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FURTHER RESULTS ON p -AUTOMORPHIC p -GROUPS

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Graham Higman [3] has shown that a finite p -group, p an odd prime, with an automorphism permuting the subgroups of order p cyclically is abelian. In [1] a p -group was defined to be p -automorphic if its automorphism group is transitive on the elements of order p . It was conjectured that a p -automorphic p -group ($p \neq 2$) is abelian and proved that a counterexample must be generated by at least four elements. In this present paper we prove that a counterexample generated by n elements must be such that $n > 5$ and, if $n \neq 6$, then $p < n3^{n^3}$ (Theorem 3). We also show that the existence of a counterexample implies the existence of a certain algebraic configuration (Theorem 1). All groups considered are finite.

Notation. $\Phi(P)$ is the Frattini subgroup of the p -group P and P' is its commutator subgroup. $\Omega_i(P)$ is the subgroup generated by the elements of P whose orders do not exceed p^i . $Z(P)$ is the center of P . $F(m, n, p)$ denotes the set of p -automorphic p -groups P which enjoy the additional properties:

1. $P' = \Omega_1(P)$ is elementary abelian of order p^n .
2. $\Phi(P) = Z(P) = \Omega_m(P)$ is the direct product of n cyclic groups of order p^m .
3. $|P: \Phi(P)| = p^n$.

In [1] it was shown that a counterexample generated by n elements has a quotient group in $F(m, n, p)$. Hence, in arguing by contradiction, we may assume that a counterexample P is in $F(m, n, p)$.

Let $A = A(P) = \text{Aut } P$ and let $A_0 = \ker(\text{Aut } P \rightarrow \text{Aut } P/\Phi(P))$. Thus $A/A_0 = B$ is faithfully represented as linear transformations of $V = P/\Phi(P)$, considered as a vector space over $GF(p)$.

Since p is odd and $cl(P) = 2$, the mapping $\eta: x \rightarrow x^{p^m}$ is an endomorphism of P which commutes with each σ of $\text{Aut } P$. Since $\Omega_m(P) = \Phi(P)$, $\ker \eta = \Phi(P)$, so η induces an isomorphism of V into $W = P'$. Since $\dim V = \dim W$, η is onto.

The commutator function induces a skew-symmetric bilinear map of $V \times V$ onto W , (onto since P is p -automorphic) and since $\Phi(P) = Z(P)$, $(,)$ is nondegenerate. Associated with $(,)$ is a nonassociative product \circ , defined as follows: If $\alpha, \beta \in V$, say $\alpha = x\Phi(P)$, $\beta = y\Phi(P)$, then $[x, y]$ is an element of W which depends only on α, β , and so $[x, y] = z^{p^m}$ where the coset $\gamma = z\Phi(P)$ depends only on α, β . We write $\alpha \circ \beta = \gamma$. An immediate consequence of this condition is the statement that $\alpha \rightarrow \alpha \circ \beta$

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is a linear map ϕ_β of V into V . Thus, \circ induces a map θ of V into $\text{End } V$, the ring of linear transformations of V to V .

If $\bar{\sigma}$ is the inner automorphism of $\text{End } V$ induced by $\sigma \in B$, then the diagram

$$\begin{array}{ccc} V & \xrightarrow{\theta} & \text{End } V \\ \sigma \downarrow & & \downarrow \bar{\sigma} \\ V & \xrightarrow{\theta} & \text{End } V \end{array}$$

commutes, that is $\phi_{\beta^\sigma} = \sigma^{-1}\phi_\beta\sigma$. Since P is p -automorphic, if α, β are nonzero elements of V , then $\alpha = \beta^\sigma$ for suitable $\sigma \in B$, so that $\phi_\alpha = \sigma^{-1}\phi_\beta\sigma$.

THEOREM 1. *If $\alpha \in V$, then ϕ_α is nilpotent.*

Proof. We can suppose $\alpha \neq 0$. Since $\alpha \circ \alpha = 0$, ϕ_α is singular. Let $f(x) = x^n + c_1x^{n-1} + c_2x^{n-2} + \dots$ be the characteristic equation of ϕ_α . $f(x)$ is independent of the nonzero element α of V , and $c_n = 0$ since ϕ_α is singular.

Let $\alpha_1, \dots, \alpha_n$ be a basis for V , and identify ϕ_α with the matrix which is associated with ϕ_α and the basis $\alpha_1, \dots, \alpha_n$. Then c_i is the sum of all i by i principal minors of ϕ_α , so if $\alpha = \lambda_1\alpha_1 + \dots + \lambda_n\alpha_n$, c_i is a homogeneous polynomial of degree i ($\leq n-1$) in the n variables $\lambda_1, \dots, \lambda_n$. By a Theorem of Chevalley [2], there are values $\lambda_1, \dots, \lambda_n$ of $GF(p)$ which are not all zero, such that $c_i = 0$. Since c_i is independent of the non-zero tuple $(\lambda_1, \dots, \lambda_n)$, it follows that $c_i = 0$ so ϕ_α is nilpotent.

Theorem 1 states that $\theta(V)$ is a linear variety of $\text{End}(V)$ consisting only of nilpotent matrices such that any two nonzero $x, y \in \theta(V)$ are similar. If one could show that the algebra generated by $\theta(V)$ were nilpotent, an easy argument would show that all p -automorphic p -groups (p odd) are abelian.

THEOREM 2. *Let r be the rank of ϕ_α . If $n > 3$, then $2 < r < n-1$.*

Proof. We assume $n > 3$ because $n \leq 3$ was treated in [1]. Clearly $r \neq 0$ because P is non-abelian and $r \neq n$ by Theorem 1.

Case I. $r \neq n-1$. Suppose $r = n-1$. Then, for $\alpha \neq 0$, $\beta \circ \alpha = \beta\phi_\alpha = 0$ implies that $\beta \in \{\alpha\}$ where $\{\alpha\}$ is the subspace of V spanned by α . If $\gamma\phi_\alpha^2 = (\gamma\phi_\alpha)\phi_\alpha = 0$, then $\gamma\phi_\alpha \in \{\alpha\}$, say $\gamma\phi_\alpha = k\alpha$. But $\gamma\phi_\alpha + \alpha\phi_\gamma = 0$ by the skew-symmetry of \circ , so $\alpha\phi_\gamma = -k\alpha$. By Theorem 1, $k = 0$ and thus $\gamma \in \{\alpha\}$. Hence $\text{rank } \phi_\alpha^2 = \text{rank } \phi_\alpha$, a contradiction to Theorem 1.

Case II. $r \neq 1$. Choose a basis of V , say $\alpha_1, \dots, \alpha_n$, and suppose

that $\phi_\alpha = (\alpha_{ij})$ with respect to this basis; $\text{End}(V)$ has the obvious matrix representation with $\phi_\alpha \in \theta(V) \subset \text{End}(V)$. Recall that $\theta(V)$ becomes an n -space of n by n nilpotent matrices over $GF(p)$ in which any two nonzero matrices are similar. If $r = 1$, then we may assume without loss of generality that ϕ_α has a 1 in the (1, 2) position and zeros elsewhere.

If every $(x_{ij}) = X \in \theta(V)$ satisfies $x_{ij} = 0$ for $i > 1$, then we are done because the nilpotency of X implies that $x_{11} = 0$ for every $X \in \theta(V)$, which implies that $\dim \theta(V) < n$. If, on the other hand, there exists $X \in \theta(V)$ with a nonzero entry below the first row, then we may use the fact that every 2 by 2 subdeterminant of every element of $\theta(V)$ vanishes to show that every X has its nonzero elements in the second column only. But the nilpotency of X implies that $x_{22} = 0$. Hence $\dim \theta(V) < n$, a contradiction.

Case III. $r \neq 2$. If $r = 2$, we may assume without loss of generality that

- (a) ϕ_α has 1's in the (1, 2), (2, 3) positions and zeros elsewhere or else
- (b) ϕ_α has 1's in the (1, 2), (3, 4) positions and zeros elsewhere.

First consider (a).

If every $(x_{ij}) = X \in \theta(V)$ satisfies $x_{ij} = 0$ for $i > 2$, then $Z(P) \cong \Phi(P)$, a contradiction. If every $X \in \theta(V)$ satisfies $x_{ij} = 0$ for $j \neq 2, 3$, then $x_{32} = 0$ because $X + k\phi_\alpha$ is nilpotent for every $k \in GF(p)$ and $p > 2$. But then $\dim \theta(V) < n$, a contradiction. Hence we need consider only the subcase of (a) in which some $X \in \theta(V)$ has a nonzero entry below the third row and a nonzero entry that is not in columns two or three. Consider such an X . Unless $x_{ij} = 0$ when $i \neq 1, 2$ and $j \neq 2, 3$, it is easy to see that there exists a nonzero 3 by 3 determinant in $X + k\phi_\alpha$ for some k . It is also easy to see that any two rows of X below the second row are dependent, and that any two columns other than the second and third are dependent. Using the fact that every 3 by 3 subdeterminant of every element of $\theta(V)$ is zero, it is straightforward to show that there exist nonsingular matrices R and S such that RXS has 1's in the (1, 4), (3, 2) positions and zeroes elsewhere and $R\phi_\alpha S$ has 1's in the (1, 3), (2, 2) positions and zeroes elsewhere.

Set $X' = RXS$, $\phi'_\alpha = R\phi_\alpha S$. It is now straightforward to show that that if $Y = (y_{ij}) \in R\theta(V)S$ is linearly independent from $\{X', \phi'_\alpha\}$, then $y_{ij} = 0$ for $i \neq 1$ and $j \neq 2$. This implies that $\dim R\theta(V)S < n$, a contradiction, since $\dim R\theta(V)S = \dim \theta(V) = n$.

Subcase (b), in which $\phi_\alpha^2 = 0$, is handled in a similar fashion except that we exclude the case in which every $X \in \theta(V)$ satisfies $x_{ij} = 0$, $j \neq 2, 4$, by noting the following: In such a case $(X + k\phi_\alpha)^2 = 0$ for every k implies that $x_{22} = 0$, which in turn implies that $\dim \theta(V) < n$.

COROLLARY. $F(m, n, p)$ is empty for all m and odd p unless $n > 5$.

Proof. Theorem 2 implies that $n > 4$ and that if $n = 5$, then $\text{rank } \phi_\alpha = 3$. Let S_n denote the projective $(n - 1)$ -space whose points are the 1-subspaces of V . If $n = 5$ and $\text{rank } \phi_\alpha = 3$, then it follows that S_5 is partitioned into lines according to the rule that $\{\alpha\}, \{\beta\}$ ($0 \neq \alpha, \beta \in V$) lie on the same line if and only if $\alpha \circ \beta = 0$. But S_5 has $p^4 + p^3 + p^2 + p + 1$ points and cannot be partitioned into disjoint subsets of $p + 1$ points each.

THEOREM 3. If $p \geq n3^{n^2}$ and $n \neq 6$, then $F(m, n, p)$ is empty for all positive integers m .

Proof. If $GL(n, p)$ denotes the invertible elements of $\text{End } V$, then

$$|GL(n, p)| = p^{n(n-1)/2} \cdot k(n, p), \text{ where } k(n, p) = (p^n - 1)(p^{n-1} - 1) \cdots (p - 1).$$

If we consider $GF(p^n)$ as a vector space over $GF(p)$, the right-regular representation shows that $GL(n, p)$ contains a cyclic group of order $p^n - 1$.

Let $\Phi_d(x)$ be the monic polynomial whose complex roots are the primitive d th roots of unity. Then $p^n - 1 = \prod_{d|n} \Phi_d(p)$. By an elementary number-theoretic theorem [4], $\Phi_n(p)$ and $k(n, p)/\Phi_n(p)$ are relatively prime, or their greatest common divisor is q where q is the largest prime divisor of n , in which case $\Phi_n(p)/q$ is relatively prime to $k(n, p)/\Phi_n(p)$. Thus, we determine $\varepsilon = 0$ or 1 so that $\Phi_n(p)/q^\varepsilon$ is relatively prime to $k(p, n)/\Phi_n(p)$.

Let $p \in F(m, n, p)$. Since P is p -automorphic, $|B|$ is divisible by $p^n - 1$ and in particular is divisible by $\Phi_n(p)/q^\varepsilon$. Let r^α be the largest power of the prime r which divides $\Phi_n(p)/q^\varepsilon$, $\alpha \geq 1$, and let S_r be a Sylow r -subgroup of B . By Sylow's theorem and the preceding paragraph, S_r is cyclic with generator σ_r .

Since P belongs to the exponent n modulo r , it follows that $\lambda, \lambda^p, \dots, \lambda^{p^{n-1}}$ are the characteristic roots of σ_r , λ being a primitive r^α th root of unity in $GF(p^n)$.

Since η commutes with σ_r , λ is also a characteristic root of σ_r on W . Since $(\alpha, \beta)^\sigma = (\alpha^\sigma, \beta^\sigma)$, the characteristic roots of σ_r on W are to be found among the $\lambda^{p^i + p^j}$, $0 \leq i < j \leq n - 1$, as can be seen by diagonalizing σ_r over $V \otimes GF(p^n)$. Hence, $\lambda = \lambda^{p^i + p^j}$ for suitable i, j and so

$$(1) \quad p^i + p^j - 1 \equiv 0 \pmod{r^\alpha}.$$

Since r was any prime divisor of $\Phi_n(p)/q^\varepsilon$, we have

$$(2) \quad \prod_{0 \leq i < j \leq n-1} (p^i + p^j - 1) \equiv 0 \pmod{\Phi_n(p)/q^\varepsilon}.$$

The polynomials $\Phi_n(x)$, $n \neq 6$, and $x^i + x^j - 1$ are relatively prime, a fact

which can be seen geometrically, as pointed out by G. Higman. Namely, if $\varepsilon, \varepsilon'$ are complex numbers of absolute value one, and $\varepsilon + \varepsilon' = 1$, then the points $0, 1, \varepsilon$ are the vertices of an equilateral triangle, so that ε is a primitive sixth root of unity. Since $n \neq 6$, we can therefore find integral polynomials $f(x), g(x)$, such that

$$(3) \quad f(x)\Phi_n(x) + g(x) \prod_{0 \leq i < j \leq n-1} (x^i + x^j - 1) = |N|,$$

where

$$(4) \quad \begin{aligned} N &= \prod_{\zeta} \prod_{i,j} (\zeta^i + \zeta^j - 1) \\ \Phi_n(\zeta) &= 0 \end{aligned}$$

is the resultant of $\Phi_n(x)$ and $\prod (x^i + x^j - 1)$.

From (4) we see that $N \leq 3^{\phi(n)n^2}$, since there are at most $\phi(n)n^2$ triples (ζ, i, j) . Now (2) and (3), the fact that $\Phi_n(p)/q^e$ divides $|N|$, imply that

$$(5) \quad \Phi_n(p)/q^e \leq 3^{\phi(n)n^2}.$$

One sees geometrically that $\Phi_n(p) \geq (p - 1)^{\phi(n)}$, so with (5) and $q^e \leq n$ we find

$$(6) \quad p \leq 1 + n^{1/\phi(n)} 3^{n^2} < n 3^{n^2}.$$

REMARK. Theorem 3 of [3] provides a certain motivation for the detailed examination of $\Phi_n(p)$ in the preceding theorem.

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