FURTHER RESULTS ON $p$-AUTOMORPHIC $p$-GROUPS

James Robert Boen, Oscar S. Rothaus and John Griggs Thompson
Graham Higman [3] has shown that a finite $p$-group, $p$ an odd prime, with an automorphism permuting the subgroups of order $p$ cyclically is abelian. In [1] a $p$-group was defined to be $p$-automorphic if its automorphism group is transitive on the elements of order $p$. It was conjectured that a $p$-automorphic $p$-group ($p \neq 2$) is abelian and proved that a counterexample must be generated by at least four elements. In this present paper we prove that a counterexample generated by $n$ elements must be such that $n > 5$ and, if $n \neq 6$, then $p < n 3^{n}$ (Theorem 3). We also show that the existence of a counterexample implies the existence of a certain algebraic configuration (Theorem 1). All groups considered are finite.

Notation. $\Phi(P)$ is the Frattini subgroup of the $p$-group $P$ and $P'$ is its commutator subgroup. $\Omega_{t}(P)$ is the subgroup generated by the elements of $P$ whose orders do not exceed $p'$. $Z(P)$ is the center of $P$. $F(m, n, p)$ denotes the set of $p$-automorphic $p$-groups $P$ which enjoy the additional properties:

1. $P' = \Omega_{t}(P)$ is elementary abelian of order $p^{n}$.
2. $\Phi(P) = Z(P) = \Omega_{m}(P)$ is the direct product of $n$ cyclic groups of order $p^{m}$.
3. $|P : \Phi(P)| = p^{n}$.

In [1] it was shown that a counterexample generated by $n$ elements has a quotient group in $F(m, n, p)$. Hence, in arguing by contradiction, we may assume that a counterexample $P$ is in $F(m, n, p)$.

Let $A = A(P) = \text{Aut} P$ and let $A_{0} = \ker(\text{Aut} P \to \text{Aut} P/\Phi(P))$. Thus $A/A_{0}$ is faithfully represented as linear transformations of $V = P/\Phi(P)$, considered as a vector space over $GF(p)$.

Since $p$ is odd and $cl(P) = 2$, the mapping $\eta: x \to x^{p^{m}}$ is an endomorphism of $P$ which commutes with each $\sigma$ of Aut $P$. Since $\Omega_{m}(P) = \Phi(P)$, ker $\eta = \Phi(P)$, so $\eta$ induces an isomorphism of $V$ into $W = P'$. Since dim $V = \dim W$, $\eta$ is onto.

The commutator function induces a skew-symmetric bilinear map of $V \times V$ onto $W$, (onto since $P$ is $p$-automorphic) and since $\Phi(P) = Z(P)$, $(,)$ is nondegenerate. Associated with $(,)$ is a nonassociative product $\circ$, defined as follows: If $\alpha, \beta \in V$, say $\alpha = x\Phi(P), \beta = y\Phi(P)$, then $[x, y]$ is an element of $W$ which depends only on $\alpha, \beta$, and so $[x, y] = z^{p^{m}}$ where the coset $\gamma = z\Phi(P)$ depends only on $\alpha, \beta$. We write $\alpha \circ \beta = \gamma$. An immediate consequence of this condition is the statement that $\alpha \to \alpha \circ \beta$
is a linear map \( \phi_\beta \) of \( V \) into \( V \). Thus, \( \circ \) induces a map \( \theta \) of \( V \) into \( \text{End} \, V \), the ring of linear transformations of \( V \) to \( V \).

If \( \tilde{\sigma} \) is the inner automorphism of \( \text{End} \, V \) induced by \( \sigma \in B \), then the diagram

\[
\begin{array}{ccc}
V & \xrightarrow{\theta} & \text{End} \, V \\
\sigma \downarrow & & \downarrow \tilde{\sigma} \\
V & \xrightarrow{\theta} & \text{End} \, V
\end{array}
\]

commutes, that is \( \phi_\beta^{\sigma} = \sigma^{-1} \phi_\beta \sigma \). Since \( P \) is \( p \)-automorphic, if \( \alpha, \beta \) are nonzero elements of \( V \), then \( \alpha = \beta^\sigma \) for suitable \( \sigma \in B \), so that \( \phi_\alpha = \sigma^{-1} \phi_\beta \sigma \).

**Theorem 1.** If \( \alpha \in V \), then \( \phi_\alpha \) is nilpotent.

**Proof.** We can suppose \( \alpha \neq 0 \). Since \( \alpha^\circ \alpha = 0 \), \( \phi_\alpha \) is singular. Let \( f(x) = x^n + c_1x^{n-1} + c_2x^{n-2} + \cdots \) be the characteristic equation of \( \phi_\alpha \). \( f(x) \) is independent of the nonzero element \( \alpha \) of \( V \), and \( c_n = 0 \) since \( \phi_\alpha \) is singular.

Let \( \alpha_1, \ldots, \alpha_n \) be a basis for \( V \), and identify \( \phi_\alpha \) with the matrix which is associated with \( \phi_\alpha \) and the basis \( \alpha_1, \ldots, \alpha_n \). Then \( c_i \) is the sum of all \( i \) by \( i \) principal minors of \( \phi_\alpha \), so if \( \alpha = \lambda_1 \alpha_1 + \cdots + \lambda_n \alpha_n \), \( c_i \) is a homogeneous polynomial of degree \( i (\leq n - 1) \) in the \( n \) variables \( \lambda_1, \ldots, \lambda_n \). By a Theorem of Chevalley [2], there are values \( \lambda_1, \ldots, \lambda_n \) of \( GF(p) \) which are not all zero, such that \( c_i = 0 \). Since \( c_i \) is independent of the non-zero tuple \( (\lambda_1, \ldots, \lambda_n) \), it follows that \( c_i = 0 \) so \( \phi_\alpha \) is nilpotent.

Theorem 1 states that \( \theta(V) \) is a linear variety of \( \text{End} \, (V) \) consisting only of nilpotent matrices such that any two nonzero \( x, y \in \theta(V) \) are similar. If one could show that the algebra generated by \( \theta(V) \) were nilpotent, an easy argument would show that all \( p \)-automorphic \( p \)-groups (\( p \) odd) are abelian.

**Theorem 2.** Let \( r \) be the rank of \( \phi_\alpha \). If \( n > 3 \), then \( 2 < r < n - 1 \).

**Proof.** We assume \( n > 3 \) because \( n \leq 3 \) was treated in [1]. Clearly \( r \neq 0 \) because \( P \) is non-abelian and \( r \neq n \) by Theorem 1.

**Case I.** \( r \neq n - 1 \). Suppose \( r = n - 1 \). Then, for \( \alpha \neq 0 \), \( \beta^\circ \alpha = \beta \phi_\alpha = 0 \) implies that \( \beta \in \{ \alpha \} \) where \( \{ \alpha \} \) is the subspace of \( V \) spanned by \( \alpha \). If \( \gamma \phi_\alpha = (\gamma \phi_\alpha) \phi_\alpha = 0 \), then \( \gamma \phi_\alpha \in \{ \alpha \} \), say \( \gamma \phi_\alpha = k \alpha \). But \( \gamma \phi_\alpha + \alpha \phi_\gamma = 0 \) by the skew-symmetry of \( \circ \), so \( \alpha \phi_\gamma = -k \alpha \). By Theorem 1, \( k = 0 \) and thus \( \gamma \in \{ \alpha \} \). Hence rank \( \phi_\alpha^2 = \text{rank} \, \phi_\alpha \), a contradiction to Theorem 1.

**Case II.** \( r \neq 1 \). Choose a basis of \( V \), say \( \alpha_1, \ldots, \alpha_n \), and suppose
that $\phi = (a_{ij})$ with respect to this basis; End $(V)$ has the obvious matrix representation with $\phi \in \mathcal{O}(V) \subset \text{End}(V)$. Recall that $\mathcal{O}(V)$ becomes an $\mathcal{n}$-space of $\mathcal{n}$ by $\mathcal{n}$ nilpotent matrices over $GF(p)$ in which any two nonzero matrices are similar. If $r = 1$, then we may assume without loss of generality that $\phi$ has a 1 in the $(1, 2)$ position and zeros elsewhere.

If every $(x_{ij}) = X \in \mathcal{O}(V)$ satisfies $x_{ij} = 0$ for $i > 1$, then we are done because the nilpotency of $X$ implies that $x_{ii} = 0$ for every $X \in \mathcal{O}(V)$, which implies that $\dim \mathcal{O}(V) < \mathcal{n}$. If, on the other hand, there exists $X \in \mathcal{O}(V)$ with a nonzero entry below the first row, then we may use the fact that every 2 by 2 subdeterminant of every element of $\mathcal{O}(V)$ vanishes to show that every $X$ has its nonzero elements in the second column only. But the nilpotency of $X$ implies that $x_{23} = 0$. Hence $\dim \mathcal{O}(V) < \mathcal{n}$, a contradiction.

Case III. $r \neq 2$. If $r = 2$, we may assume without loss of generality that

(a) $\phi$ has 1’s in the $(1, 2), (2, 3)$ positions and zeros elsewhere or else

(b) $\phi$ has 1’s in the $(1, 2), (3, 4)$ positions and zeros elsewhere.

First consider (a).

If every $(x_{ij}) = X \in \mathcal{O}(V)$ satisfies $x_{ij} = 0$ for $i > 2$, then $Z(P) \not\subseteq \mathcal{O}(P)$, a contradiction. If every $X \in \mathcal{O}(V)$ satisfies $x_{ij} = 0$ for $j \neq 2, 3$, then $x_{23} = 0$ because $X + k\phi$ is nilpotent for every $k \in GF(p)$ and $p > 2$. But then $\dim \mathcal{O}(V) < \mathcal{n}$, a contradiction. Hence we need consider only the subcase of (a) in which some $X \in \mathcal{O}(V)$ has a nonzero entry below the third row and a nonzero entry that is not in columns two or three. Consider such an $X$. Unless $x_{ij} = 0$ when $i \neq 1, 2$ and $j \neq 2, 3$, it is easy to see that there exists a nonzero 3 by 3 determinant in $X + k\phi$ for some $k$. It is also easy to see that any two rows of $X$ below the second row are dependent, and that any two columns other than the second and third are dependent. Using the fact that every 3 by 3 subdeterminant of every element of $\mathcal{O}(V)$ is zero, it is straightforward to show that there exist nonsingular matrices $R$ and $S$ such that $RXS$ has 1’s in the $(1, 4), (3, 2)$ positions and zeros elsewhere and $R\phi S$ has 1’s in the $(1, 3), (2, 2)$ positions and zeros elsewhere.

Set $X' = RXS, \phi' = R\phi S$. It is now straightforward to show that if $Y = (y_{ij}) \in \mathcal{O}(V)S$ is linearly independent from $\{X', \phi'\}$, then $y_{ij} = 0$ for $i \neq 1$ and $j \neq 2$. This implies that $\dim R\mathcal{O}(V)S < \mathcal{n}$, a contradiction, since $\dim R\mathcal{O}(V)S = \dim \mathcal{O}(V) = \mathcal{n}$.

Subcase (b), in which $\phi^2 = 0$, is handled in a similar fashion except that we exclude the case in which every $X \in \mathcal{O}(V)$ satisfies $x_{ij} = 0, j \neq 2, 4$, by noting the following: In such a case $(X + k\phi)^2 = 0$ for every $k$ implies that $x_{23} = 0$, which in turn implies that $\dim \mathcal{O}(V) < \mathcal{n}$.
COROLLARY. \( F(m, n, p) \) is empty for all \( m \) and odd \( p \) unless \( n > 5 \).

\textbf{Proof.} Theorem 2 implies that \( n > 4 \) and that if \( n = 5 \), then rank \( \phi_n = 3 \). Let \( S_n \) denote the projective \((n-1)\)-space whose points are the 1-subspaces of \( V \). If \( n = 5 \) and rank \( \phi_n = 3 \), then it follows that \( S_5 \) is partitioned into lines according to the rule that \( \{\alpha\}, \{\beta\} (0 \neq \alpha, \beta \in V) \) lie on the same line if and only if \( \alpha \circ \beta = 0 \). But \( S_5 \) has \( p^4 + p^3 + p^2 + p + 1 \) points and cannot be partitioned into disjoint subsets of \( p + 1 \) points each.

\textbf{THEOREM 3.} If \( p \geq n3^n \) and \( n \neq 6 \), then \( F(m, n, p) \) is empty for all positive integers \( m \).

\textbf{Proof.} If \( GL(n, p) \) denotes the invertible elements of \( \text{End} \ V \), then
\[
|GL(n, p)| = p^{n(n-1)/2} \cdot k(n, p),
\]
where \( k(n, p) = (p^n - 1)(p^{n-1} - 1) \cdots (p - 1) \).

If we consider \( GF(p^n) \) as a vector space over \( GF(p) \), the right-regular representation shows that \( GL(n, p) \) contains a cyclic group of order \( p^n - 1 \).

Let \( \Phi_d(x) \) be the monic polynomial whose complex roots are the primitive \( d \)-th roots of unity. Then \( p^n - 1 = \prod_{d|n} \Phi_d(p) \). By an elementary number-theoretic theorem [4], \( \Phi_d(p) \) and \( k(n, p)/\Phi_n(p) \) are relatively prime, or their greatest common divisor is \( q \) where \( q \) is the largest prime divisor of \( n \), in which case \( \Phi_n(p)/q \) is relatively prime to \( k(p, n)/\Phi_n(p) \).

Let \( p \in F(m, n, p) \). Since \( P \) is \( p \)-automorphic, \( |B| \) is divisible by \( p^n - 1 \) and in particular is divisible by \( \Phi_n(p)/q^s \). Let \( r^* \) be the largest power of the prime \( r \) which divides \( \Phi_n(p)/q^s \), \( \alpha \geq 1 \), and let \( S_r \) be a Sylow \( r \)-subgroup of \( B \). By Sylow’s theorem and the preceding paragraph, \( S_r \) is cyclic with generator \( \sigma_r \).

Since \( P \) belongs to the exponent \( n \) modulo \( r \), it follows that \( \lambda, \lambda^r, \ldots, \lambda^{p^{n-1}} \) are the characteristic roots of \( \sigma_r \), \( \lambda \) being a primitive \( r^* \)-th root of unity in \( GF(p^n) \).

Since \( \eta \) commutes with \( \sigma_r \), \( \lambda \) is also a characteristic root of \( \sigma_r \) on \( W \). Since \( (\alpha, \beta)^r = (\alpha^r, \beta^r) \), the characteristic roots of \( \sigma_r \) on \( W \) are to be found among the \( \lambda^{p^i+p^j} \), \( 0 \leq i < j \leq n - 1 \), as can be seen by diagonalizing \( \sigma_r \) over \( V \otimes GF(p^n) \). Hence, \( \lambda = \lambda^{p^i+p^j} \) for suitable \( i, j \) and so
\[
(1) \quad p^i + p^j - 1 \equiv 0 \pmod{r^s}.
\]

Since \( r \) was any prime divisor of \( \Phi_n(p)/q^s \), we have
\[
(2) \quad \prod_{0 \leq i < j \leq n - 1} (p^i + p^j - 1) \equiv 0 \pmod{\Phi_n(p)/q^s}.
\]
The polynomials \( \Phi_n(x), n \neq 6 \), and \( x^i + x^j - 1 \) are relatively prime, a fact
which can be seen geometrically, as pointed out by G. Higman. Namely, if \( \varepsilon, \varepsilon' \) are complex numbers of absolute value one, and \( \varepsilon + \varepsilon' = 1 \), then the points 0, 1, \( \varepsilon \) are the vertices of an equilateral triangle, so that \( \varepsilon \) is a primitive sixth root of unity. Since \( n \neq 6 \), we can therefore find integral polynomials \( f(x), g(x) \), such that

\[
(3) \quad f(x)\Phi_n(x) + g(x) \prod_{0 \leq i < j \leq n-1} (x^i + x^j - 1) = |N|,
\]

where

\[
(4) \quad N = \prod_\xi \prod_{i,j} (\xi^i + \xi^j - 1)
\]

\[
\Phi_n(\xi) = 0
\]

is the resultant of \( \Phi_n(x) \) and \( \prod(x^i + x^j - 1) \).

From (4) we see that \( N \leq 3^{\phi(n)n^2} \), since there are at most \( \phi(n)n^2 \) triples \((\xi, i, j)\). Now (2) and (3), the fact that \( \Phi_n(p)/q^e \) divides \(|N|\), imply that

\[
(5) \quad \Phi_n(p)/q^e \leq 3^{\phi(n)n^2}.
\]

One sees geometrically that \( \Phi_n(p) \geq (p - 1)^{\phi(n)} \), so with (5) and \( q^e \leq n \) we find

\[
(6) \quad p \leq 1 + n^{1/\phi(n)}3^{n^2} < n3^{n^2}.
\]

**Remark.** Theorem 3 of [3] provides a certain motivation for the detailed examination of \( \Phi_n(p) \) in the preceding theorem.

**Bibliography**

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