ON DIRECT SUMS AND PRODUCTS OF MODULES

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A well-known theorem of the theory of abelian groups states that the direct product of an infinite number of infinite cyclic groups is not free ([6], p. 48.) Two generalizations of this result to modules over various rings have been presented in earlier papers of the author ([3], [4].) In this note we exhibit a broader generalization which contains the preceding ones as special cases.

Moreover, it has other applications. For example, it yields an easy proof of a part of a result of Baumslag and Blackburn [2] which gives necessary conditions under which the direct sum of a sequence of abelian groups is a direct summand of their direct product. We also use it to prove the following variant of a result of Baer [1]: If a torsion group \( T \) is an epimorphic image of a direct product of a sequence of finitely generated abelian groups, then \( T \) is the direct sum of a divisible group and a group of bounded order. Finally, we derive a property of modules over a Dedekind ring which, for the ring \( \mathbb{Z} \) of rational integers, reduces to the following recent theorem of Rotman [10] and Nunke [9]: If \( A \) is an abelian group such that \( \text{Ext}_\mathbb{Z}(A, T) = 0 \) for any torsion group \( T \), then \( A \) is slender.

In this note all rings have identities and all modules are unitary.

1. The main theorem. Our discussion will be based on the following technical device.

**Definition 1.1.** Let \( \mathcal{F} \) be a collection of principal right ideals of a ring \( R \). \( \mathcal{F} \) will be called a filter of principal right ideals if, whenever \( aR \) and \( bR \) are in \( \mathcal{F} \), there exists \( c \in aR \cap bR \) such that \( cR \) is in \( \mathcal{F} \).

We proceed immediately to the principal result of this note.

**Theorem 1.2.** Let \( A^{(1)}, A^{(2)}, \ldots \) be a sequence of left modules over a ring \( R \), and set \( A = \prod_{i=1}^\infty A^{(i)} \), \( A_n = \prod_{i=n+1}^\infty A^{(i)} \). Let \( C = \bigcup_{\alpha} C_\alpha \), where \( \{C_\alpha\} \) is a family of left \( R \)-modules and \( \alpha \) traces an index set \( I \). Let \( f: A \to C \) be an \( R \)-homomorphism, and denote by \( f_\alpha: A \to C_\alpha \) the composition of \( f \) with the projection of \( C \) onto \( C_\alpha \). Finally, let \( \mathcal{F} \) be a filter of principal right ideals of \( R \). Then there exists \( \alpha R \) in \( \mathcal{F} \) and an integer \( n > 0 \) such that \( f_\alpha(aA_n) \subseteq \bigcap_{b \in \mathcal{F}} bC_\alpha \) for all but a finite number of \( \alpha \) in \( I \).

**Proof.** Assume that the statement is false. We shall first construct
inductively sequences \( \{x_n\} \subseteq A \), \( \{a_nR\} \subseteq \mathcal{F} \), and \( \{\alpha_n\} \subseteq I \) such that the following conditions hold:

(i) \( a_nR \supseteq a_{n+1}R \).

(ii) \( x_n \in a_nA_n \).

(iii) \( f_{\alpha_n}(x_n) \not\equiv 0 \pmod{a_{n+1}C_{\alpha_n}} \).

(iv) \( f_{\alpha_n}(x_k) = 0 \) for \( k < n \).

We proceed as follows. Select any \( a_1R \in \mathcal{F} \). Then there exists \( \alpha_1 \in I \) such that \( f_{\alpha_1}(a_1A_1) \not\subseteq \bigcap_{bR \in \mathcal{F}} bC_{\alpha_1} \), and hence we may select \( bR \in \mathcal{F} \) such that \( f_{\alpha_1}(a_1A_1) \not\subset bC_{\alpha_1} \). Since \( \mathcal{F} \) is a filter of principal right ideals, there exists \( a_2 \in a_1R \cap bR \) such that \( a_2R \in \mathcal{F} \), in which case \( f_{\alpha_1}(a_1A_1) \not\subseteq a_2C_{\alpha_1} \). Hence there exists \( x_1 \in a_1A_1 \) such that \( f_{\alpha_1}(x_1) \not\equiv 0 \pmod{a_2C_{\alpha_1}} \). Then conditions (i)–(iv) above are satisfied for \( n = 1 \).

Proceed by induction on \( n \); assume that the sequences \( \{x_k\} \) and \( \{\alpha_k\} \) have been constructed for \( k < n \) and the sequence \( \{a_kR\} \) has been constructed for \( k \leq n \) such that conditions (i)–(iv) are satisfied. Now, there exist \( \beta_1, \ldots, \beta_r \in I \) such that, if \( \alpha \neq \beta_1, \ldots, \beta_r \), then \( f_{\alpha}(x_k) = 0 \) for all \( k < n \). We may then select \( \alpha_n \neq \beta_1, \ldots, \beta_r \) such that \( f_{\alpha_n}(a_nA_n) \not\subseteq \bigcap_{bR \in \mathcal{F}} bC_{\alpha_n} \); for, if we could not do so, then the theorem would be true. Hence there exists \( bR \in \mathcal{F} \) such that \( f_{\alpha_n}(a_nA_n) \not\subseteq bC_{\alpha_n} \). Since \( \mathcal{F} \) is a filter of principal right ideals, there exists \( a_n+1A_n \subseteq a_nR \cap bR \) such that \( a_n+1R \in \mathcal{F} \), in which case \( f_{\alpha_n}(a_nA_n) \not\subseteq a_n+1C_{\alpha_n} \). Thus we may select \( x_n \in a_nA_n \) such that \( f_{\alpha_n}(x_n) \not\equiv 0 \pmod{a_n+1C_{\alpha_n}} \). It is then clear that the sequences \( \{x_k\} \) and \( \{\alpha_k\} \) for \( k \leq n \) and \( \{a_kR\} \) for \( k \leq n + 1 \) satisfy conditions (i)–(iv), and hence the construction of all three sequences is complete.

Now write \( x_k = (x_k^{(i)}) \), where \( x_k^{(i)} \in A^{(i)} \). Since \( x_k \in a_kA_k \), \( x_k^{(i)} = 0 \) for \( k > i \), and \( x^{(i)} = \sum_{k=1}^{n} x_k^{(i)} \) is a well-defined element of \( A^{(i)} \). Also, since \( a_nR \supseteq a_n+1R \supseteq \cdots \), it follows that there exists \( y_n^{(i)} \in A^{(i)} \) such that \( x^{(i)} = x_1^{(i)} + \cdots + x_n^{(i)} + a_{n+1}y_n^{(i)} \). Therefore, setting \( x = (x^{(i)}) \) and \( y_n = (y_n^{(i)}) \), we see that \( x = x_1 + \cdots + x_n + a_{n+1}y_n \) for all \( n \geq 1 \).

It follows immediately from inspection of conditions (iii) and (iv) above that \( \alpha_i \neq \alpha_j \) if \( i \neq j \). Hence there exists \( n \) such that \( f_{\alpha_n}(x) = 0 \). Writing \( x = x_1 + \cdots + x_n + a_{n+1}y_n \) as above, we may then apply \( f_{\alpha_n} \) and use condition (iv) to conclude that \( f_{\alpha_n}(x_n) = -a_{n+1}f_{\alpha_n}(y_n) \equiv 0 \pmod{a_{n+1}C_{\alpha_n}} \), contradicting condition (iii). The proof of the theorem is hence complete.

In the following discussion we shall use the symbol \( |X| \) to denote the cardinality of the set \( X \).

**Corollary 1.3** ([3], Theorem 3.1, p. 464). Let \( R \) be a ring, and \( A = \prod_{\alpha \in J} R^{(\alpha)} \), where \( R^{(\alpha)} \sim R \) as a left \( R \)-module and \( |J| \geq \aleph_0 \). Suppose that \( A \) is a pure submodule of \( C = \sum_{\beta} \bigoplus C_{\beta} \), where each \( C_{\beta} \) is a left \( R \)-
module and \( |C_\beta| \leq |J| \). Then \( R \) must satisfy the descending chain condition on principal right ideals.

**Proof.** Since \( J \) is an infinite set, it is easy to see that \( A \cong \prod_{i=1}^\infty A^{(i)} \), where \( A^{(i)} \approx A \), and so without further ado we shall identify \( A \) with \( \prod_{i=1}^\infty A^{(i)} \). Let \( f: A \to C \) be the inclusion mapping, and \( f_\beta: A \to C_\beta \) be the composition of \( f \) with the projection of \( C \) onto \( C_\beta \). Finally, set \( A_n = \prod_{i=n}^\infty A^{(i)} \).

Suppose that the statement is false. Then there exists a strictly descending infinite chain \( a_1R \supsetneq a_2R \supsetneq \cdots \) of principal right ideals of \( R \). These ideals obviously constitute a filter of principal right ideals of \( R \), and so we may apply Theorem 1.2 to conclude that there exists \( n \geq 1 \) and \( \beta_1, \cdots, \beta_r \) such that \( f_\beta(a_nA_n) \subseteq a_{n+1}C_\beta \) for \( \beta \neq \beta_1, \cdots, \beta_r \).

Now let \( C' = C_{\beta_1} \oplus \cdots \oplus C_{\beta_r} \); then the projection of \( C \) onto \( C' \) induces a \( Z \)-homomorphism \( g: a_nC/a_{n+1}C \to a_nC'/a_{n+1}C' \), where \( Z \) is the ring of rational integers. Also, the restriction of \( f \) to \( A_n \) induces a \( Z \)-homomorphism \( h: a_nA_n/a_{n+1}A_n \to a_nC/a_{n+1}C \). \( A_n \) is a direct summand of \( A \), which is a pure submodule of \( C \), and so \( A_n \) is likewise a pure submodule of \( C \). Hence \( h \) is a monomorphism. We may then apply the conclusion of the preceding paragraph to obtain that the composition \( gh \) is a monomorphism. In particular, \( |a_nA_n/a_{n+1}A_n| \leq |a_nC'/a_{n+1}C'| \leq |C'| \).

Observe that \( |C'| \leq |J| \), since \( J \) is infinite and \( |C_\beta| \leq |J| \) for all \( \beta \). However, since \( a_nR \neq a_{n+1}R \), \( a_nR/a_{n+1}R \) contains at least two elements; therefore \( |a_nA_n/a_{n+1}A_n| = |a_nA/a_{n+1}A| \geq 2^{|J|} > |J| \). We have thus reached a contradiction, and the corollary is proved.

2. Applications to integral domains. Throughout this section \( R \) will be an integral domain. If \( C \) is an \( R \)-module, we shall denote the maximal divisible submodule of \( C \) by \( d(C) \). In addition, we shall write \( R^\omega C = \bigcap aC \), where \( a \) traces the nonzero elements of \( R \).

Our principal result concerning modules over integral domains is the following theorem.

**Theorem 2.1.** Let \( \{A^{(i)}\} \) be a sequence of \( R \)-modules, and set \( A = \prod_{i=1}^\infty A^{(i)} \), \( A_n = \prod_{i=n}^\infty A^{(i)} \). Let \( C = \sum_a C_a \), where each \( C_a \) is an \( R \)-module. Let \( f: A \to C \) be an \( R \)-homomorphism, and \( f_a: A \to C_a \) be the composition of \( f \) with the projection of \( C \) onto \( C_a \). Then there exists an integer \( n \geq 1 \) and \( a \in R \), \( a \neq 0 \), such that \( af_a(A_n) \subseteq R^\omega C_a \) for all but finitely many \( a \).

**Proof.** Let \( \mathcal{F} \) be the set of all nonzero principal ideals of \( R \). Since \( R \) is an integral domain, it is clear that \( \mathcal{F} \) is a filter of principal ideals. The theorem then follows immediately from Theorem 1.2.

---

1 A is a pure submodule of \( C \) if \( A \cap aC = aA \) for all \( a \in R \).
Corollary 2.2 (see [4].) Same hypotheses and notation as in Theorem 2.1, with the exception that now each $C_\alpha$ is assumed to be torsion-free. Then there exists an integer $n \geq 1$ such that $f_\alpha(A_\alpha) \subseteq d(C_\alpha)$ for all but finitely many $\alpha$. In particular, if each $C_\alpha$ is reduced (i.e., has no divisible submodules) then $f_\alpha(A_\alpha) = 0$ for all but finitely many $\alpha$.

Proof. This follows immediately from Theorem 2.1 and the trivial observation that, since each $C_\alpha$ is torsion-free, $R^\omega C_\alpha = d(C_\alpha)$.

Next we present our proof of the afore-mentioned result of Baumslag and Blackburn concerning direct summands of direct products of abelian groups ([2], Theorem 1, p. 403.)

Theorem 2.3. Let $\{A^{(i)}\}$ be a sequence of modules over an integral domain $R$, and set $A = \prod_{i=1}^{\infty} A^{(i)}$, $C = \sum_{i=1}^{\infty} A^{(i)}$ (then $C$ is, in the usual way, a submodule of $A$.) If $C$ is a direct summand of $A$, then there exists $n \geq 1$ and $a \neq 0$ in $R$ such that $aA^{(i)} \subseteq d(A^{(i)})$ for $i > n$.

Proof. Assume that $C$ is a direct summand of $A$, and let $f: A \rightarrow C$ be the projection. Then the composition of $f$ with the projection of $C$ onto $A^{(i)}$ is an epimorphism $f_i: A \rightarrow A^{(i)}$. We then obtain from an easy application of Theorem 2.1 that there exists $n \geq 1$ and $a \neq 0$ in $R$ such that $af_i(A) \subseteq R^\omega A^{(i)}$. Since each $f_i$ is an epimorphism, it follows that $aA^{(i)} \subseteq R^\omega A^{(i)}$ for $i > n$.

Now let $z \in R^\omega A^{(i)}$, where $i > n$. If $b \neq 0$ is in $R$, then there exists $x \in A^{(i)}$ such that $abx = z$. Hence, setting $y = ax$, we have that $y \in R^\omega A^{(i)}$ and $by = z$. It then follows that $R^\omega A^{(i)}$ is divisible, and so $R^\omega A^{(i)} \subseteq d(A^{(i)})$. Therefore $aA^{(i)} \subseteq R^\omega A^{(i)} \subseteq d(A^{(i)})$ for $i > n$, completing the proof of the theorem.

We end this section with a proposition which will be useful in the proof of some later results.

Proposition 2.4. Let $\{A^{(i)}\}$ be a sequence of finitely generated modules over an integral domain $R$, and set $A = \prod_{i=1}^{\infty} A^{(i)}$. Let $C = \sum_{\alpha} C_\alpha$, where each $C_\alpha$ is a finitely generated torsion $R$-module. If $f: A \rightarrow C$ is an $R$-homomorphism, then there exists $c \in R$ such that $cf(A) = 0$ but $c \neq 0$.

Proof. As before we let $\mathcal{F}$ be the filter of all nonzero principal ideals of $R$. Clearly $R^\omega C_\alpha = 0$ for all $\alpha$, and so we may apply Theorem 2.1 to obtain $a \neq 0$ in $R$ and an integer $n > 0$ such that $af_\alpha(A_\alpha) = 0$ for all but finitely many $\alpha$, where $A_n = \prod_{i=n+1}^{\infty} A^{(i)}$ and $f_\alpha: A \rightarrow C_\alpha$ is defined as before. Say this condition holds for $\alpha \neq \alpha_1, \ldots, \alpha_r$; then, since each $C_\alpha$ is finitely generated and torsion, there exists $\alpha' \neq 0$ in $R$ such that $a'C_{\alpha_i} = 0$ for $i = 1, \ldots, r$, in which case $aaf(A_\alpha) = 0$. Since
each $A^{(i)}$ is finitely generated and $C$ is a torsion module, there exists $a'' \neq 0$ in $R$ such that $a''f(A^{(i)}) = 0$ for $i \leq n$. Set $c = aa'a''$; then $c \neq 0$ and, since $A = A^{(1)} \oplus \cdots \oplus A^{(n)} \oplus A_n$, it is clear that $cf(A) = 0$, completing the proof of the proposition.

3. Applications to Abelian groups. This section is devoted to a discussion of the results of Baer, Rotman, and Nunke mentioned in the introduction.

**Theorem 3.1** (see [1], Lemma 4.1, p. 231). Let $\{A^{(i)}\}$ be a sequence of finitely generated modules over a principal ideal domain $R$, and set $A = \prod_{i=1}^{\infty} A^{(i)}$. If $C$ is a torsion $R$-module which is an epimorphic image of $A$, then $C$ is the direct sum of a divisible module and a module of bounded order.

**Proof.** For each prime $p$ in $R$, let $C_p$ be the $p$-primary component of $C$ and $C'_p$ be a basic submodule of $C_p$ (see [5], p. 98;) i.e., $C'_p$ is a direct sum of cyclic modules and is a pure submodule of $C_p$, and $C_p/C'_p$ is divisible. Set $C'' = \sum_p C'_p$; then, since $C = \sum_p \oplus C_p$, $C'$ is a pure submodule of $C$ and $C/C'$ is divisible. Also, $C'$ is a direct sum of cyclic modules.

We now apply the fundamental result of Szele ([5], Theorem 32.1, p. 106) to conclude that $C'_p$ is an endomorphic image of $C_p$ for each prime $p$, from which it follows that $C'$ is an endomorphic image of $C$. Since by hypothesis $C$ is an epimorphic image of $A$, we then see that there exists an epimorphism $f: A \to C'$. By Proposition 2.4, there exists $c \neq 0$ in $R$ such that $cC = cf(A) = 0$; i.e., $C'$ has bounded order. Since $C'$ is a pure submodule of $C$, we may apply Theorem 7 of [6] (p. 18) to conclude that $C'$ is a direct summand of $C$. Since $C/C'$ is divisible, the proof is complete.

For the case in which $R$ is the ring of rational integers, the assertion of Theorem 3.1 follows from the work of Nunke [9].

In the remainder of this note, $R$ will be a Dedekind ring which is not a field. If $A$ and $C$ are $R$-modules, we shall write $\text{Ext}(A, C)$ for $\text{Ext}_R(A, C)$. The following two lemmas are well-known, but to our knowledge have not appeared explicitly in the literature.

**Lemma 3.2.** Let $a \neq 0$ be a nonunit in $R$, and let $A$ and $C$ be $R$-modules. Assume that $aC = 0$, and $a$ operates faithfully on $A$ (i.e., $ax = 0$ for $x \in A$ only if $x = 0$.) Then $\text{Ext}(A, C) = 0$.

---

The definition and properties of basic submodules used here, as well as the theorem of Szele applied in the following paragraph, are in [5] given only for the special case in which $R$ is the ring of rational integers. However, it is well-known that these results can be trivially extended to modules over an arbitrary principal ideal domain.
Proof. Since \( a \) operates faithfully on \( A \), we obtain the exact sequence—

\[
0 \longrightarrow A \xrightarrow{m_a} A \longrightarrow A/aA \longrightarrow 0
\]

where \( m_a \) is defined by \( m_a(x) = ax \). This gives rise to the exact cohomology sequence—

\[
\text{Ext}(A, C) \xrightarrow{m_a^*} \text{Ext}(A, C) \longrightarrow 0
\]

where \( m_a^*(u) = au \) for \( u \) in \( \text{Ext}(A, C) \). But, since \( aC = 0 \), we have that \( m_a^* = 0 \), and so it follows from exactness that \( \text{Ext}(A, C) = 0 \), completing the proof.

**Lemma 3.3.** Let \( a \neq 0 \) be a nonunit in \( R \), and \( A, C \) be \( R \)-modules. Assume that \( a \) operates faithfully on \( A \). Then the following statements are equivalent:

(a) \( a \) operates faithfully on \( \text{Ext}(A, C) \).

(b) The natural mapping \( \text{Hom}(A, C) \to \text{Hom}(A, C/aC) \) is an epimorphism.

Proof. Consider the exact sequence—

\[
0 \longrightarrow C_a \longrightarrow C \xrightarrow{m_a} C \longrightarrow C/aC \longrightarrow 0
\]

where \( C_a = \{x \in C/ax = 0\} \) and \( m_a \) is defined as in Lemma 3.2. This sequence may be broken up into the following short exact sequences:

\[
0 \longrightarrow C_a \longrightarrow C \xrightarrow{\mu} aC \longrightarrow 0
\]

\[
0 \longrightarrow aC \xrightarrow{\nu} C \longrightarrow C/aC \longrightarrow 0
\]

where \( \nu \) is the inclusion mapping and \( \mu \) differs from \( m_a \) only by the obvious contraction of the range. Since \( aC_a = 0 \) and \( a \) operates faithfully on \( A \), we obtain from Lemma 3.2 that \( \text{Ext}(A, C_a) = 0 \), and so the relevant portions of the resulting cohomology sequences are as follows:

\[
0 \longrightarrow \text{Ext}(A, C) \xrightarrow{\mu_*} \text{Ext}(A, aC) \longrightarrow 0
\]

\[
\text{Hom}(A, C) \longrightarrow \text{Hom}(A, C/aC) \longrightarrow \text{Ext}(A, aC) \xrightarrow{\nu_*} \text{Ext}(A, C).
\]

Since \( m_a = \nu \mu \), we have that \( m_a = \nu \mu \), where \( m_a^* : \text{Ext}(A, C) \to \text{Ext}(A, C) \) is defined by \( m_a^*(u) = au \) for \( u \) in \( \text{Ext}(A, C) \). Hence (a) holds if and only if \( m_a^* \) is a monomorphism. But this is true if and only if \( \nu^* \) is a monomorphism, since \( \mu^* \) is an isomorphism. But it is clear from the second exact sequence above that \( \nu^* \) is a monomorphism if and only if (b) holds. The proof is hence complete.
In the remainder of this section we shall set $\Pi = \prod_{i=1}^{n} R^{(i)}$, where $R^{(i)} \cong R$.

**Theorem 3.4.** Let $R$ be a Dedekind ring, and $a \neq 0$ be a nonunit in $R$. Set $C = \sum_{n=1}^{\infty} \oplus R/a^nR$. Let $A$ be a torsion-free $R$-module satisfying the following conditions:

(a) Every submodule of $A$ of finite rank is projective.

(b) $a$ operates faithfully on $\text{Ext}(A, C)$.

Then, if $f \in \text{Hom}(II, A)$, $f(\Pi)$ has finite rank.

**Proof.** Assume that the statement is false for some $f \in \text{Hom}(II, A)$. Then $f(\Pi)$ contains a submodule $F_0$ of countably infinite rank. Let $F = \{x \in A | a^n x \in F_0 \text{ for some } n\}$. Then $F$ likewise has countably infinite rank. We may then apply condition (a) and a result of Nunke ([8], Lemma 8.3, p. 239) to obtain that $F$ is projective, and then a result of Kaplansky ([7], Theorem 2, p. 330) to conclude that $F$ is free. Let $x_1, x_2, \cdots$ be a basis of $F$. Then there exist nonnegative integers $\nu_1, \nu_2, \cdots$ such that $y_n = a^{\nu_n} x_n$ is in $F_0$.

Let $z_n$ generate the direct summand of $C$ isomorphic to $R/a^nR$, and let $\bar{z}_n$ be the image of $z_n$ under the natural mapping of $C$ onto $\bar{C} = C/aC$. Define an $R$-homomorphism $\theta_1 : F \to \bar{C}$ by $\theta_1(x_n) = \bar{z}_{n+\nu_n}$. Observe that $\theta_1(aF') = 0$, and so $\theta_1$ induces a homomorphism $\theta_2 : F/aF \to \bar{C}$. Now, it follows easily from the construction of $F$ that the sequence $0 \to F/aF \to A/aF \to A/F \to 0$ is exact, and $a$ operates faithfully on $A/F$. We may then apply Lemma 3.2 to conclude that this sequence splits. It is then clear that $\theta_2$ can be extended to a homomorphism $\theta : A \to \bar{C}$. We emphasize the fact that $\theta(x_n) = \bar{z}_{n+\nu_n}$.

Since $a$ operates faithfully on $\text{Ext}(A, C)$, we may now apply Lemma 3.3 to obtain $\varphi \in \text{Hom}(A, C)$ such that the diagram:

$$
\begin{array}{ccc}
A & \xrightarrow{\varphi} & C \\
\theta \downarrow & \downarrow & \\
C & & \\
\end{array}
$$

is commutative. Observe that, since $\theta(x_n) = \bar{z}_{n+\nu_n}$, $\varphi(x_n) \equiv z_{n+\nu_n} \pmod{aC}$. That is, the coefficient of $z_{n+\nu_n}$ in the expansion of $\varphi(x_n)$ is $1 + at_n$ for some $t_n \in R$. Since $y_n = a^{\nu_n} x_n$, the coefficient of $z_{n+\nu_n}$ in the expansion of $\varphi(y_n)$ is $a^{\nu_n} + a^{\nu_n+1} t_n$.

Set $g = \varphi f$; then $g \in \text{Hom}(II, C)$, and so we may apply Proposition 2.4 to conclude that $cg(II) = 0$ for some $c \neq 0$ in $R$. Since each $y_n$ is in $f(II)$, and $z_n$ generates a direct summand of $C$ isomorphic to $R/a^nR$, it then follows from the preceding paragraph that $c(a^{\nu_n} + a^{\nu_n+1} t_n)$ is in $a^{n+\nu_n}R$ for all $n$, in which case $c(1 + at_n)$ is in $a^nR$ for all $n$. Let $P$
be any prime ideal in $R$ containing $a$; then $1 + at^*_n$ is a unit modulo $P^*$ for all $n > 0$, and so $c \in P^n$ for all $n$. Therefore $c = 0$, a contradiction. This completes the proof of the theorem.

**Corollary 3.5.** Let $R$ be a Dedekind ring (not a field,) and let $A$ be an $R$-module with the property that $\text{Ext}(A, C) = 0$ for any torsion module $C$. Then, if $f \in \text{Hom}(\Pi, A)$, $f(\Pi)$ is a projective module of finite rank.

**Proof.** We may apply a result of Nunke ([8], Theorem 8.4, p. 239) to obtain that $A$ is torsion-free and every submodule of $A$ of finite rank is projective. The corollary then follows immediately from Theorem 3.4.

The following special case of Theorem 3.4 was first proved by Rotman ([10], Theorem 3, p. 250) under an additional hypothesis which was later removed by Nunke ([9], p. 275.)

**Corollary 3.6.** Let $A$ be an abelian group such that $\text{Ext}(A, C) = 0$ for any torsion group $C$. Then $A$ is slender.

**Proof.** We need only show that, for any $f \in \text{Hom}(\Pi, A)$, $f(\Pi)$ is slender. By Corollary 3.5, $f(\Pi)$ is free of finite rank. But it is well-known that a free abelian group is slender (see [5], Theorems 47.3 and 47.4, pp. 171-172.) The proof is hence complete.

**References**

2. G. Baumslag and N. Blackburn, *Direct summands of unrestricted direct sums of Abelian groups*, Arkiv Der Mathematik, 10 (1959), 403-408.

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* For the definition of a slender Abelian group we refer the reader to [9].
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