

Pacific Journal of Mathematics

A CHARACTERIZATION OF $C(X)$

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It is a classical fact that there exist harmonic functions u in the unit disk with conjugate harmonic function v such that u has continuous boundary values on the unit circumference, while v does not. Let us restate this fact as follows:

Denote by A_0 the algebra of functions analytic in $|z| < 1$ with continuous boundary values on $|z| = 1$ and write $\mathbf{Re} A_0$ for the space of all real parts of functions in A_0 . Then we may say: there exists a harmonic function u in $|z| < 1$ with continuous boundary values such that u does not lie in $\mathbf{Re} A_0$. On the other hand, u is certainly a uniform limit of functions in $\mathbf{Re} A_0$ on $|z| = 1$, for all finite real trigonometric polynomials on $|z| = 1$ are in $\mathbf{Re} A_0$. Thus we see: $\mathbf{Re} A_0$ is not closed under uniform convergence on $|z| = 1$. In this paper, we shall show that this phenomenon is a special case of a very general property of algebras of functions.

Let X be a compact Hausdorff space and $C(X)$ the algebra of all continuous complex-valued functions on X . Let A be a complex linear subalgebra of $C(X)$ such that

- (1) A is closed under uniform convergence;
- (2) A contains the constant functions;
- (3) A separates the points of X .

We write $\mathbf{Re} A$ for the set of functions $\mathbf{Re} f$ with f in A , that is, for the set of real parts of the functions in A . Clearly $\mathbf{Re} A$ is a (real) vector space of real-valued continuous functions on X . The purpose of this paper is to prove the following.

THEOREM. *If $\mathbf{Re} A$ is closed under uniform convergence, then $A = C(X)$.*

COROLLARY 1. *If $\mathbf{Re} A$ contains every real-valued continuous function on X , then $A = C(X)$.*

COROLLARY 2. *(Stone-Weierstrass) If A is closed under complex conjugation, then $A = C(X)$.*

Corollary 1 is an evident consequence of the theorem, and Corollary 2 follows upon observing that, if A is closed under complex conjugation,

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tion, $\mathbf{Re} A$ is simply the collection of real-valued functions which are contained in A . The proof of the theorem proceeds by reducing it to the case when A is *anti-symmetric*, i.e., every real-valued function in A is constant. Let us first settle this case.

LEMMA. *If $\mathbf{Re} A$ is closed and A is anti-symmetric, then the space X contains not more than one point.*

Proof. Suppose that X contains at least two points. Fix a point x_0 in X , and let $(\mathbf{Re} A)_0$ be the class of all u in $\mathbf{Re} A$ with $u(x_0) = 0$.

Suppose u is in $(\mathbf{Re} A)_0$. Let f be a function in A such that $u = \mathbf{Re} f$. Since the constants are in A , we may assume that $v = \mathbf{Im} f$ vanishes at x_0 . Since $v = \mathbf{Re}(-if)$, we then have $v \in (\mathbf{Re} A)_0$. Now given u , the function v in $(\mathbf{Re} A)_0$ such that $(u + iv)$ is in A is uniquely determined. For, if v' is another such function, $(v - v')$ is a real-valued function in A . Since A is anti-symmetric $v - v'$ is constant, and the condition $v(x_0) = v'(x_0) = 0$ tells us that $v = v'$. Put $v = Tu$.

Then T is a linear transformation of $(\mathbf{Re} A)_0$ into itself. Since we are assuming that $\mathbf{Re} A$ is closed under uniform convergence, $(\mathbf{Re} A)_0$ is a Banach space with the norm

$$\|u\| = \sup_x |u|.$$

We claim that T is a bounded operator on this Banach space. To prove this, it will suffice to show that the graph of T is closed. Suppose we have a sequence of elements u_n in $(\mathbf{Re} A)_0$ such that $u_n \rightarrow u$ and $Tu_n \rightarrow v$ uniformly. Then the functions $(u_n + iTu_n)$ lie in A and converge uniformly to $(u + iv)$. Thus $(u + iv)$ is in A , and since it is apparent that $v(x_0) = 0$, we have $v = Tu$. We conclude that T is bounded.

Since X contains at least two points, we may choose a nonconstant function $g = s + it$ in A such that $g(x_0) = 0$. Let R denote the rectangle in the complex plane defined by

$$-\|s\| \leq x \leq \|s\|, \quad -\|t\| \leq y \leq \|t\|.$$

Then $g(X)$ is a compact subset of R . Since g is nonconstant, we cannot have $s = 0$ or $t = 0$. In particular, there is a point $x_1 \neq x_0$ in X such that $|t(x_1)| = \|t\|$. Let $z_0 = g(x_1)$, so that z_0 is a boundary point of R .

Fix any integer $N > 0$. There exists a conformal map ϕ of the interior of R onto the interior of the rectangle R_N :

$$-\|s\| \leq x \leq \|s\|, \quad -N \leq y \leq N$$

such that $\phi(0) = 0$ and $\theta(z_0) = iN$. Since R and R_N are rectangles, the conformal map ϕ extends to a homeomorphism of the boundaries of R and R_N . In particular, ϕ is a uniform limit of polynomials on R . There-

fore, the function $h = \phi(g)$ is in the algebra A , and $h(x_0) = \phi(0) = 0$. If $h = u + iv$ we have

$$\begin{aligned} \|u\| &\leq \|s\| \\ \|v\| &= N. \end{aligned}$$

Since N was arbitrary and $v = Tu$, we have contradicted the fact that T is bounded. Thus X cannot contain more than one point.

Proof of theorem. A theorem of Bishop [1] states the following. If A is a subalgebra of $C(X)$ satisfying (1), (2), (3), there exists a partition P of the space X into nonempty disjoint closed sets, such that

(i) for each S in P the algebra A_s , obtained by restricting A to S , is anti-symmetric;

(ii) A_s is a uniformly closed subalgebra of $C(S)$;

(iii) the algebra A consists of all continuous functions f on the space X such that the restriction of f to S is in A_s for each S in the partition P .

Glicksberg [2] proved that we may also arrange that

(iv) if S is a fixed element of P and T is a closed subset of X disjoint from S , there exists a function g in A such that

$$\|g\| \leq 1, \quad g = 1 \text{ on } S, \quad |g| < 1 \text{ on } T.$$

Actually, (ii) is a consequence of (iv). What we shall show now is that (iv), together with the assumption that $\text{Re } A$ is closed, implies that $\text{Re } A_s$ is uniformly closed for each set S in the partition P . This will prove the theorem. For A_s is an anti-symmetric closed algebra on the space S , and the above lemma shows that S consists of one point. By (iii) we then have $A = C(X)$.

Fix S in P . We show that $\text{Re } A_s$ is closed. We first assert the following. If $f \in A$ and $\varepsilon > 0$, we can find $F \in A$ such that

$$(4) \quad \sup_x |\text{Re } F| \leq \sup_s |\text{Re } f| + 2\varepsilon, \quad \text{and} \quad \text{Re } F = \text{Re } f \text{ on } S.$$

Let Ω be the region in the w -plane ($w = u + iv$) defined by

$$|w| < 1, \quad -\varepsilon < v < \varepsilon.$$

Let τ be a conformal map of $|z| < 1$ on Ω with $\tau(0) = 0$ and $\tau(1) = 1$. Choose $\delta > 0$ such that τ maps $|z| < \delta$ into $|w| < \varepsilon$. Choose a neighborhood U of S in X with

$$|\text{Re } f| \leq \sup_s |\text{Re } f| + \varepsilon, \quad \text{on } U.$$

By (iv) above there is a $g \in A$ such that $\|g\| \leq 1$, $g = 1$ on S , $|g| < 1$ on $X - U$. Choose a positive integer n large enough that $|g^n| < \delta$ on

$X - U$. Put $h = \tau(g^n)$. Then $h \in A$, $h = 1$ on S , and $|\operatorname{Im} h| \leq \varepsilon$ on all of X . Also $|\operatorname{Re} h| < \varepsilon$ on $X - U$ and $|\operatorname{Re} h| \leq 1$ on all of X . Now define $F = fh$. Then $F \in A$ and

$$\operatorname{Re} F = \operatorname{Re} f \operatorname{Re} h - \operatorname{Im} f \operatorname{Im} h .$$

Therefore

$$(5) \quad \operatorname{Re} F = \operatorname{Re} f \text{ on } S$$

$$(6) \quad |\operatorname{Re} F| \leq (\sup_S |\operatorname{Re} f| + \varepsilon) + \varepsilon, \text{ on } U$$

$$(7) \quad |\operatorname{Re} F| \leq \varepsilon + \varepsilon, \text{ on } X - U .$$

In particular, F satisfies (4). (For (6) and (7) we have used $\|f\| \leq 1$.)

We finish the proof with a standard closure argument. Let R_S denote the subspace of $\operatorname{Re} A$ consisting of all functions in $\operatorname{Re} A$ which vanish on S . With norm given by maximum modulus over X , $\operatorname{Re} A$ is a Banach space, and R_S is a closed subspace. The quotient space $Q = \operatorname{Re} A / R_S$ is therefore complete in the norm

$$\|\operatorname{Re} f + R_S\| = \inf_F \|\operatorname{Re} F\| , \quad \operatorname{Re} F = \operatorname{Re} f \text{ on } S .$$

But by (4)

$$\sup |\operatorname{Re} f| = \inf \|\operatorname{Re} F\| , \quad \operatorname{Re} F = \operatorname{Re} f \text{ on } S .$$

We conclude that $\operatorname{Re} A_S$, which is clearly isomorphic to Q , is complete in the maximum norm on S . We are done.

The theorem of this paper was proved independently by H. Rossi and H. Bear.

BIBLIOGRAPHY

1. Errett Bishop, *A generalization of the Stone-Weierstrass theorem*, to appear.
2. Irving Glicksberg, *Measures orthogonal to algebras and sets of anti-symmetry*, to appear.

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