frac{\chi(ax, b)}{\chi(a, b)} \frac{\chi(ay, b)}{\chi(a, b)} = \frac{\chi(axy, b)\chi(a, b)}{\chi(a, b)\chi(a, b)} = \phi(xy) \text{ for all } x, y \in S.$$

We let

$$\psi(u) = \frac{\chi(a, bu)}{\chi(a, b)} \text{ for all } u \in T.$$

Like  $\phi, \psi$  is multiplicative. Let  $(x, u)$  be any element of  $S \times T$ . By (1), we have

$$\begin{aligned}\chi(ax, b)\chi(a, bu) &= \chi(a, bu)\chi(ax, b) = \chi(a, bu)\chi(xa, b) = \chi(axa, bub) \\ &= \chi(a, b)\chi(x, u)\chi(a, b)\end{aligned}$$

and thus

$$\phi(x)\psi(u) = \frac{\chi(ax, b)}{\chi(a, b)} \frac{\chi(a, bu)}{\chi(a, b)} = \frac{\chi(a, b)\chi(x, u)\chi(a, b)}{\chi(a, b)\chi(a, b)} = \chi(x, u).$$

Therefore

$$(2) \quad \chi = (\phi, \psi).$$

Since  $\chi(a, b)$  is a constant,  $\phi$  is bounded, and since  $\phi(a) \neq 0$ , we conclude that  $\phi \in \hat{S}$ . A similar argument shows that  $\psi \in \hat{T}$ .

It only remains to prove uniqueness of  $\phi$  and  $\psi$ . Suppose now that  $(\phi, \psi) = (\phi_1, \psi_1)$ . Then  $\phi(x)\psi(u) = \phi_1(x)\psi_1(u)$  for all  $x \in S$ ,  $u \in T$ . There exists an element  $u_0 \in T$  such that  $\psi(u_0) \neq 0$ . Hence

$$\phi(x) = \frac{\psi_1(u_0)}{\psi(u_0)} \phi_1(x) \text{ for all } x \in S.$$

Let  $K = \psi_1(u_0)/\psi(u_0)$ . If  $x_0$  is an element of  $S$  such that  $\phi(x_0) \neq 0$ , then  $\phi(x_0^2) = [\phi(x_0)]^2 = [K\phi_1(x_0)]^2 = K[K\phi_1(x_0^2)] = K\phi(x_0^2)$  and thus  $K = 1$  since  $\phi(x_0) \neq 0$ . Therefore  $\phi = \phi_1$ . One shows similarly that  $\psi = \psi_1$ .

2.4. COROLLARY. *If  $S$  and  $T$  are semigroups, then  $\widehat{S \times T} = \widehat{S} \circ \widehat{T}$ .*

*Proof.* If  $\phi \in \widehat{S}$  and  $\psi \in \widehat{T}$ , it is easy to show that  $(\phi, \psi) \in \widehat{S \times T}$ . Therefore  $\widehat{S \times T} \supseteq \widehat{S} \circ \widehat{T}$ . The reverse inclusion follows from 2.3.

The following lemma has been proved by Št. Schwarz for several classes of semigroups (Lemma 3, [2] and Lemma 3.2, [3]).

2.5. LEMMA. *Let  $S$  be a semigroup and let  $\chi \in \widehat{S}$ . Then the set  $I = \{x \in S \mid \chi(x) = 0\}$  is a prime ideal of  $S$ . Conversely, if  $I$  is a prime ideal of  $S$ , then there exists a semicharacter  $\chi \in \widehat{S}$  such that*

$$I = \{x \in S \mid \chi(x) = 0\}.$$

*Proof.* The proof of the first statement is routine and is omitted. For the converse, let  $I$  be a prime ideal of  $S$ . Define the function  $\chi$  on  $S$  by

$$\chi(x) = \begin{cases} 1 & \text{if } x \in S - I \\ 0 & \text{if } x \in I \end{cases}$$

Then  $\chi \in \widehat{S}$  and  $I = \{x \in S \mid \chi(x) = 0\}$ .

2.6. THEOREM. *Let  $S$  and  $T$  be semigroups. Then a set  $L$  is a prime ideal of  $S \times T$  if and only if  $L = (I \times T) \cup (S \times J)$  where  $I$  and  $J$  are prime ideals of  $S$  and  $T$ , respectively.*

*Proof.* Let  $L$  be a prime ideal of  $S \times T$ . By the second part of 2.5, there is a semicharacter  $\chi \in \widehat{S \times T}$  vanishing exactly on  $L$ . From 2.4 it follows that  $\chi = (\phi, \psi)$  for some  $\phi \in \widehat{S}$ ,  $\psi \in \widehat{T}$ . Clearly  $\chi(x, u) = \phi(x)\psi(u) = 0$  if and only if either  $\phi(x) = 0$  or  $\psi(u) = 0$ . Hence  $L = \{(x, u) \in S \times T \mid \chi(x, u) = 0\} = (I \times T) \cup (S \times J)$ , where  $I = \{x \in S \mid \phi(x) = 0\}$  and  $J = \{u \in T \mid \psi(u) = 0\}$ . By the first part of 2.5,  $I$  and  $J$  are prime ideals of  $S$  and  $T$ , respectively.

Conversely, let  $I$  and  $J$  be prime ideals of  $S$  and  $T$ , respectively. By

the second part of 2.5, there are semicharacters  $\phi \in \widehat{S}$ ,  $\psi \in \widehat{T}$  vanishing exactly on  $I$  and  $J$ , respectively. From 2.4 it follows that  $(\phi, \psi) = \chi$  for some  $\chi \in \widehat{S \times T}$ . Clearly  $\chi(x, u) = \phi(x)\psi(u) = 0$  if and only if either  $\phi(x) = 0$  or  $\psi(u) = 0$ , and this happens if and only if either  $x \in I$  or  $u \in J$ . Thus  $L = (I \times T) \cup (S \times J) = \{(x, u) \in S \times T \mid \chi(x, u) = 0\}$ , and hence by the first part of 2.5,  $L$  is a prime ideal of  $S \times T$ .

3. We next consider continuous semicharacters of topological semigroups.

3.1. DEFINITION. A semigroup  $S$  is called a *topological semigroup* if  $S$  is also a topological space and the mapping of  $S \times S$  into  $S$  defined by  $(x, y) \rightarrow xy$  is a continuous mapping of  $S \times S$  into  $S$ . The set of all continuous semicharacters of  $S$  will be denoted by  $\widehat{S}_c$ .

It is straightforward to prove that if  $S$  and  $T$  are topological semigroups, then  $S \times T$  is a topological semigroup under the product topology.

3.2. DEFINITION. If  $S$  and  $T$  are topological semigroups, we define  $\widehat{S}_c \circ \widehat{T}_c = \{\chi \mid \chi = (\phi, \psi) \text{ for some } \phi \in \widehat{S}_c, \psi \in \widehat{T}_c\}$ .

3.3. THEOREM. If  $S$  and  $T$  are topological semigroups, then  $(\widehat{S \times T})_c = \widehat{S}_c \circ \widehat{T}_c$ .

*Proof.* If  $\phi \in \widehat{S}_c$  and  $\psi \in \widehat{T}_c$ , then  $(\phi, \psi) \in \widehat{S \times T}$  by 2.4. Hence to show that  $(\phi, \psi) \in (\widehat{S \times T})_c$ , it suffices to show that  $(\phi, \psi)$  is continuous in both variables at an arbitrary point of  $S \times T$ . Using the fact that  $\phi$  and  $\psi$  are bounded, the proof of this fact is a standard continuity argument and is omitted. Therefore  $(\widehat{S \times T})_c \supseteq \widehat{S}_c \circ \widehat{T}_c$ . The reverse inclusion follows from 2.4 and the fact that joint continuity implies continuity in each variable.

3.4. THEOREM. Let  $S$  and  $T$  be topological semigroups and let  $\chi \in \widehat{S \times T}$ . Then the following statements are true.

(a) Let  $\phi \in \widehat{S}$  be such that  $(\phi, \psi) = \chi$  for some  $\psi \in \widehat{T}$ . If there exists  $(a, b) \in S \times T$  such that  $\chi(a, b) \neq 0$  and  $\chi(y, b)$  is a continuous function of  $y$  either in  $aS$  or in  $Sa$ , then  $\phi \in \widehat{S}_c$ .

(b)  $\chi(x, d)$  is continuous in  $S$  for each  $d \in T$  if and only if for some  $(a, b) \in S \times T$  such that  $\chi(a, b) \neq 0$  and  $\chi(y, b)$  is continuous either in  $aS$  or in  $Sa$ .

(c)  $\chi \in (\widehat{S \times T})_c$  if and only if for some  $(a, b) \in S \times T$  such that  $\chi(a, b) \neq 0$ ,  $\chi(y, b)$  is continuous either in  $aS$  or in  $Sa$ , and for some  $(c, d) \in S \times T$  such that  $\chi(c, d) \neq 0$ ,  $\chi(c, v)$  is continuous either in  $dT$

or in  $Td$ .

*Proof.* (a) By 2.3, we have  $\phi(x) = \chi(ax, b)/\chi(a, b)$  for all  $x \in S$ . Since  $(a, b)$  is fixed, it suffices to show that  $\chi(ax, b)$  is a continuous function of  $x$  in  $S$ . Suppose that  $\chi(y, b)$  is continuous in  $aS$ . Let  $m(x) = ax$  for all  $x \in S$  and  $l(y) = \chi(y, b)$  for all  $y \in aS$ . Then  $m$  is continuous by continuity of multiplication and  $l$  is continuous by hypothesis. We have  $l \circ m(x) = \chi(ax, b)$  for all  $x \in S$ . Since  $l \circ m$  is continuous,  $\chi(ax, b)$  is continuous in  $x$ . Hence  $\phi \in \hat{S}_c$ .

Suppose now that  $\chi(y, b)$  is continuous in  $Sa$ . By (1) of 2.3, we have  $\chi(ax, b) = \chi(xa, b)$  and consequently  $\phi(x) = \chi(xa, b)/\chi(a, b)$  for all  $x \in S$ . Defining  $m(x) = xa$  for all  $x \in S$ , we show that  $\phi \in \hat{S}_c$  in a similar way as above.

(b) Necessity is obvious; we prove sufficiency. Let  $d$  be any element of  $T$ . If  $\chi(x, d) = 0$  for all  $x \in S$ , then  $\chi(x, d)$  is continuous in  $S$ . Suppose that  $\chi(c, d) \neq 0$  for some  $c \in S$ . Continuity of  $\chi(y, b)$  in  $aS$  or in  $Sa$  implies that  $\phi \in \hat{S}_c$ , where  $\phi(x) = \chi(ax, b)/\chi(a, b)$  for all  $x \in S$ , by part (a) of the present theorem and 2.3. By 2.3,  $\phi$  is unique and thus  $\chi(ax, b)/\chi(a, b) = \chi(cx, d)/\chi(c, d)$  for all  $x \in S$ . Consequently,  $\chi(cx, d)/\chi(c, d)$  is continuous in  $x$ . We have

$$\begin{aligned} \chi(x, d) &= \frac{\chi(c^2, d)\chi(x, d)}{\chi(c^2, d)} = \frac{\chi(c^2x, d^2)}{\chi(c^2, d)} \\ &= \frac{\chi(c, d)\chi(cx, d)}{\chi(c^2, d)} = \frac{\chi(c^2, d^2)\chi(cx, d)}{\chi(c^2, d)\chi(c, d)} \end{aligned}$$

for all  $x \in S$ . Since  $\chi(c^2, d^2)/\chi(c^2, d)$  is a constant,  $\chi(x, d)$  is continuous in  $S$ .

(c) Necessity is obvious; we prove sufficiency. By 2.3,  $\chi = (\phi, \psi)$  for some  $\phi \in \hat{S}$ ,  $\psi \in \hat{T}$ , and by part (a) of the present theorem,  $\phi \in \hat{S}_c$  and similarly  $\psi \in \hat{T}_c$ . From 3.3 it follows that  $\chi = (\phi, \psi) \in (\widehat{S \times T})_c$ .

## BIBLIOGRAPHY

1. Edwin Hewitt and Herbert S. Zuckerman, *The  $l_1$ -algebra of a commutative semigroup*, Trans. Amer. Math. Soc., **83**, no. 1 (1956), 70-97.
2. Štefan Schwarz, *The theory of characters of commutative semigroups*, Czechoslovak Math. J., **4** (79) (1954), 219-247 (in Russian).
3. ———, *The theory of characters of commutative Hausdorff bicomact semigroups*, Czechoslovak Math. J., **6** (81) (1956), 330-364.



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The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insa'susha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.



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