CONGRUENCE PROPERTIES OF $\sigma_r(N)$

Vincent C. Harris and M. V. Subba Rao
CONGRUENCE PROPERTIES OF $\sigma_r(N)$

V. C. HARRIS AND M. V. SUBBA RAO

1. Introduction. Let $\sigma_r(N)$ denote as usual the sum of the $r$th powers of the divisors of $N$. Let $d$ be a divisor of $N$ with $1 \leq d \leq \sqrt{N}$ and $d'$ its conjugate, so that $dd' = N$. By a component of $\sigma_r(N)$ we mean the quantity $d^r + d'^r$ or $d^r$ according as $1 \leq d < \sqrt{N}$ or $d = \sqrt{N}$. Components corresponding to distinct divisors $d \leq \sqrt{N}$ are distinct and $\sigma_r(N)$ is their sum.

If every component of $\sigma_r(N)$ is congruent to the integer $a$, modulo $K$, we say that $\sigma_r(N)$ is componently congruent to $a$ (mod $K$) and indicate this by writing

$$\sigma_r(N) \equiv a \pmod{K}.$$ 

This does not necessarily imply that also $\sigma_r(N) \equiv a \pmod{K}$. For example $\sigma_4(8) \equiv 2 \pmod{3}$ but $\sigma_4(8) \equiv 1 \pmod{3}$. Similarly ordinary congruence does not imply component congruence, as the same example shows.

2. Theorem 1. If $r$, $K$, $L$ are fixed positive integers with $K \geq 3$ and $(L, K) = 1$, and if $a$ is a nonnegative integer, then a necessary and sufficient condition that

$$\sigma_r(nK + L) \equiv a \pmod{K} \quad \text{for all integral values of } n \geq 0$$

is that

(1) $L$ is a quadratic nonresidue of $K$
(2) $1 + L^r \equiv a \pmod{K}$
(3) $(w^r - 1)(w^r + 1 - a) \equiv 0 \pmod{K}$ for all $w$ such that $(w, K) = 1$ all hold.

We first show necessity. Assume that $\sigma_r(nK + L) \equiv a \pmod{K}$ and $L$ is a quadratic residue of $K$. Then there exists $q$ such that $q^2 \equiv L \pmod{K}$ and consequently $n_i$ such that $n_iK + L = q^2$. Consider $q^2$ and $n_2K + L = (n_1K + n_1 + L)K + L = (K + 1)q^2$, both occurring in the sequence $nK + L$. Since $\sigma_r(q^2) \equiv a \pmod{K}$ we have with $d = q$ that $q^r \equiv a \pmod{K}$ and since $\sigma_r[(K + 1)q^2] \equiv a \pmod{K}$ we have with $d = q$ and $d' = (K + 1)q$ that $q^r + (K + 1)q^r \equiv a \pmod{K}$. Thus $q^r + (K + 1)q^r \equiv q^r$, or, $2 \equiv 1 \pmod{K}$.

This is a contradiction and (2) is necessary. Assume next (1) holds. Then in particular for $n = 0$ we have $\sigma_r(L) \equiv a \pmod{K}$. By condition (2) just proved $L \neq 1$ and the component with $d = 1$ and $d' = L$ gives $1 + L^r \equiv a \pmod{K}$ which is (3).

Received May 11, 1961, and in revised form November 1, 1961.
Next to show (4). Given any \( w \) such that \( (w, K) = 1 \), there exists an \( x \equiv w \pmod{K} \) such that \( wx = L \pmod{K} \). Let this \( x \) be denoted by \( w_1 \).

Then

\[ w_1 = L \pmod{K} \]

and by our assumption \( \sigma_r(nK + L) \equiv a \pmod{K} \) applied to \( w_1 \), it follows that

\[ 1 + w^r_1 = a \pmod{K} \quad \text{and} \quad w^r + w^r_1 = a \pmod{K} \, . \]

Eliminating \( w^r_1 \) gives

\[ 1 + w^r(a - w^r) = a \pmod{K} \]

Rewriting this gives (4) and shows (4) is necessary.

To show sufficiency, we need to show for any divisor \( d \) of \( N = nK + L \) with \( 1 \leq d \leq \sqrt{N} \) and conjugate divisor \( d' \) that \( d^r + d'^r \equiv a \pmod{K} \) or \( d^r \equiv a \pmod{K} \) according as \( 1 \leq d < \sqrt{N} \) or \( d = \sqrt{N} \) provided (2), (3) and (4) hold. But (2) insures that \( N \) cannot be a square, so the second alternative cannot occur. Now

\[ d^r(d^r + d'^r) = d^r + (dd')^r \]

\[ = (1 + ad^r - a) + L^r \]

by (4) and the fact that \( dd' \equiv L \pmod{K} \). Then using (3),

\[ d^r(d^r + d'^r) \equiv (1 + ad^r - a) + a - 1 \equiv ad^r \pmod{K} \, . \]

Since \( (d, K) = 1 \) it follows that

\[ d^r + d'^r \equiv a \pmod{K} \]

for each \( d \) as specified. But this shows (1) holds and completes the proof.

3. Examples and some special cases. It is not difficult to show that when \( K = p \) is an odd prime, all component congruences are obtained with \( r = (p - 1)/2 \) and \( a = 0 \) or \( r = (p - 1) \) and \( a = 2 \). Thus for example:

\[ \sigma_6(13n + L) \equiv 0 \pmod{13} \quad \text{for} \quad L = 2, 5, 6, 7, 8, 11 \]

\[ \sigma_{10}(13n + L) \equiv 2 \pmod{13} \quad \text{for} \quad L = 2, 5, 6, 7, 8, 11 \, . \]

When \( K \) is composite we have \( \sigma_{\varphi(K)}(nK + L) \equiv 2 \pmod{K} \) for any non-quadratic residue \( L \) of \( K \).

In the special case \( r = 1 \) we show

**THEOREM 2.** For all integral \( n \geq 0 \), \( \sigma_1(nK + L) \equiv a \pmod{K} \) holds for suitable \( L \) and \( a \) if and only if \( K \) is one of \( 3, 4, 6, 8, 12 \) and \( 24 \).

The equation in condition (4) becomes

\[ \sigma_1(nK + L) \equiv a \pmod{K} \]

for suitable \( L \) and \( a \) if and only if \( K \) is one of \( 3, 4, 6, 8, 12 \) and \( 24 \).
The congruence (5) is equivalent to
\[ 4x^2 - 4ax + a^2 \equiv (2x - a)^2 \equiv (a - 2)^2 \pmod{4K}. \]

With \( y = 2x - a \) we have
\[ y^2 \equiv (a - 2)^2 \pmod{4K} \]
subject to \( y \equiv -a \pmod{2} \). But this last condition is no restriction so that the number of solutions of (5) is the same as that of (6). Let \( S(4K) \) be the number of solutions of (6) and let \( 4K = p_1^{\epsilon_1}p_2^{\epsilon_2} \cdots p_t^{\epsilon_t} \) where \( p_1 = 2, \ p_2 = 3, \cdots \) are distinct primes. Then
\[ S(4K) = S(p_1^{\epsilon_1})S(p_2^{\epsilon_2}) \cdots S(p_t^{\epsilon_t}) \text{ and } S(p_i^{\epsilon_i}) \leq 2 \text{ for } \epsilon_i = 0; \]
\[ S(p_i^{\epsilon_i}) \leq 4 \text{ for } \epsilon_i > 0; S(p_i^{\epsilon_i}) \leq 2 \text{ for } p_i > 2. \]

Since (5) is to hold for all \( w \) such that \( (w, K) = 1 \), we must have \( \phi(p_i^{\epsilon_i}) \leq S(4p_i^{\epsilon_i}) \) or
\[ p_i^{\epsilon_i - 1}(p_i - 1) = \phi(p_i^{\epsilon_i}) \leq \begin{cases} 2 & p_i = 2, \epsilon_i = 0 \\ 4 & p_i = 2, \epsilon_i > 0 \\ 2 & p_i > 2 \end{cases}. \]

But the only values of \( p_i^{\epsilon_i} \) satisfying these are 1, 2, 4, 8 and 1, 3. Since \( K \geq 3 \) these give \( K = 3, 4, 6, 8, 12, 24 \). The converse can be proved by enumeration. The results are listed:

\[
\begin{array}{cccccccccc}
K & 3 & 4 & 6 & 8 & 12 & 12 & 24 & 24 \\
L & 2 & 3 & 5 & 3 & 7 & 5 & 11 & 11 & 23 \\
a & 0 & 0 & 0 & 4 & 0 & 6 & 0 & 12 & 0 \\
\end{array}
\]

4. Relation between component congruence and congruence. We have

**Theorem 3.** If \( \sigma_r(nK + L) \equiv a (\pmod{K}) \) for all integral \( n \geq 0 \), then \( \sigma_r(nK + L) \equiv a (\pmod{K}) \) for all integral \( n \geq 0 \) if and only if \( a \equiv 0 (\pmod{K}) \).

If \( a \equiv 0 (\pmod{K}) \) then each component is congruent to zero and the sum of the components—that is, \( \sigma_r(nK + L) \)—is congruent to zero. Conversely, if \( \sigma_r(nK + L) \equiv a (\pmod{K}) \) as well as \( \sigma_r(nK + L) \equiv a (\pmod{K}) \), then \( \tau(n) \) standing for the number of divisors of \( n \), we have
\[ \lceil \tau(nK + L)/2 \rceil a \equiv a (\pmod{K}) \]
since there are \( \tau(nK + L)/2 \) components each congruent to \( a (\pmod{K}) \). By Dirichlet’s theorem, \( w \) and \( w_1 \) in the proof of Theorem 1 may be
taken as primes \( p \) and \( p_k \). Then for \( nK + L = pp_1 \), \( \tau(nK + L) = 4 \). We must have \( 2a \equiv a \) or \( a \equiv 0 \) (mod \( K \)).

In the particular case \( a = 0 \), conditions (2), (3) and (4) reduce to conditions which Gupta [1] and Ramanathan [2] found to be necessary and sufficient in order that \( \sigma_r(nK + L) \equiv 0 \) (mod \( K \)) for \( r, n, K \) and \( L \) as above. Thus we have the remarkable result:

**Theorem 4.** Let \( r, K \) and \( L \) be positive integers with \( (K, L) = 1 \) and \( K \geq 3 \). Then \( \sigma_r(nK + L) \equiv 0 \) (mod \( K \)) for all \( n \geq 0 \) if and only if \( \sigma_r(nK + L) \equiv 0 \) (mod \( K \)) for all \( n \geq 0 \).

**References**


**San Diego State College**

**Sri Venkateswara University and University of Missouri**
Mathematical papers intended for publication in the Pacific Journal of Mathematics should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunkei Insatsu-sha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.
Alfred Aeppli, Some exact sequences in cohomology theory for Kähler manifolds ............................................................... 791
Paul Richard Beesack, On the Green's function of an N-point boundary value problem ....................................................... 801
James Robert Boen, On p-automorphic p-groups ......................................................... 813
James Robert Boen, Oscar S. Rothaus and John Griggs Thompson, Further results on p-automorphic p-groups ................................................................. 817
James Henry Bramble and Lawrence Edward Payne, Bounds in the Neumann problem for second order uniformly elliptic operators ................. 823
Chen Chung Chang and H. Jerome (Howard) Keisler, Applications of ultraproducts of pairs of cardinals to the theory of models .................. 835
Stephen Urban Chase, On direct sums and products of modules ......................... 847
Paul Civin, Annihilators in the second conjugate algebra of a group algebra ......... 855
J. H. Curtiss, Polynomial interpolation in points equidistributed on the unit circle ................................................................. 863
Marion K. Fort, Jr., Homogeneity of infinite products of manifolds with boundary ................................................................. 879
James G. Glimm, Families of induced representations ......................................................... 885
Daniel E. Gorenstein, Reuben Sandler and William H. Mills, On almost-commuting permutations ................................................................. 913
Vincent C. Harris and M. V. Subba Rao, Congruence properties of σr(N) ............. 925
Harry Hochstadt, Fourier series with linearly dependent coefficients ................. 929
Kenneth Myron Hoffman and John Wermer, A characterization of C(X) ............. 941
Robert Weldon Hunt, The behavior of solutions of ordinary, self-adjoint differential equations of arbitrary even order ...................... 945
Edward Takashi Kobayashi, A remark on the Nijenhuis tensor ................................. 963
David London, On the zeros of the solutions of w''(z) + p(z)w(z) = 0 ................. 979
Gerald R. Mac Lane and Frank Beall Ryan, On the radial limits of Blaschke products ................................................................. 993
T. M. MacRobert, Evaluation of an E-function when three of its upper parameters differ by integral values ........................................... 999
Robert W. McKelvey, The spectra of minimal self-adjoint extensions of a symmetric operator ................................................................. 1003
Adegoke Obulumbo, Operators of finite rank in a reflexive Banach space ............. 1023
David Alexander Pope, On the approximation of function spaces in the calculus of variations ................................................................. 1029
Bernard W. Roos and Ward C. Sangren, Three spectral theorems for a pair of singular first-order differential equations .............................. 1047
Arthur Argyle Sagle, Simple Malcev algebras over fields of characteristic zero .... 1057
Leo Sario, Meromorphic functions and conformal metrics on Riemann surfaces ...... 1079
Richard Gordon Swan, Factorization of polynomials over finite fields ................. 1099
S. C. Tang, Some theorems on the ratio of empirical distribution to the theoretical distribution ................................................................. 1107
Robert Charles Thompson, Normal matrices and the normal basis in abelian number fields ................................................................. 1115
Howard Gregory Tucker, Absolute continuity of infinitely divisible distributions .. 1125
Elliot Carl Weinberg, Completely distributed lattice-ordered groups ................. 1131
James Howard Wells, A note on the primes in a Banach algebra of measures ...... 1139
Horace C. Wiser, Decomposition and homogeneity of continua on a 2-manifold .... 1145