Pacific Journal of Mathematics

A SPECIAL CLASS OF MATRICES

K. Rogers and Ernst Gabor Straus

Vol. 12, No. 2 February 1962

A SPECIAL CLASS OF MATRICES

K. ROGERS AND E. G. STRAUS

1. Introduction. Let D be an integral domain, K its quotient field, D^n the set of all n-by-1 matrices over D, and A an n-by-n matrix over a field containing K. We say that A has *property* P_D if and only if, for all nonzero u in D^n, the vector Au has at least one component in $D^* = D - \{0\}$. The setting in which this property arose is detailed in [**1**], where we investigated the case where D was either Z, the rational integers, or the ring of integers of an algebraic number field of class-number one. Now, if P is a permutation matrix, T is lower triangular with only ones in the diagonal, and N is nonsingular and over D, then $A = PTN$ has property P_D. It was shown in [**1**] that for $D = Z$ there are matrices not of the form PTN which have property P_D; but, at least in the case of the ring of integers of an algebraic number field of class-number one, we found the necessary but far from sufficient condition, that $\det A$ be in D^*. Our present purpose is to extend this to all algebraic number fields and also to prove necessary and sufficient conditions for property P_D in certain cases.

THEOREM I. *Let D be a domain whose quotient field K is algebraic over its prime field. Let A be an n-by-n matrix, where $n \leqq \sharp(K)$.*[1] *Then*:

(i) *If K is of prime characteristic, then A has property P_D if and only if $A = PTN$, where P, T and N are as above*:

(ii) *If D is Dedekind and K is a finite algebraic extension of the rationals, then for A to have P_D we must have* $\det A \in D^*$.

THEOREM II. *If $D = D_1[t]$, where t is transcendental over D_1, if $\sharp(D_1) > n$, and if A has P_D, then the rows of A can be so ordered that the matrices A_r of the first r rows of A have all r-by-r minors in D and not all zero, for $r = 1, 2, \cdots, n$. In particular, the first row is over D, and* $\det A \in D^*$.

If in addition we have only principal ideals, then we can reduce all but one element of the first row to zero and prove by induction:

COROLLARY. *If $D = F[t]$, where $\sharp(F) > n$, so K is a simple transcendental extension, then A has P_D if and only if $A = PTN$, where P, T and N are as above.*

Received August 11, 1961.

[1] $\sharp(K)$ = cardinality of K.

We can improve Theorem II to an if and only if statement, as long as $D_1[t]$ is a Gaussian domain.

THEOREM III. *If* $D = F[t_1, t_2 \cdots, t_k]$, *where the* t_i *are algebraically independent over the field* F, *and* $\sharp(F) > n$, *then a matrix* A *has* P_D *if and only if* $A = PLV$, *where* P *is a permutation matrix*, L *is a diagonal matrix over* D^*, *while* V *is nonsingular and such that for* $r = 1, 2, \cdots, n$, *the first* r *rows of* V *have their r-by-r minors in* D *and without common divisor.*

2. We try to reduce down to the case that A is over K.

LEMMA. I. *Let* B *be an r-by-n matrix over a field containing* K, *where* $\sharp(K) \geqq n \geqq r$, *and assume that there is a subspace* V *of* K^n *of dimension* r *such that, for all nonzero* $\underline{u}$ *in* V, $B\underline{u}$ *has a component in* K^*. *Then* $B = PTB_1$, *where* P *is a permutation matrix*, T *is triangular with only ones on the diagonal, and* B_1 *is r-by-n and such that, for all* u *in* V, *the product* B_1u *has all its components in* K *and is* 0 *only when* $u = 0$.

Proof. Let L_i note the subspace of V consisting of those u in V such that the ith component of Bu is in K. Then the relation between B and V implies that $V = \bigcup' L_i$, the union over those i such that for u in L_i the component $(Bu)_i$ is not always zero. We first show that some $L_i = V$. Assume that to be false: hence V is the union of at most r proper subspaces, say $V = H_1 \cup \cdots \cup H_m, m \leqq r \leqq n, m$ minimal. By choosing u, v so that $u \in H_1$, $v \in H_2 \cup \cdots \cup H_m$, $u \notin H_2 \cdots \cup H_m$, $v \notin H_1$, we ensure that the plane $Ku + Kv$ equals the union of at most m lines through the origin. This is clearly impossible if the field K is infinite. If $\sharp(K) = q$, then we should require that $q^2 \leqq n(q-1)+1$, that is, $q + 1 \leqq m \leqq n$, whereas we assumed that $q \geqq n$. Hence some row of B has all its inner products with V in K and not all zero. Permute the rows so that the first row, R_1^t, has this property. Then the lemma is proved for $r = 1$, and we are ready for induction on r; the matrix C of the last $r-1$ rows of B has the correct inner product property relative to $W = V \cap (KR_1)^\perp$, a space of dimension $r - 1$. Hence, $C = T_1C_1$, where T_1 is triangular of order $r-1$ with only ones on the diagonal, while the rows $S_2^t, \cdots, S_r^t$ of C_1 are such that all S_j^tu are in K whenever $u \in W$. Since we have not yet chosen the first column of our final T, we can still modify the S_j by multiples of R: for all a_j in any field containing K, the row $S_j^t - a_jR_1^t$ has the same inner product on W as S_j^t. Let S_1 be a vector in V but not in W, so that R and S_1 are not perpendicular. We can then choose a_j so that $(S_j^t a_j R_1^t)S_1 - 0$, so that the rows $R_j^t = S_j^t - a_jR_1^t$ have all inner products in K with a basis for V over K, hence the same with all vectors in V. The result now follows, with T obtained from T_1 by putting the row $(1, 0, \cdots, 0)$ on top and the column $(1, a_2, \cdots, a_n)^t$ to the left, while B_1 has rows $R_1^t, \cdots, R_r^t$. Finally, if some nonzero u in V were perpendicular to all the R, it would be perpendicular to all the rows of B and thus violate the hypothesis.

COROLLARY 1. *If* $\sharp(K) \geqq n$, *and if* A *has property* P_K, *then* $A = PTA_1$, *where* T *is lower triangular with only ones on the diagonal, while* A_1 *is nonsingular over* K. *As usual,* P *is a permutation matrix.*

Proof. This is the case $r = n$, so $V = K^n$ and the deduction is immediate.

COROLLARY 2. *If* $\sharp(K) \geqq n$, *then* A *has* P_D *implies* $\det A \in K^*$.

3. *Proof of Theorem* I. We note first that, if A has P_D and R is any sub-domain of D, then A has property P relative to the intersection of D with the ring obtained from R by adjoining the elements of A. Hence we can take D to be a sub-domain of a finite extension of the prime field. In case K is purely algebraic, this intersection is a finite algebraic extension of the prime field. However, this procedure may spoil the Dedekind property, so we only use this for *part* (*i*). There, we are now down to the case where D is a sub-domain of a finite field and therefore is itself a finite field. This part of Theorem I follows now from Corollary 1 above, with $D = K$. For *part* (*ii*) we proceed as follows. In the preceding section we saw that if A has P_D then $\det A \in K^*$, and now we shall show that $\det A \in D^*$ in the case that D is a Dedekind ring and K is an algebraic number field. The usual case is when D is the ring of integers of K, of course. First, we shall replace A by a matrix over K. Permute the rows so that $A = TA_1$, as in Corollary 1. Now, if $1, \xi_1, \cdots, \xi_N$ is a basis for the K-module obtained by adjoining to K all the elements of T, then $A = (T_1 + \xi_2 T_2 + \cdots + \xi_N T_N)A_1$, where the T_i are over K, are strictly lower triangular for $i \geqq 2$, and T_1 is lower triangular with only ones on the diagonal. The matrix $T_1 A_1$ is over K, has the same determinant as A, and it has P_D. For, by the independence of $1, \xi_2, \cdots, \xi_N$ over K, $(Au)_i \in K$ if and only if $(Au)_i = (T_1 A_1 u)_i \in K$, for $u \in K^n$. So we are down to the case that A is over K. If $\det A$ is not in D, some prime ideal $\mathfrak{P}$ must occur to a negative power in the factorisation of the ideal $(\det A)$. Since every element of D can be expressed as $\pi^\nu u/v$, where $\pi \in \mathfrak{P}$, $\pi \notin \mathfrak{P}^2$, u and v are in D but not in $\mathfrak{P}$, while ν is a rational integer, the ring $D_{\mathfrak{P}} = \{a/b \mid a, b \in D, b \notin \mathfrak{P}\}$ is a discrete valuation ring in which every element is a unit times a power of π the only ideals being $D \supset (\pi) \supset (\pi^2) \supset$ etc. Since it is easily shown that A has property P relative to $D_{\mathfrak{P}}$, we are now down to the

case that D is a discrete valuation ring with prime element π, and $\det A$ is a unit times a negative power of π. By multiplying a row of A by an appropriate element of D^*, we can ensure that $\det A = \pi^{-1}$, if we wish. Things now proceed as in Lemma 3 of [1]. Multiply the ith row of A by π^{d_i}, where the d_i are such that the ensuing matrix is over D. Since D is a principal ideal ring, we can triangularize this new matrix B. It has the property that for all nonzero u in D^n, some component $(B\underline{u})_i$ is a nonzero multiple of π^{d_i}; also, $\det B = \pi^{\Sigma d_i - 1}$. These properties are shown to be contradictory. If the residue class field $D/\mathfrak{P}$ has degree f over $Z/\mathfrak{P} \cap Z = Z/pZ$, it has p^f elements. Then, the mumber of residue classes mod $\mathfrak{P}^a$ is p^{af}. By absorbing unit factors, we can assume that the diagonal elements of B are $\pi^{a_i}, i = 1, \cdots, n$, so that $\Sigma a_i < \Sigma d_i$. We let α_i, δ_i run over complete residue systems mod π^{a_i} and mod π^{d_i}, respectively: then the number of vectors α is $(p^f)^{\Sigma a_i}$ and the number of δ is $(p^f)^{\Sigma d_i}$. Hence there are more δ than α. As in [1], one now shows that for given δ there is one and only one α such that the equation $Bu = \alpha + \delta$ is solvable with u in D^n. Then, we find distinct δ, δ' and some α such that $Bu = \delta + \alpha$ and $Bu' = \delta' + \alpha$, where u and u' are in D^n. Hence, $B(u - u') = \delta - \delta'$, and each component of $\delta - \delta'$ is either zero or indivisible by π^{d_i}. This contradicts the P-property for B and establishes at last that we must have had $\det A \in D^*$.

4. *The case* $D = D_1[t]$. We saw in Lemma I, Corollary 2, that if A has P_D then we can permute the rows and reduce A to the form TA_1, where T is lower triangular with only ones on the diagonal, while A_1 is nonsingular and over K. We now note that $TA_1 = TEEA_1$, where E is any elementary matrix with $E^2 = I$; hence we can add K-multiples of columns of T to other columns, doing the corresponding row-operation on A_1. Hence, we may assume that the sub-diagonal elements of T are either zero or outside K.

LEMMA II. *If A has P_D, where $D = D_1[t]$, $\sharp(D_1) > n$ and t is transcendental over D_1, then some row of A must have all its elements in D.*

Proof. We have $A = TA_1$, as above. Some rows of T, such as the first, have only one nonzero component, and it is 1. By permutation of the columns of T (and hence of the rows of A_1) and also the rows of T, we can put things in the form:

$$A = \begin{pmatrix} I_s & & 0 \\ t_{s+1,1} \cdots 1 & & 0 \\ & 1 & \\ t_{n1} & & 1 \end{pmatrix} A_1 \,.$$

Thus, the first s rows of A_1 are also rows of A, and the last $n-s$ rows of T involve elements outside K. We shall show that if none of the first s rows in over D, then we can find a vector $u \in D^n$ such that the first s components of Au are in K but not in D, while the last $n-s$ components are not even in K. In general, if we want an element $\underline{u}$ of K^n to be such that the last $n-s$ components of Au are not in K, we want $b = A_1u$ to be in K^n but such that none of $t_{i1}b_1 + \cdots t_{i,i-1}b_{i-1} + b_1$ is in K, for $s < i \leqq n$. Since the coefficients $t_{i1}, \cdots, t_{i,i-1}$ are not all zero and the nonzero ones are outside K, these conditions amount to making b avoid $n-s$ subspaces of K^n. Thus, $u = A_1^{-1}b$ must avoid at most $n-1$ hyperplanes of K^n. So we are finished as soon as we have found u in D^n such that the first s components of A_1u are outside D, and with u avoiding a given set of hyperplanes. There are two cases, according as the matrix A_s of the first s rows of A has a common denominator out of D_1 or not.

(1) *Case when*

$$A_s = \begin{pmatrix} \dfrac{a_{11}(t)}{d}, & \cdots, & \dfrac{a_{1n}(t)}{d} \\ \dfrac{a_{s1}(t)}{d}, & \cdots, & \dfrac{a_{sn}(t)}{d} \end{pmatrix}$$

where $d \in D_1$, $a_{ij}(t) \in D_1[t]$, for $1 \leqq 1 \leqq s$, $1 \leqq j \leqq n$, and d is not a divisor of all the coefficients of $a_{i1}, \cdots, a_{in}$, for each i from 1 to s. We choose $u^t = (t, t^{N_2}, \cdots, t^{N_n})$, where 1 $N_2, \cdots, N_n$ are in ascending order and so far apart that the terms in $\sum_j a_{ij}(t)t^{N_j}$ do not combine, since their terms are of vastly different degrees. Hence, d does not divide all the coefficients of $\sum_j a_{ij}t^{N_j}$, as required.

(ii) *Case when*

$$A_s = \begin{pmatrix} \dfrac{a_{11}(t)}{a(t)} \cdots & \\ & \dfrac{a_{sn}(t)}{a(t)} \end{pmatrix}, \quad s \leqq n,$$

where for no value of i does $d(t)$ divide all of $a_{i1}(t), \cdots, a_{in}(t)$. The approach in (i) needs modification, since $d(t)$ might be just a power of t. We begin by showing that if $\sum_{i=1}^n a_i(t)(t-\alpha)^{N_i}$ is divisible by $d(t)$, then, for $N_1, \cdots, N_n$ sufficiently spaced, each $a_i(t)(t-\alpha)^{N_i}$ is divisible by $d(t)$. Since we could change to the new transcendental $t-\alpha$ over D_1, we need only treat the case $\alpha = 0$. Let $d =$ max degree among $d(t), a_1(t), \cdots$. If

$$d(t)(q_i(t) + \cdots + q_n(t)) = \sum_{i=1}^n a_i(t)t^{N_i}, \cdots_{(*)}$$

where $N_{i-1} + d < N_i$, $i = 2, \cdots, n$, and $q_\nu(t)$ involves only terms of degree greater then $N_{\nu-1}$ but not greater than $N_\nu + d$, then:

The terms on the right side of (*) of degree not greater than $N_1 + d$

$$= a_1(t)t^{N_1}$$
$$= \text{terms on left side of (*) of degree less than } N_2$$
$$= d(t)q_1(t).$$

Thus $d(t) \mid a_1(t)t^{N_1}$, and so on. Hence, if for some i we have $\sum_{\nu=1}^{n} a_{i\nu}(t)(t-\alpha)^{N_\nu}$ divisible by $d(t)$, then $d(t) \mid a_{i\nu}(t)(t-\alpha)^{N_\nu}$, $1 \leqq \nu \leqq n$. By cancelling the factors $t - \alpha$ which may occur in $d(t)$, we deduce that the complementary factor in $d(t)$ must divide some row of the $a_{i\nu}$. So, if we can pick more α than there are rows, we'd need some row divisible by so much that $d(t)$ would have to divide each $a_{i\nu}(t)$. We assumed that $\#(D_1) > n$ for exactly this reason. So, for some $\alpha \in D_1$ and for all $N_1, \cdots, N_n$ sufficiently large and far apart, all of $\sum_{\nu=1}^{n} a_{i\nu}(t)(t-\alpha)^{N_\nu}$ are indivisible by $d(t)$. As to avoiding hyperplanes of K^n: these have the form $h_1x_1 + \cdots + h_nx_n = 0$, where $h_i \in D_1[t]$. Since for $N_1, \cdots, N_n$ far enough apart, the terms of the $h_i(t)t^{N_i}$ don't overlap, we cannot have $\sum h_i(t)t^{N_i} = 0$. As usual, the change $t \to t - \alpha$ is no problem, so Lemma II is proved.

For our purpose, somewhat more than the above is needed. A mild generalisation of Lemma II is now proved.

LEMMA III. *Let B be an r-by-n matrix over a field containing $D_1(t)$, and assume that there is an r-dimensional subspace V of K^n, where $K = D_1(t)$, such that for all nonzero u in $D_1[t]^n \cap V$ some component of Bu is in $D_1[t]$ and is nonzero. Then, some row of B is such that its inner product with $D_1[t]^n \cap V$ is always in $D_1[t]$ and is not always zero.*

Proof. Since every nonzero element of V goes into D^n, where $D = D_1[t]$, on being multiplied by a suitable element of D, we know that Lemma I applies to B and V. Hence, as in the remarks immediately before Lemma II, we know that by permuting the rows of B we can put it in the form $B = TB_1$, where T is r-by-r, is triangular with only ones on the diagonal and every sub-diagonal entry is either 0 or outside K, while B_1 is such that for all nonzero u in V, the product B_1u is nonzero and in K^r. As in Lemma II, we can order the rows of T so that the ones in K come first:

$$B = \begin{pmatrix} I & & & \\ t_{s1}, & 1 & & \\ & & 1 & \\ t_{r1}, & & & 1 \end{pmatrix} B_1 \,,$$

where the last $r - s$ (posssibly 0) rows of T involves elements outside K.

The first $s(\geqq 1)$ rows of B_1 coincide with those of B, and we now show that one of these has the desired property. If not, then for the ith row R_i^t, $1 \leqq 1 \leqq s$, we can find a nonzero u_i in $D^n \cap V$, such that $R_i^t u_i$ is not in D. Consider now the matrix $B_1 U$, where U is n-by-s, consisting of the columns $u_1, \cdots, u_s$; $B_1 U$ is r-by-s, is over K, and the first s rows each contain an element outside D. Hence, as before, we can choose $N_1, \cdots, N_s$ so far apart that $B_1 U((t-\alpha)^{N_1}, \cdots, (t-\alpha)^{N_s})^t$ has its first s components outside D and such that the last $r-s$ components of $TB_1 U((t-\alpha)^{N_1}, \cdots, (t-\alpha)^{N_s})_t$ are not even in K. But the vector $u = \sum_{i=1}^s (t-\alpha)^{N_1} u_i \in D^n \cap V$, and we've just shown that Bu has no component in D. This contradiction shows that one of the first s rows of B has its inner product with $D^n \cap V$ always in D. It cannot be perpendicular to V, as there are nonzero elements of V perpendicular to all the other rows of B, by dimensions, and we excluded having all rows of B perpendicular to some nonzero element of V.

COROLLARY. *If A has property P_D, where $D = D_1[t]$ and $\sharp(D_1) > n$, as before, then the rows of A can be so arranged that R_1^t is over D, and for $k = 1, \cdots, n-1$, for all u in D^n and perpendicular to the first k rows of A, we have $R_{k+1}^t \cdot u$ in D, not always zero.*

Proof. By Lemma II we may assume the first row is over D. Assume that the first k rows have been arranged as desired, for some $k \geqq 1$; we can then proceed to the choice of R_{k+1}^t by applying Lemma III to the matrix of the last $n-k$ rows of A, with V the subspace of K^n orthogonal to the first k rows of A.

This necessary condition for P_D, in the simple transcendental case, has the virtue of being patently sufficient. It also makes evident the Corollary to Theorem II: when $D = F[t]$, so that all ideals are principal, matrices with P_D are essentially just nonsingular matrices over D, apart from permuting the rows and pre-multiplying by the usual triangular T. However, it is not easy to see how this criterion for general $D_1[t]$ would be checked, nor does it seem an obvious deduction that $\det A \in D^*$.

Theorem II will now be deduced. Since we already know that det $A \neq 0$, the r-by-r minors of the first r rows of A cannot all be zero. Hence, we need only show that if the rows have been arranged as in the corollary above, then all the r-by-r minors of the first r rows are in D, for $1 \leqq r \leqq n$. By looking at an r-by-r sub-matrix of the first r rows of A, we see that its orthogonality properties should imply that its determinant is in D, and so it will suffice to prove:

LEMMA IV. *Let B be r-by-r over some field containing K, such that the first row is over D and, for $k = 1, \cdots, r-1$, all u perpen-*

dicular to the first k rows of B and in D^r have an inner product with the $k+1$st row in D. Then $\det B \in D$.

Proof. The case $r = 1$ is trivial, so induction can begin. By the case $r-1$, all the minors of the last row are in D. Since these numbers give a vector in D^r perpendicular to the first $r-1$ rows and having inner product $\det B$ with the last row, we are done. The proof of Theorem II is now complete.

It is not a sufficient condition on A for P_D, to have all these r-by-r minors in D and not all zero, for $1 \leqq r \leqq n$, as the example

$$A = \begin{pmatrix} x^2 & -xy \\ 0 & x^{-2} \end{pmatrix}$$

soon shows. In preparation for the proof of the last theorem, we shall show that the extra condition, that the r-by-r minors be in D and without common divisor, is sufficient in the cases when $D = D_1[t]$ is a unique factorisation ring, for example when $D = F[t_1, t_2, \cdots, t_k]$.

LEMMA V. *Let D be a unique factorisation domain with quotient field K, and let A be an r-by-n matrix of rank r such that, for $1 \leqq k \leqq r$, the k-by-k minors of the first k rows of A are all in D and without common divisor. Then the first row is, of course, over D and, for $1 \leqq k < r$, and for all u in D^n and perpendicular to the first k rows of A, the inner product $R^t_{k+1} \cdot u$ is in D.*

Proof. Since we use induction on r, it is necessary only to deal with the case of u perpendicular to the first $r-1$ rows of A. Consider the equations

$$\begin{pmatrix} a_{11}, & \cdots, & a_{1n} \\ \cdot & \cdots & \cdot \\ a_{r1}, & \cdots, & a_{rn} \\ & & \\ & I_{n-r} & \\ & & \end{pmatrix} \begin{pmatrix} u_1 \\ \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ p \\ u_{r+1} \\ u_n \end{pmatrix}$$

To show $p \in D$, we multiply both sides by $(C_1, \cdots C_n)$, these being the co-factors of the rth column of the n-by-n matrix: hence

$$\begin{vmatrix} a_{11} & \cdots & a_{1r} \\ a_{ri} & \cdots & a_{rr} \end{vmatrix} u_r = C_r p + C_{r+1}u_{r+1} + \cdots + C_n u_n \,.$$

But C_r equals the minor formed with the first $r-1$ rows and columns, while $C_{r+1}, \cdots, C_n$ are also equal to cofactors from the first $r-1$ rows. Thus, $C_r p \in D$. Since changing the order of the columns of A does not alter the truth of the hypotheses, we know that for all the minors C at the $(r-1)$st stage, $Cp \in D$. But these minors are without common divisor. Hence $p \in D$, as required.

COROLLARY. *Every matrix of the form* PLV, *as in Theorem* III, *has property* P_D.

Proof. Since P serves only to permute the rows, we may ignore it. Then we observe that since L is triangular with elements of D^* on the diagonal, the orthogonality property for V of Lemma III, Corollary, is not changed by going to LV. Thus, it is enough to use Lemma V with $r = n$.

Proof of Theorem III. We have just proved the "sufficienty" part of the theorem. So now assume A has P_D. By Lemma III we can order the rows of A so that for all $u \in D^n$ and perpendicular to the first k rows, $R_{k+1}^t \cdot u \in D$ and is not always zero. By using only those u with $n-k$ entries $u_{i1}, \cdots, u_{i_{n-k}}$ equal to zero, we see that the matrix obtained by erasing columns $i_i, \cdots, i_{n-k}$ and the last $n-k$ rows of A has the orthogonality property. By Lemma IV we deduce that the first k rows of A have all k-by-k minors in D. We now put A in the form LV by taking common factors as follows. We examine the first row of A: it is over D, so we take out the common factors. Proceed inductively: assume that factors have been take out so that the co-factors for the first k rows are without common divisor, for $1 \leqq k < r$, and the new matrix still has the orthogonality property. If the minors of the r rows are not relatively prime, divide the rth row by the common factor. Lemma V shows that the orthogonaltiy property is not lost by this process, so we can continue. This completes the proof of Theorem III.

REFERENCE

1. K. Rogers and E. G. Straus, *A class of geometric lattices*, Bull. Amer. Math. Soc., **66** (1960), 118-123.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 12, No. 2 February, 1962