SEMICHARACTERS OF THE CARTESIAN PRODUCT OF TWO SEMIGROUPS

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OF TWO SEMIGROUPS

MARIO PETRICH

1. If $S$ and $T$ are semigroups, then by $S \times T$ we mean the semigroup consisting of the Cartesian product $S \times T$ of the sets $S$ and $T$ with coordinatewise multiplication. The semigroup $S \times T$ is called the Cartesian product of the semigroups $S$ and $T$. A complex-valued multiplicative function on a semigroup $S$ is called a semicharacter of $S$ if it is different from 0 at some point and is bounded (1.3, [1]). The set of all semicharacters of $S$ is denoted by $\hat{S}$.

We show that $S \times T = \{\chi(x, u) = \phi(x)\psi(u) \text{ for some } \phi \in \hat{S}, \psi \in \hat{T}\}$ (2.4). We obtain a similar result for continuous semicharacters of topological semigroups (3.3). One of the most interesting consequences of the above results is a theorem on prime ideals (2.6). A subset $I$ of a semigroup $S$ is called a prime ideal of $S$ if $I$ is a proper (i.e., $\neq S$) two-sided ideal of $S$ whose complement in $S$ is a semigroup. For convenience we also call the empty set a prime ideal (cf. Definitions 2, 2a, [2]). We also prove a theorem concerning continuity of the semicharacters of the Cartesian product $S \times T$ of two topological semigroups (3.4).

If $A$ and $B$ are sets, then $A - B$ will denote the set of all elements of $A$ which are not contained in $B$. A semigroup will always be non-empty. A nonempty subset $I$ of $S$ is said to be an (two-sided) ideal of $S$ if $xy, yx \in I$ for all $x \in S, y \in I$.

All results in this paper are stated for the Cartesian product of two semigroups. However, a simple inductive argument shows that all of them generalize to the Cartesian product of any finite number of semigroups.

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2. If $S$ and $T$ are semigroups with two-sided identities, then semicharacters of $S \times T$ are obtained easily from the semicharacters of $S$ and $T$. (If $e$ and $f$ are identities of $S$ and $T$, respectively, then each element $(x, u)$ of $S \times T$ can be written as $(x, f)(e, u)$.) In 5, [3], Št. Schwarz considers this case for commutative semigroups. We first introduce two definitions.

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2.1. **DEFINITION.** Let \( f \) and \( g \) be arbitrary complex-valued functions defined on sets \( S \) and \( T \), respectively. We define the function \((f, g)\) on \( S \times T \) by \((f, g)(x, u) = f(x)g(u)\) for all \( x \in S, u \in T \).

2.2. **DEFINITION.** Let \( S \) and \( T \) be semigroups. We define \( \hat{S} \circ \hat{T} = \{ \chi | \chi = (\phi, \psi) \text{ for some } \phi \in \hat{S}, \psi \in \hat{T} \} \).

2.3. **THEOREM.** Let \( S \) and \( T \) be semigroups and let \( \chi \in \hat{S} \times \hat{T} \). Then \( \chi \) can be written uniquely as \((\phi, \psi)\), where \( \phi \in \hat{S} \) and \( \psi \in \hat{T} \). If \((a, b)\) is any element of \( S \times T \) such that \( \chi(a, b) \neq 0 \), then

\[
\phi(x) = \frac{\chi(ax, b)}{\chi(a, b)} \quad \text{for all } x \in S \quad \text{and} \\
\psi(u) = \frac{\chi(a, bu)}{\chi(a, b)} \quad \text{for all } u \in T.
\]

**Proof.** Let \((a, b)\) be any element of \( S \times T \) such that \( \chi(a, b) \neq 0 \) and let \( x \) and \( y \) be elements of \( S \). Then \( \chi(ax, b)\chi(a, b) = \chi(axa, b^2) = \chi(a, b)\chi(xa, b) \) and after dividing this identity by \( \chi(a, b) \), we obtain

\[
(1) \quad \chi(ax, b) = \chi(xa, b) \quad \text{for all } x \in S.
\]

Let

\[
\phi(x) = \frac{\chi(ax, b)}{\chi(a, b)} \quad \text{for all } x \in S.
\]

From (1) we obtain

\[
\chi(ax, b)\chi(ay, b) = \chi(ax, b)\chi(ya, b) = \chi(axya, b^2) = \chi(axy, b)\chi(a, b)
\]

and consequently

\[
\phi(x)\phi(y) = \frac{\chi(ax, b)}{\chi(a, b)} \frac{\chi(ay, b)}{\chi(a, b)} = \frac{\chi(ax, b)\chi(a, b)}{\chi(a, b)\chi(a, b)} = \phi(xy) \quad \text{for all } x, y \in S.
\]

We let

\[
\psi(u) = \frac{\chi(a, bu)}{\chi(a, b)} \quad \text{for all } u \in T.
\]

Like \( \phi, \psi \) is multiplicative. Let \((x, u)\) be any element of \( S \times T \). By (1), we have

\[
\chi(ax, b)\chi(a, bu) = \chi(a, bu)\chi(ax, b) = \chi(a, bu)\chi(xa, b) = \chi(axa, bu) = \chi(a, b)\chi(x, u)\chi(a, b)
\]

and thus

\[
\phi(x)\psi(u) = \frac{\chi(ax, b)}{\chi(a, b)} \frac{\chi(a, bu)}{\chi(a, b)} = \frac{\chi(a, b)\chi(x, u)\chi(a, b)}{\chi(a, b)\chi(a, b)} = \chi(x, u).
\]

Therefore
(2) \( \chi = (\phi, \psi) \).

Since \( \chi(a, b) \) is a constant, \( \phi \) is bounded, and since \( \phi(a) \neq 0 \), we conclude that \( \phi \in \hat{S} \). A similar argument shows that \( \psi \in \hat{T} \).

It only remains to prove uniqueness of \( \phi \) and \( \psi \). Suppose now that \( (\phi, \psi) = (\phi_1, \psi_1) \). Then \( \phi(x)\psi(u) = \phi_1(x)\psi_1(u) \) for all \( x \in S, u \in T \). There exists an element \( u_0 \in T \) such that \( \psi(u_0) \neq 0 \). Hence

\[
\phi(x) = \frac{\psi_1(u_0)}{\psi(u_0)} \phi_1(x) \quad \text{for all} \quad x \in S.
\]

Let \( K = \psi_1(u_0)/\psi(u_0) \). If \( x_0 \) is an element of \( S \) such that \( \phi(x_0) \neq 0 \), then

\[
\phi(x_0) = [\phi(x_0)]^2 = [K\phi_1(x_0)]^2 = K[K\phi_1(x_0)] = K\phi(x_0)
\]

and thus \( K = 1 \) since \( \phi(x_0) \neq 0 \). Therefore \( \phi = \phi_1 \). One shows similarly that \( \psi = \psi_1 \).

2.4. **Corollary.** If \( S \) and \( T \) are semigroups, then \( S \times T = \hat{S} \circ \hat{T} \).

**Proof.** If \( \phi \in \hat{S} \) and \( \psi \in \hat{T} \), it is easy to show that \( (\phi, \psi) \in \hat{S} \times \hat{T} \). Therefore \( \hat{S} \times \hat{T} \supseteq \hat{S} \circ \hat{T} \). The reverse inclusion follows from 2.3.

The following lemma has been proved by Št. Schwarz for several classes of semigroups (Lemma 3, [2] and Lemma 3.2, [3]).

2.5. **Lemma.** Let \( S \) be a semigroup and let \( \chi \in \hat{S} \). Then the set \( I = \{x \in S | \chi(x) = 0 \} \) is a prime ideal of \( S \). Conversely, if \( I \) is a prime ideal of \( S \), then there exists a semicharacter \( \chi \in \hat{S} \) such that

\[
I = \{x \in S | \chi(x) = 0 \}.
\]

**Proof.** The proof of the first statement is routine and is omitted. For the converse, let \( I \) be a prime ideal of \( S \). Define the function \( \chi \) on \( S \) by

\[
\chi(x) = \begin{cases} 
1 & \text{if } x \in S - I \\
0 & \text{if } x \in I
\end{cases}
\]

Then \( \chi \in \hat{S} \) and \( I = \{x \in S | \chi(x) = 0 \} \).

2.6. **Theorem.** Let \( S \) and \( T \) be semigroups. Then a set \( L \) is a prime ideal of \( S \times T \) if and only if \( L = (I \times T) \cup (S \times J) \) where \( I \) and \( J \) are prime ideals of \( S \) and \( T \), respectively.

**Proof.** Let \( L \) be a prime ideal of \( S \times T \). By the second part of 2.5, there is a semicharacter \( \chi \in \hat{S} \times \hat{T} \) vanishing exactly on \( L \). From 2.4 it follows that \( \chi = (\phi, \psi) \) for some \( \phi \in \hat{S}, \psi \in \hat{T} \). Clearly \( \chi(x, u) = \phi(x)\psi(u) = 0 \) if and only if either \( \phi(x) = 0 \) or \( \psi(u) = 0 \). Hence \( L = \{(x, u) \in S \times T | \chi(x, u) = 0 \} = (I \times T) \cup (S \times J) \), where \( I = \{x \in S | \phi(x) = 0 \} \) and \( J = \{u \in T | \psi(u) = 0 \} \). By the first part of 2.5, \( I \) and \( J \) are prime ideals of \( S \) and \( T \), respectively.

Conversely, let \( I \) and \( J \) be prime ideals of \( S \) an \( T \), respectively. By
the second part of 2.5, there are semicharacters $\phi \in \hat{S}$, $\psi \in \hat{T}$ vanishing exactly on $I$ and $J$, respectively. From 2.4 it follows that $(\phi, \psi) = \chi$ for some $\chi \in \hat{S} \times \hat{T}$. Clearly $\chi(x, u) = \phi(x)\psi(u) = 0$ if and only if either $\phi(x) = 0$ or $\psi(u) = 0$, and this happens if and only if either $x \in I$ or $u \in J$. Thus $L = (I \times T) \cup (S \times J) = \{(x, u) \in S \times T \mid \chi(x, u) = 0\}$, and hence by the first part of 2.5, $L$ is a prime ideal of $S \times T$.

3. We next consider continuous semicharacters of topological semigroups.

3.1. Definition. A semigroup $S$ is called a topological semigroup if $S$ is also a topological space and the mapping of $S \times S$ into $S$ defined by $(x, y) \rightarrow xy$ is a continuous mapping of $S \times S$ into $S$. The set of all continuous semicharacters of $S$ will be denoted by $\hat{S}$.

It is straightforward to prove that if $S$ and $T$ are topological semigroups, then $S \times T$ is a topological semigroup under the product topology.

3.2. Definition. If $S$ and $T$ are topological semigroups, we define $\hat{S} \circ \hat{T} = \{\chi \mid \chi = (\phi, \psi)$ for some $\phi \in \hat{S}$, $\psi \in \hat{T}\}$.

3.3. Theorem. If $S$ and $T$ are topological semigroups, then $(S \times T)_c = \hat{S} \circ \hat{T}$.

Proof. If $\phi \in \hat{S}$ and $\psi \in \hat{T}$, then $(\phi, \psi) \in S \times T$ by 2.4. Hence to show that $(\phi, \psi) \in (S \times T)_c$, it suffices to show that $(\phi, \psi)$ is continuous in both variables at an arbitrary point of $S \times T$. Using the fact that $\phi$ and $\psi$ are bounded, the proof of this fact is a standard continuity argument and is omitted. Therefore $(S \times T)_c \supseteq \hat{S} \circ \hat{T}$. The reverse inclusion follows from 2.4 and the fact that joint continuity implies continuity in each variable.

3.4. Theorem. Let $S$ and $T$ be topological semigroups and let $\chi \in \hat{S} \times \hat{T}$. Then the following statements are true.

(a) Let $\phi \in \hat{S}$ be such that $(\phi, \psi) = \chi$ for some $\psi \in \hat{T}$. If there exists $(a, b) \in S \times T$ such that $\chi(a, b) \neq 0$ and $\chi(y, b)$ is a continuous function of $y$ either in $aS$ or in $Sa$, then $\phi \in \hat{S}$.

(b) $\chi(x, d)$ is continuous in $S$ for each $d \in T$ if and only if for some $(a, b) \in S \times T$ such that $\chi(a, b) \neq 0$ and $\chi(y, b)$ is continuous either in $aS$ or in $Sa$.

(c) $\chi \in (S \times T)_c$ if and only if for some $(a, b) \in S \times T$ such that $\chi(a, b) \neq 0$, $\chi(y, b)$ is continuous either in $aS$ or in $Sa$, and for some $(c, d) \in S \times T$ such that $\chi(c, d) \neq 0$, $\chi(c, u)$ is continuous either in $dT$
Proof. (a) By 2.3, we have $\phi(x) = \chi(ax, b)/\chi(a, b)$ for all $x \in S$. Since $(a, b)$ is fixed, it suffices to show that $\chi(ax, b)$ is a continuous function of $x$ in $S$. Suppose that $\chi(y, b)$ is continuous in $aS$. Let $m(x) = ax$ for all $x \in S$ and $l(y) = \chi(y, b)$ for all $y \in aS$. Then $m$ is continuous by continuity of multiplication and $l$ is continuous by hypothesis. We have $l \circ m(x) = \chi(ax, b)$ for all $x \in S$. Since $l \circ m$ is continuous, $\chi(ax, b)$ is continuous in $x$. Hence $\phi \in \hat{S}_c$.

Suppose now that $\chi(y, b)$ is continuous in $Sa$. By (1) of 2.3, we have $\chi(ax, b) = \chi(xa, b)$ and consequently $\phi(x) = \chi(xa, b)/\chi(a, b)$ for all $x \in S$. Defining $m(x) = xa$ for all $x \in S$, we show that $\phi \in \hat{S}_c$ in a similar way as above.

(b) Necessity is obvious; we prove sufficiency. Let $d$ be any element of $T$. If $\chi(x, d) = 0$ for all $x \in S$, then $\chi(x, d)$ is continuous in $S$. Suppose that $\chi(c, d) \neq 0$ for some $c \in S$. Continuity of $\chi(y, b)$ in $aS$ or in $Sa$ implies that $\phi \in \hat{S}_c$, where $\phi(x) = \chi(ax, b)/\chi(a, b)$ for all $x \in S$, by part (a) of the present theorem and 2.3. By 2.3, $\phi$ is unique and thus $\chi(ax, b)/\chi(a, b) = \chi(cx, d)/\chi(c, d)$ for all $x \in S$. Consequently, $\chi(cx, d)/\chi(c, d)$ is continuous in $x$. We have

$$
\frac{\chi(x, d)}{\chi(c^2, d)} = \frac{\chi(c^2x, d)}{\chi(c^2, d)} = \frac{\chi(c^2x, d^2)}{\chi(c^2, d)}
$$

for all $x \in S$. Since $\chi(c^2, d^2)/\chi(c^2, d)$ is a constant, $\chi(x, d)$ is continuous in $S$.

(c) Necessity is obvious; we prove sufficiency. By 2.3, $\chi = (\phi, \psi)$ for some $\phi \in \hat{S}$, $\psi \in \hat{T}$, and by part (a) of the present theorem, $\phi \in \hat{S}_c$ and similarly $\psi \in \hat{T}_c$. From 3.3 it follows that $\chi = (\phi, \psi) \in (\hat{S} \times \hat{T})_c$.

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