

Pacific Journal of Mathematics

**HOMOGENEITY OF INFINITE PRODUCTS OF MANIFOLDS
WITH BOUNDARY**

MARION K. FORT, JR.

HOMOGENEITY OF INFINITE PRODUCTS OF MANIFOLDS WITH BOUNDARY

M. K. FORT, JR.

1. Introduction. In 1931, O. H. Keller [2] proved that the Hilbert cube Q is homogeneous. V. L. Klee, Jr., proved [3] in 1955 that Q is homogeneous with respect to finite sets, and in 1957 strengthened this result [4] by showing that Q is homogeneous with respect to countable closed sets. Our Theorem 1 extends this latter result to spaces which are the product of a countably infinite number of manifolds with boundary. Our method of proof exploits the notion of category for the space of self-homeomorphisms of the product space, and differs considerably from the methods of Keller and Klee, who made use of convexity properties of linear spaces.

In Theorem 2 we prove that if P is the product of a countably infinite number of manifolds with boundary and U and V are countable dense subsets of P , then there is a homeomorphism h of P onto itself such that $h[U] = V$. This theorem is analogous to a well known theorem about Euclidean spaces (see [1], p. 44). In a corollary to our Theorem 2, we show that if U is a countable subset of the Hilbert cube Q , then there is a contraction h_t , $0 \leq t \leq 1$, on Q such that if $0 < t < 1$, then h_t is a homeomorphism and $h_t[Q] \cap U = \phi$.

2. Notation and lemmas. For each positive integer n , we let M_n be a compact manifold with boundary, and we let B_n be the boundary of M_n . We let P be the cartesian product space $M_1 \times M_2 \times M_3 \times \dots$. The projection mapping of P into M_n is denoted by π_n . If $x \in P$, we denote $\pi_n(x)$ by x_n . An admissible metric d_n for M_n is chosen so that M_n has diameter less than 2^{-n} , and we then define an admissible metric d for P by letting

$$d(x, y) = \sum_{n=1}^{\infty} d_n(x_n, y_n).$$

If f and g are mappings on a compact metric space X into a metric space Y , we let $\rho(f, g)$ denote the least upper bound of the distances between $f(x)$ and $g(x)$ for x in X .

The set of all homeomorphisms of P onto P is denoted by H . Although the metric space (H, ρ) is not complete, it is topologically complete (i.e. homeomorphic to a complete metric space) and hence is

Received October 3, 1961. Written during a period in which the author was an Alfred P. Sloan Research Fellow. This work was also partially supported by National Science Foundation grant NSF-G12972.

a second category space.

The following two lemmas can be proved using standard techniques, and the proofs are merely outlined.

LEMMA 1. *If M is a manifold with boundary B , α is an arc lying in B , u and v are the end points of α , and W is an open subset of M which contains α , then there is a homeomorphism ψ of M onto M such that $\psi(u) = v$ and $\psi(x) = x$ for $x \in M - W$.*

Proof. Let S be the set of all points t of α for which there exists a homeomorphism ψ of M onto M such that $\psi(u) = t$ and $\psi(x) = x$ for $x \in M - W$. It is easy to see that S is both open and closed relative to α .

LEMMA 2. *If M is a manifold with boundary B , the dimension of M is at least 2, C is a countable and compact subset of $M - B$, and φ is a homeomorphism on C into $M - B$, then φ can be extended to a homeomorphism Φ on M onto M .*

Proof. For each positive integer n , we can obtain compact sets J_n and K_n such that:

(i) C is contained in the interior of J_n and $\varphi[C]$ is contained in the interior of K_n ;

(ii) each component of J_n and of K_n has diameter less than $1/n$ and is homeomorphic to a spherical ball of dimension equal to that of M ;

(iii) for each component D of J_n , $\varphi[D \cap C]$ is contained in a single component of K_n ; and

(iv) $J_n \supset J_{n+1}$ and $K_n \supset K_{n+1}$.

Using the sets J_n and K_n , it is possible to construct homeomorphisms Φ_n of M onto M such that:

(i) if D is a component of J_n and E is a component of K_n , then $\Phi_n[D] \subset E$ if and only if $\varphi[D \cap C] \subset E$; and

(ii) $\Phi_{n+1}(x) = \Phi_n(x)$ for all $x \in M - J_n$.

The sequence $\Phi_1, \Phi_2, \Phi_3, \dots$ converges to the desired homeomorphism.

LEMMA 3. *If $p \in P$, there is a residual subset R of H such that if $h \in R$, then $h(p)_n \in M_n - B_n$ for each n .*

Proof. Let $K_n = \{h \mid h \in H \text{ and } h(p)_n \in B_n\}$. It is obvious that each K_n is closed. We want to prove that K_n is nowhere dense. Thus, suppose $h \in K_n$ for some n and that $\varepsilon > 0$. We seek $g \in H - K_n$ such that $\rho(g, h) < \varepsilon$.

Choose an integer $m \neq n$ such that M_m has diameter less than ε . We define $M = M_n \times M_m$. M is also a manifold with boundary, and the boundary B of M is the set $(M_n \times B_m) \cup (B_n \times M_m)$. Since $h \in K_n$, the point $(h(p)_n, h(p)_m)$ is a member of $B_n \times M_m$. Let q be a point of B_m

such that $q \neq h(p)_m$. There is an arc β in $B_n \times M_m$ which joins $(h(p)_n, h(p)_m)$ to $(h(p)_n, q)$ and has diameter less than ε (since M_m has diameter less than ε). We may now choose a point $r \in M_n - B_n$ and an arc γ joining (r, q) to $(h(p)_n, q)$ such that $\beta \cup \gamma$ is an arc and has diameter less than ε . We let $\alpha = \beta \cup \gamma$.

We now use Lemma 1 to obtain a homeomorphism ψ of M onto M such that ψ maps the point $(h(p)_n, h(p)_m)$ onto (r, q) and the distance from x to $\psi(x)$ is less than ε for all $x \in M$.

Now, we define $g \in H$ by letting $g(y)_k = h(y)_k$ if $n \neq k \neq m$, and letting

$$(g(y)_n, g(y)_m) = \psi((h(y)_n, h(y)_m)) .$$

Since $g(p)_n = r$ and $r \notin B_n$, $g \in H - K_n$. It is easy to see that $\rho(g, h) < \varepsilon$, and hence we have proved that K_n is nowhere dense. We define $R = H - \bigcup_{n=1}^{\infty} K_n$. R is a residual set and if $h \in R$, then $h(p)_n \notin B_n$ for all n .

LEMMA 4. *If p and q are points of P , then there is a residual subset R of H such that if $h \in R$, then $h(p)_n \neq h(q)_n$ for all n .*

Proof. We define $J_n = \{h \mid h \in H \text{ and } h(p)_n = h(q)_n\}$. Each J_n is closed. We want to prove that J_n is nowhere dense. Suppose $h \in J_n$ and $\varepsilon > 0$. We seek $g \in H - J_n$ such that $\rho(g, h) < \varepsilon$.

It follows from Lemma 3, and the fact that residual subsets of H are dense in H , that there exists $f \in H$ such that $\rho(f, h) < \varepsilon/2$ and for all k , $f(p)_k \notin B_k$ and $f(q)_k \notin B_k$. If $f(p)_n \neq f(q)_n$ we can let $g = f$. Otherwise, we choose $m \neq n$ so that $f(p)_m \neq f(q)_m$ and define $M = M_n \times M_m$. Since $(f(p)_n, f(p)_m)$ and $(f(q)_n, f(q)_m)$ are not equal and neither is on the boundary of M , there is a homeomorphism φ of M onto M such that the distance from x to $\varphi(x)$ is less than $\varepsilon/2$ for all $x \in M$ and such that the points $\varphi((f(p)_n, f(p)_m))$ and $\varphi((f(q)_n, f(q)_m))$ have different first coordinates. We now define $g \in H$ by letting $g(y)_k = f(y)_k$ if $n \neq k \neq m$, and $(g(y)_n, g(y)_m) = \varphi((f(y)_n, f(y)_m))$. It is easy to see that $\rho(g, f) < \varepsilon/2$ and hence $\rho(g, h) < \varepsilon$. Moreover, $g(p)_n \neq g(q)_n$ and hence $g \in H - J_n$.

We obtain the desired residual set R by letting $R = H - \bigcup_{n=1}^{\infty} J_n$.

THEOREM 1. *If A is a closed and countable subset of P and f is a homeomorphism on A into P , then f can be extended to a homeomorphism F on P onto P .*

Proof. There is no loss in generality in assuming that each M_n has dimension at least 2, for otherwise we could define $S_n = M_{2n-1} \times M_{2n}$ and represent P as $S_1 \times S_2 \times S_3 \times \dots$.

It follows from Lemma 3 and Lemma 4 that there is a homeomorphism

$h \in H$ such that for each n , the projection mapping π_n maps both $h[A]$ and $hf[A]$ in a one-to-one manner into $M_n - B_n$. The mapping $\varphi_n = \pi_n h f h^{-1} \pi_n^{-1}$ is one-to-one on $\pi_n h[A]$ onto $\pi_n h f[A]$ and can be extended by Lemma 2 to a homeomorphism Φ_n on M_n onto M_n . We obtain $\Phi \in H$ by letting $\Phi(x)_n = \Phi_n(x_n)$. The desired extension F of f is obtained by defining $F = h^{-1} \Phi h$.

Let h be a homeomorphism on a compact space X into a compact space Y , and let n be a positive integer. We define

$$\eta(h, n) = 2^{-n} \cdot \inf \{d(h(x), h(y)) \mid x, y \in X \text{ and } d(x, y) \geq 1/n\}.$$

LEMMA 5. *If h_1, h_2, h_3, \dots is a sequence of homeomorphisms on X onto Y such that $\rho(h_n, h_{n+1}) < \eta(h_n, n)$, then the sequence converges uniformly to a homeomorphism h on X into Y .*

Proof. It is clear that the sequence converges uniformly to a continuous function h on X into Y . We must prove that h is one-to-one.

Suppose u and v are distinct points of X . We choose $n > 1$ so that $d(u, v) > 1/n$. Then, for $k \geq n$,

$$\begin{aligned} d(h_{k+1}(u), h_{k+1}(v)) &\geq d(h_k(u), h_k(v)) - d(h_k(u), h_{k+1}(u)) - d(h_k(v), h_{k+1}(v)) \\ &\geq d(h_k(u), h_k(v)) - 2\eta(h_k, k) \\ &\geq d(h_k(u), h_k(v)) - 2 \cdot 2^{-k} d(h_k(u), h_k(v)) \\ &\geq d(h_k(u), h_k(v)) \cdot (1 - 2^{-k+1}). \end{aligned}$$

Thus,

$$\begin{aligned} d(h(u), h(v)) &= \lim_{k \rightarrow \infty} d(h_k(u), h_k(v)) \\ &\geq d(h_n(u), h_n(v)) \cdot \prod_{j=n}^{\infty} (1 - 2^{-j+1}) \\ &\geq d(h_n(u), h_n(v))/4, \quad (\text{since } n > 1). \end{aligned}$$

This proves that h is one-to-one and hence a homeomorphism.

THEOREM 2. *If U and V are countable dense subsets of P , then there is a homeomorphism h of P onto P such that $h[U] = V$.*

Proof. As we have remarked in the proof of Theorem 1, there is no loss in generality in assuming that each M_n has dimension at least 2. In view of Lemma 3 and Lemma 4, we may also assume that U and V are so situated in P that each π_n maps both U and V in a one-to-one manner into $M_n - B_n$.

We are going to arrange the points of U and V into sequences u_1, u_2, u_3, \dots and v_1, v_2, v_3, \dots and choose homeomorphisms h_{ij} for all

positive integers i and j . This is done by a fairly complicated inductive process, the first four steps of which are given below. We let $U_1 = U$, $V_1 = V$, and as soon as u_1, \dots, u_n and v_1, \dots, v_n are defined, we let $U_{n+1} = U_n - \{u_1, \dots, u_n\}$, $V_{n+1} = V_n - \{v_1, \dots, v_n\}$. We assume that U and V are well ordered so as to have the order type of the positive integers. We let H_n be the set of homeomorphisms of M_n onto itself.

Step 1. u_1 is chosen to be the first point of U and v_1 is chosen to be the first point of V . $h_{11} \in H_1$ is chosen so that $h_{11}\pi_1(u_1) = \pi_1(v_1)$. $h_{1j} \in H_j$ is the identity for $j > 1$.

Step 2. v_2 is the first point of V_2 . $u_2 \in U_2$ is chosen near enough to v_2 for us to obtain $h_{21} \in H_1$ so that: $\rho(h_{21}, h_{11}) < \eta(h_{11}, 1)$ and $h_{21}\pi_1(u_2) = \pi_1(v_2)$ for $j = 1, 2$. $h_{22} \in H_2$ is chosen so that $h_{22}\pi_2(u_j) = \pi_2(v_j)$ for $j = 1, 2$. $h_{2j} \in H_j$ is the identity for $j > 2$.

Step 3. u_3 is the first point of U_3 . $v_3 \in V_3$ is chosen near enough to u_3 for us to obtain $h_{3i} \in H_i$ so that: $\rho(h_{3i}, h_{2i}) < \eta(h_{2i}, 2)$ and $h_{3i}\pi_i(u_j) = \pi_i(v_j)$ for $i = 1, 2$ and $j = 1, 2, 3$. $h_{33} \in H_3$ is chosen so that $h_{33}\pi_3(u_j) = \pi_3(v_j)$ for $j = 1, 2, 3$. $h_{3j} \in H_j$ is the identity for $j > 3$.

Step 4. v_4 is the first point of V_4 . $u_4 \in U_4$ is chosen near enough to v_4 for us to obtain $h_{4i} \in H_i$ so that: $\rho(h_{4i}, h_{3i}) < \eta(h_{3i}, 3)$ and $h_{4i}\pi_i(u_j) = \pi_i(v_j)$ for $i = 1, 2, 3$ and $j = 1, \dots, 4$. $h_{44} \in H_4$ is chosen so that $h_{44}\pi_4(u_j) = \pi_4(v_j)$ for $j = 1, \dots, 4$. $h_{4j} \in H_j$ is the identity for $j > 4$.

We continue this process. By Lemma 5, the homeomorphisms $h_{j1}, h_{j2}, h_{j3}, \dots$ converge uniformly to a homeomorphism $g_j \in H_j$. It is easy to see that $g_j\pi_j(u_i) = \pi_j(v_i)$ for all i and j . There is determined uniquely a homeomorphism $h \in H$ for which $\pi_j h = g_j \pi_j$ for all j . Since $h(u_i) = v_i$ for all i , and $U = \{u_1, u_2, \dots\}$, $V = \{v_1, v_2, \dots\}$, h is the desired homeomorphism.

COROLLARY. *If C is a countable subset of the Hilbert cube Q , then there is a contraction h_t , $0 \leq t \leq 1$, defined on Q such that:*

- (i) h_1 is the identity,
- (ii) h_0 is a constant mapping, and
- (iii) if $0 < t < 1$, h_t is a homeomorphism of Q into Q and $h_t[Q] \cap C = \phi$.

Proof. We let M_n be the closed interval $[-5^{-n}, 5^{-n}]$. The resulting space P may then be thought of as the Hilbert cube Q . (This representation is used since M_n was assumed to have diameter less than 2^{-n} .) We let D be the set of all points x in P such that $\pi_i(x)$ is rational for

all i , and $\pi_i(x) = 5^{-i}$ for all but a finite number of values of i :

Both $C \cup D$ and D are countable and dense in P , so by Theorem 2 there is a homeomorphism G of P onto P such that $G[C \cup D] = D$. We define $g_t(x) = tx$ for $0 \leq t \leq 1$ and $x \in P$. Finally, we let $h_t = G^{-1}g_tG$. It is easy to see that the desired contraction is h_t , $0 \leq t \leq 1$.

REFERENCES

1. W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton 1941.
2. Ott-Heinrich Keller, *Die Homöomorphie der kompakten konvexen Mengen im Hilbertschen Raum*, Math. Ann., **105** (1931), 748-758.
3. V. L. Klee, Jr., *Some topological properties of convex sets*, Trans. Amer. Math. Soc., **78** (1955), 30-45.
4. ———, *Homogeneity of infinite-dimensional parallelotopes*, Annals of Math., **66** (1957), 454-460.

UNIVERSITY OF GEORGIA

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RALPH S. PHILLIPS

Stanford University
Stanford, California

M. G. ARSOVE

University of Washington
Seattle 5, Washington

A. L. WHITEMAN

University of Southern California
Los Angeles 7, California

LOWELL J. PAIGE

University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH

T. M. CHERRY

D. DERRY

M. OHTSUKA

H. L. ROYDEN

E. SPANIER

E. G. STRAUS

F. WOLF

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA

CALIFORNIA INSTITUTE OF TECHNOLOGY

UNIVERSITY OF CALIFORNIA

MONTANA STATE UNIVERSITY

UNIVERSITY OF NEVADA

NEW MEXICO STATE UNIVERSITY

OREGON STATE UNIVERSITY

UNIVERSITY OF OREGON

OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY

UNIVERSITY OF TOKYO

UNIVERSITY OF UTAH

WASHINGTON STATE UNIVERSITY

UNIVERSITY OF WASHINGTON

* * *

AMERICAN MATHEMATICAL SOCIETY

CALIFORNIA RESEARCH CORPORATION

SPACE TECHNOLOGY LABORATORIES

NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Alfred Aeppli, <i>Some exact sequences in cohomology theory for Kähler manifolds</i>	791
Paul Richard Beesack, <i>On the Green's function of an N-point boundary value problem</i>	801
James Robert Boen, <i>On p-automorphic p-groups</i>	813
James Robert Boen, Oscar S. Rothaus and John Griggs Thompson, <i>Further results on p-automorphic p-groups</i>	817
James Henry Bramble and Lawrence Edward Payne, <i>Bounds in the Neumann problem for second order uniformly elliptic operators</i>	823
Chen Chung Chang and H. Jerome (Howard) Keisler, <i>Applications of ultraproducts of pairs of cardinals to the theory of models</i>	835
Stephen Urban Chase, <i>On direct sums and products of modules</i>	847
Paul Civin, <i>Annihilators in the second conjugate algebra of a group algebra</i>	855
J. H. Curtiss, <i>Polynomial interpolation in points equidistributed on the unit circle</i>	863
Marion K. Fort, Jr., <i>Homogeneity of infinite products of manifolds with boundary</i>	879
James G. Glimm, <i>Families of induced representations</i>	885
Daniel E. Gorenstein, Reuben Sandler and William H. Mills, <i>On almost-commuting permutations</i>	913
Vincent C. Harris and M. V. Subba Rao, <i>Congruence properties of $\sigma_r(N)$</i>	925
Harry Hochstadt, <i>Fourier series with linearly dependent coefficients</i>	929
Kenneth Myron Hoffman and John Wermer, <i>A characterization of $C(X)$</i>	941
Robert Weldon Hunt, <i>The behavior of solutions of ordinary, self-adjoint differential equations of arbitrary even order</i>	945
Edward Takashi Kobayashi, <i>A remark on the Nijenhuis tensor</i>	963
David London, <i>On the zeros of the solutions of $w''(z) + p(z)w(z) = 0$</i>	979
Gerald R. Mac Lane and Frank Beall Ryan, <i>On the radial limits of Blaschke products</i>	993
T. M. MacRobert, <i>Evaluation of an E-function when three of its upper parameters differ by integral values</i>	999
Robert W. McKelvey, <i>The spectra of minimal self-adjoint extensions of a symmetric operator</i>	1003
Adegoke Olubummo, <i>Operators of finite rank in a reflexive Banach space</i>	1023
David Alexander Pope, <i>On the approximation of function spaces in the calculus of variations</i>	1029
Bernard W. Roos and Ward C. Sangren, <i>Three spectral theorems for a pair of singular first-order differential equations</i>	1047
Arthur Argyle Sagle, <i>Simple Malcev algebras over fields of characteristic zero</i>	1057
Leo Sario, <i>Meromorphic functions and conformal metrics on Riemann surfaces</i>	1079
Richard Gordon Swan, <i>Factorization of polynomials over finite fields</i>	1099
S. C. Tang, <i>Some theorems on the ratio of empirical distribution to the theoretical distribution</i>	1107
Robert Charles Thompson, <i>Normal matrices and the normal basis in abelian number fields</i>	1115
Howard Gregory Tucker, <i>Absolute continuity of infinitely divisible distributions</i>	1125
Elliot Carl Weinberg, <i>Completely distributed lattice-ordered groups</i>	1131
James Howard Wells, <i>A note on the primes in a Banach algebra of measures</i>	1139
Horace C. Wiser, <i>Decomposition and homogeneity of continua on a 2-manifold</i>	1145