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The vanishing of the Nijenhuis tensor of the almost complex structure is known to give the integrability of the almost complex structure [3, 7]. In order to generalize this fact, we consider a vector 1-form h on a manifold M[4], whose Jordan canonical form at all points on M is equal to a fixed matrix μ. Following the idea of E. Cartan, we say that such a vector 1-form is 0-deformable [2]. The frames z at x such that $z^{-1}h_x z = \mu$ define a subbundle of the frame bundle over M, as x runs through M, and the subbundle is called a G-structure defined by h [1]. We find that for a certain type of 0-deformable h, the vanishing of the Nijenhuis tensor of h is sufficient for the G-structure to be integrable (Theorem, § 2). In § 5 we give an example of a 0-deformable derogatory nilpotent vector 1-form, whose Nijenhuis tensor vanishes, but whose G-structure is not integrable.

1. Vector forms and distributions. As usual, we begin by stating, that all the objects we encounter in this paper are assumed to be C^∞.

Let M be a manifold, T_x the tangent space at point x of M, T the tangent bundle over M, $T^{(p)}$ the vector bundle of tangential covariant p-vectors of M. A vector p-form is a cross-section of $T \otimes T^{(p)}$. The collection of all vector p-forms over M is denoted by Ψ_p. We notice that a vector 1-form is nothing but a law that assigns a linear transformation to each tangent space T_x at point x of M.

We list some definitions and lemmas of the theory of vector forms [4], which we use in the sequel.

If $P \in \Psi_p$, $Q \in \Psi_q$, then $P \barwedge Q \in \Psi_{p+q-1}$ is defined by

$$(1) \qquad (p \barwedge Q)(u_1, \cdots, u_{p+q-1}) = \frac{1}{(p-1)!\, q!} \sum_\alpha |\alpha| P(Q(u_{\alpha_1}, \ldots, u_{\alpha_p}), u_{\alpha_{p+1}}, \ldots, u_{\alpha_{p+q-1}})$$

where α runs through all the permutations of $(1, 2, \cdots, p+q-1)$, and $|\alpha|$ denotes the signature of the permutation α.

If h is a vector 1-form and P is a vector p-form, we write hP instead of $h \barwedge p$. In particular if $p = h$, we write $h \barwedge h$ as h^2. In general, $\underbrace{h \barwedge h \cdots \barwedge h}_{k \text{ times}}$ is written as h^k, and this agrees with the usual notation,

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when we consider h as a linear transformation of the tangent space at each point of the manifold M.

Let h and k be two vector 1-forms. The bracket $[h, k]$ of h and k is a vector 2-form defined by

$$(2)\qquad \begin{aligned}[h, k](u, v) = [hu, kv] + [ku, hv] - k[hu, v] - h[ku, v] \\ -k[u, hv] - h[u, kv] + kh[u, v] + hk[u, v]\,,\end{aligned}$$

where u and v are vector fields over M. If $h = k$, we obtain the tensor $[h, h]$, generally known as the Nijenhuis tensor:

$$(3)\qquad \frac{1}{2}[h, h](u, v) = [hu, hv] - h[hu, v] - h[u, hv] + h^2[u, v]\,.$$

If h, k and l are vector 1-forms, using (2), we can obtain

$$(4)\qquad [hl, k] + [h, kl] - [h, k] \barwedge l = h[l, k] + k[l, h]$$

(cf. (6.7) [4]).

LEMMA 1.1. *Let h be a vector 1-form, then*

$$(5)\qquad [h^k, h^l] = \frac{1}{2} \sum_{\substack{a+b+c+=k+l-2 \\ 0\leqq b\leqq k-1 \\ 0\leqq c\leqq l-1}} h^a\{([h, h] \barwedge h^b) \barwedge h^c - [h, h] \barwedge h^{b+c}\}\,.$$

Proof. By replacing h, k and l by h, h and h^k in (4), we obtain

$$(6)\qquad [h^k, h] = h[h^{k-1}, h] + \frac{1}{2}[h, h] \barwedge h^{k-1}\,,$$

which gives us

$$(7)\qquad [h^k, h] = \frac{1}{2}\sum_{i=1}^{k} h^{i-1}[h, h] \barwedge h^{k-i}\,.$$

Again, replacing h, k and l in (4) by h^k, h and h^{l-1}, we obtain

$$(8)\qquad [h^{k+l-1}, h] + [h^k, h^l] - [h^k, h] \barwedge h^{l-1} = h^k[h^{l-1}, h] + h[h^{l-1}, h^k]\,.$$

Using (7) and (8) yields

$$(9)\qquad \begin{aligned}[h^k, h^l] &= h[h^k, h^{l-1}] \\ &\quad + \frac{1}{2}\sum_{i=1}^{k} h^{i-1}\{([h, h] \barwedge h^{k-i}) \barwedge h^{l-1} - [h, h] \barwedge h^{k-i+l-1}\}\,,\end{aligned}$$

and repeating the reduction we obtain (5).

LEMMA 1.2. *Let h be a vector 1-form on M, whose rank is constant* *in a neighbourhood of each point x of M. If $[h, h] = 0$, the distribution $x \to h_x T_x$ is completely integrable.*

Proof. By Frobenius' theorem we have to show that the bracket of any two vector fields of the form hu, hv belongs to the distribution. This follows from $[h, h] = 0$ and (3):

$$[hu, hv] = h[hu, v] + h[u, hv] - h^2[u, v] .$$

We recall that a necessary and sufficient condition for a distribution to be completely integrable can be given as follows:

Let θ be an r-dimensional distribution $x \to \theta(x)$ on an m-dimensional manifold M. For each $x_0 \in M$, let U be a neighbourhood of x_0 and $L_1, \cdots, L_r$ be vector fields on U such that $(L_1)_x, \cdots, (L_r)_x$ span $\theta(x)$ for each $x \in U$. Then θ is completely integrable if and only if for each $x_0 \in M$, there exist $m - r$ independent functions $\psi^1, \cdots, \psi^{m-r}$ defined on a neighbourhood $V \subset U$ of x_0 such that

$$L_i \psi^j = 0, \text{ for } 1 \leqq i \leqq r,\ 1 \leqq j \leqq m - r \text{ on } V .$$

Using this it is easy to prove,

LEMMA 1.3. *If $\theta_1, \cdots, \theta_g$ are completely integrable distributions of dimensions $r_1, \cdots, r_g$ on M, such that*

$$\theta_1(x) + \theta_2(x) + \cdots + \theta_g(x) = T_x \ (\textit{direct sum})$$

for each $x \in M$, then for each point $x_0 \in M$, there exists a coordinate neighbourhood U of x_0 with coordinate functions $x^1, \cdots, x^m$ such that for each j

$$x^1 = \xi^1, \cdots, x^{r_1+\cdots+r_{j-1}} = \xi^{r_1+\cdots+r_{j-1}}, x^{r_1+\cdots+r_j+1} = \xi^{r_1+\cdots+r_j+1}, \cdots, x^m = \xi^m$$

gives an integral manifold of θ_j contained in U.

2. The integrability of a 0-deformable vector 1-form. Let h be a vector 1-form, defined on M, whose characteristic polynomial has constant coefficients on M. Let the decomposition of the characteristic polynomial be

$$\{p_1(\lambda)\}^{d_1}\{p_2(\lambda)\}^{d_2} \cdots \{p_g(\lambda)\}^{d_g}$$

where $p_i(\lambda)$, $i = 1, \cdots, g$ are polynomials in λ, irreducible over the reals, and $(p_i(\lambda), p_j(\lambda)) = 1$, if $i \neq j$. It is easy to verify [5, pp 130–132], that we can get polynomials $e_1(\lambda), e_2(\lambda), \cdots, e_g(\lambda)$ in λ, with constant coefficients, such that $\sum_{i=1}^{g} e_i(h) = I$, $\{e_i(h)\}^2 = e_i(h)$, $e_i(h) \cdot e_j(h) = 0$ for $i \neq j$, and

$$e_i(h_x)T_x = \{u_x \in T_x \mid \{p_i(h_x)\}^{a_i} u_x = 0\} .$$

Let θ_i denote the distribution $x \to e_i(h_x)T_x$. If we assume $[h, h] = 0$, then by Lemma 1.1, because $e_i(h)$ is a polynomial in h with constant coefficients, we see that $[e_i(h), e_i(h)] = 0$. Hence, by Lemma 1.2, θ_i is completely integrable.

DEFINITION. A vector 1-form h on M is said to be *0-deformable*, if for all $x \in M$, the Jordan canonical form of h_x is equal to a fixed matrix μ [2].

Note that a 0-deformable vector 1-form has a characteristic polynomial with constant coefficients.

A frame at $x \in M$ is an isomorphism z from R^m onto T_x, where m is the dimension of M. For a 0-deformable vector 1-form h, the frames z at x such that $z^{-1}h_x z = \mu$ define a subbundle H of the frame bundle over M, as x runs through M. H is called the G-structure defined by h [1].

DEFINITION. A G-structure H defined by h is said to be *integrable*, if for each point x of M there exists a coordinate neighbourhood U of x with a coordinate system $\{x^1, \cdots, x^m\}$ such that the frame $\{(\partial/\partial x^1)_{x'}, \cdots, (\partial/\partial x^m)_{x'}\}$ belongs to the subbundle H for all $x' \in U$. We shall say that these coordinate functions are associated with the integrable G-structure H.

Clearly, H is integrable if and only if, for each point x of M, we can find a local coordinate system around x, in which the coordinate expression of h is μ.

We are interested in finding a sufficient condition for a G-structure defined by a 0-deformable vector 1-form h to be integrable. We now assume $[h, h] = 0$. By the argument above we know that the distributions θ_i associated to the irreducible factors $p_i(\lambda)$ are all completely integrable, so by Lemma 1.3, for each point x_0 of M there is a coordinate system $\{x^1, \cdots, x^m\}$ on a neighbourhood U of x_0, and the integral manifolds of θ_i contained in U are given by coordinate slices.

In U take a point given by coordinates $(\xi^1, \cdots, \xi^m)$. For each i, let $x^1 = \xi^1, \cdots, x^{r_{i-1}} = \xi^{r_{i-1}}, x^{r_i+1} = \xi^{r_i+1}, \cdots, x^m = \xi^m$ give an integral manifold M_i of θ_i in U, where $r_i = m_1 + m_2 + \cdots + m_i$ and $m_i =$ dimension of θ_i. Consider the restriction h_i of h on M_i. Notice that we can view h_i as a vector 1-form on an open set of M_i, depending on $m - m_i$ parameters $x^1, \cdots, x^{r_{i-1}}, x^{r_i+1}, \cdots, x^m$ in such the way that h_i is C^∞ with respect to the coordinates on M_i and the parameters together. The characteristic polynomial of h_i is $\{p_i(\lambda)\}^{a_i}$ and the minimum polynomial of h_i is $\{p_i(\lambda)\}^{v_i}$, where $\prod_{i=1}^{g}\{p_i(\lambda)\}^{v_i}$ is the minimum polynomial of h; h_i is a 0-deformable vector 1-form on M_i, and $[h_i, h_i] = 0$. If for each i, the G_i-structure defined by h_i on M_i is integrable, and if coordinate functions $y^{r_{i-1}+1}, \cdots, y^{r_i}$ associated to the integrable G_i-structure around the point $(x^{r_{i-1}+1}, \cdots, x^{r_i}) = (\xi^{r_{i-1}+1}, \cdots, \xi^{r_i})$ are dependent on coordinates $x^{r_{i-1}+1}, \cdots, x^{r_i}$ and on parameters $x^1, \cdots, x^{r_{i-1}}, x^{r_i+1}, \cdots, x^m$ jointly in a C^∞-manner, then we can replace $\{x^1, \cdots, x^m\}$ in a neighbourhood of the point $(x^1, \cdots, x^m) = (\xi^1, \cdots, \xi^m)$ by a new coordinate system $\{y^1, \cdots, y^m\}$, so that h takes the matrix form μ, i.e. H is integrable.

Hence we consider the case where h has characteristic polynomial $\{p(\lambda)\}^d$ and minimum polynomial $\{p(\lambda)\}^v$, where $p(\lambda)$ is irreducible over the reals, and suppose that h jointly depends on the coordinates of M and some parameters in a C^∞-manner. We have the following results:

Case I. $\deg p(\lambda) = 1$.

(i) If $v = 1$, then h is a constant multiple of the identity vector 1-form I on M, hence the G-structure is integrable.

(ii) If $v = d = m$, consider the nilpotent part n of h. n is a polynomial in h with constant coefficients on M, so from $[h, h] = 0$, we get $[n, n] = 0$, by Lemma 1.1. Moreover $n^m = 0$ but $n^l \neq 0$ for $l < m$, for all points of M. In § 3 we prove a proposition which shows that the G-structure defined by n (which is the same as that defined by h) is integrable, and that the associated coordinate functions depend on the parameters of h and on the point in M jointly in a C^∞-manner.

Case II. $\deg p(\lambda) = 2$. In § 4 we shall show that the semi-simple part s of h gives rise to a complex manifold structure $\tilde{M}$ in this case, and that for the $\tilde{G}$-structure given by h which is induced from h on $\tilde{M}$, (i) and (ii) of Case I has a straightforward parallel on $\tilde{M}$; hence coming back to the real manifold, we have: if $v = 1$, or $v = d = m/2$, then the G-structure defined by h is integrable, and the associated coordinate functions are C^∞ with respect to the coordinates on M and the parameters jointly.

By the preceding arguments and the results in § 3 and 4, we can conclude the following:

THEOREM. *Let h be a 0-deformable vector 1-form on a manifold M, with characteristic polynomial*

$$\prod_{i=1}^{g} p_i(\lambda)^{d_i}$$

where $p_i(\lambda)$ are polynomials in λ, irreducible over the reals, and $(p_i(\lambda), p_j(\lambda)) = 1$ for $i \neq j$, and the minimum polynomial

$$\prod_{i=1}^{g} p_i(\lambda)^{v_i} .$$

Suppose for each i, $v_i = 1$ or d_i. Then the G-structure defined by h is integrable if $[h, h] = 0$.

REMARK. If $v_i = 1$ for all i, we say that h is semi-simple. If $v_i = d_i$ for all i, we say that h is nonderogatory, and otherwise derogatory [6, p. 21].

3. The integrability of a nonderogatory nilpotent vector 1-form.

PROPOSITION. *Let h be a nilpotent vector 1-form on an m-dimensional manifold M, and suppose $h^m = 0$ but $h^l \neq 0$ for $l < m$, for all points on M. Then $[h, h] = 0$ implies that the G-structure defined by h is integrable. Moreover, if h depends on some parameters and is C^∞ with respect to the local coordinates $x^1, \cdots, x^m$ on M and the parameters jointly, then the local coordinates $y^1, \cdots, y^m$ associated to the integrable G-structure are C^∞ with respect to $x^1, \cdots, x^m$ and the parameters jointly.*

Proof. (1) Let $m = 2$. Denoting the tangent space at $x \in M$ by T_x, we have a one dimensional distribution given by $x \to h_x T_x$. For each point x_0 of M we can find a neighbourhood U of x_0 and a coordinate system $\{x^1, x^2\}$ on U, such that $x^2 = \xi^2$ is an integral manifold of this distribution in U. Let h take the matrix form in this coordinate system

$$\begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$$

β_{ij} being functions of x^1, x^2. As $\partial/\partial x^1$ at $x \in U$ spans $h_x T_x$, we have $\beta_{21} = \beta_{22} = 0$, and as h restricted to integral manifold $x^2 = \xi^2$ is given by β_{11}, and as $h^2 = 0$, we have $\beta_{11} = 0$. We claim, that we can choose a new coordinate system $\{y^1, y^2\}$ such that in this new coordinate system h takes the matrix form

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In fact, let the vector fields $\partial/\partial x^1$ and $\partial/\partial x^2$ be denoted by X_1 and X_2, and choose new vector fields Y_1 and Y_2 by

$$\begin{cases} Y_1 = \alpha_1 X_1 \\ Y_2 = \alpha_0 X_1 + X_2 \end{cases}$$

where α_1 and α_0 are to be determined so that $hY_2 = Y_1$ and $[Y_1, Y_2] = 0$. Let then π^1, π^2 be the 1-forms dual to Y_1, Y_2; we have $d\pi^1 = 0, d\pi^2 = 0$, so that y^1, y^2 can be determined from $dy^1 = \pi^1, dy^2 = \pi^2$. To prove that Y_1 and Y_2 can be found we observe that the condition $hY_2 = Y_1$ leads to

$$\alpha_1 = \beta_{12}$$

and that the condition $[Y_1, Y_2] = 0$ leads to

$$(\alpha_0 X_1 + X_2)\alpha_1 - \alpha_1 X_1 \alpha_0 = 0$$

which is a first order linear differential equation for α_0:

$$\alpha_1 \frac{\partial}{\partial x^1}\alpha_0 - \alpha_0\left(\frac{\partial}{\partial x^1}\alpha_1\right) - \frac{\partial}{\partial x^2}\alpha_1 = 0 .$$

α_1 is clearly C^∞ with respect to x^1, x^2 and the parameters. α_0 is obtained as a solution of the above differential equation, so α_0 depends on x^2 and the parameters in a C^∞ manner. By differentiating this differential equation repeatedly, we see that α_0 is C^∞ with respect to x^1, x^2 and the parameters. Hence π^1 and π^2 are C^∞ with respect to x^1, x^2 and the parameters, and finally y^1 and y^2 are C^∞ with respect to x^1, x^2 and the parameters.

(2) We assume that our proposition is true for $(m-1)$-dimensional manifolds and proceed to prove it for an m-dimensional manifold ($m \geqq 3$).

Because $[h, h] = 0$, we know that the distribution $x \to h_x T_x$, given by the image of h at each point x of M is integrable; hence, locally, there exists a coordinate system $\{x^1, \cdots, x^m\}$ such that

(i) $x^m = \xi^m$ gives the integral manifolds of this distribution, and

(ii) in this coordinate system h takes the matrix form

$$\text{(1)} \qquad \begin{pmatrix} & & \beta_{1\,m} \\ & & \cdot \\ & H & \cdot \\ & & \cdot \\ & & \beta_{m-1\,m} \\ 0 & \cdots\ 0 & 0 \end{pmatrix}$$

We further claim that $x^1, \cdots, x^{m-1}, x^m$ can be chosen so that

(iii) H takes the form

$$\text{(2)} \qquad \begin{pmatrix} 0 & 1 & 0 & \cdot & 0 \\ \cdot & 0 & 1 & \cdot & 0 \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & & 0 & 1 \\ 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

In fact, if H is not in the form (2) already, we view the restriction h_1 of h to an integral manifold $x^m = \xi^m$ as a vector 1-form on an open set V of R^{m-1}, depending on parameter x^m, and consider H to be the matrix form of h_1 with respect to the coordinate system $\{x^1, \cdots, x^{m-1}\}$. From the inductive assumption, there are coordinate functions $z^1, \cdots, z^{m-1}$ on an open set $V_1 \subset V$ depending on $x^1, \cdots, x^{m-1}$ and x^m in a C^∞-manner, such that h_1 has matrix form (2) with respect to the coordinate system

$\{z^1, \cdots, z^{m-1}\}$. Now, if we take $\{z^1, \cdots, z^{m-1}, x^m\}$ as the local coordinate system on M, then (iii) will be satisfied.

So let us suppose that we are in a coordinate system where (i) (ii) and (iii) are satisfied. For simplicity we write $\beta_1, \beta_2, \cdots, \beta_{m-1}$ instead of $\beta_{1m}, \beta_{2m}, \cdots, \beta_{m-1m}$. Note that $\beta_{m-1} \neq 0$. We want to prove that we can find a new coordinate system $\{y^1, \cdots, y^m\}$ such that in this coordinate system h takes the matrix form (1), H being of the form (2) and $\beta_1 = \beta_2 = \cdots = \beta_{m-2} = 0$, $\beta_{m-1} = 1$. In order to do this, as in the case $m = 2$, we find vector fields $Y_1, \cdots, Y_m$ satisfying $hY_i = Y_{i-1}$ $(i = 2, \cdots, m)$, $hY_1 = 0$ and $[Y_i, Y_j] = 0$ for all i, j; let the dual of $Y_1, \cdots, Y_m$ be $\pi^1, \cdots, \pi^m$ and obtain $y^1, \cdots, y^m$ from $dy^1 = \pi^1, \cdots, dy^m = \pi^m$. If we denote by $X_1, \cdots, X_m$ the vector fields $\partial/\partial x^1, \cdots, \partial/\partial x^m$ and set

$$
\tag{3}
\begin{cases}
Y_1 = \alpha_{m-1}X_1 \\
Y_2 = \alpha_{m-2}X_1 + \alpha_{m-1}X_2 \\
\quad \cdots\cdots\cdots\cdots\cdots \\
Y_{m-1} = \alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_{m-1}X_{m-1} \\
Y_m = \alpha_0 X_1 + (\alpha_1 - \beta_1)X_2 + \cdots + (\alpha_{m-2} - \beta_{m-2})X_{m-1} + X_m
\end{cases}
$$

where $\alpha_{m-1} = \beta_{m-1}$, then the problem reduces to finding the α's so that $[Y_i, Y_j] = 0$ are satisfied for all i, j.

First we shall obtain all the relations on the derivatives of $\beta_1, \cdots, \beta_{m-1}$ imposed by the condition $[h, h] = 0$. We see that

$$[h, h](X_i, X_j) = 0$$

gives us no relations for $i, j \leqq m - 1$, but

$$\frac{1}{2}[h, h](X_i, X_m) = [X_{i-1}, \beta_1 X_1 + \cdots + \beta_{m-1}X_{m-1}]$$

$$- h[X_i, \beta_1 X_1 + \cdots + \beta_{m-1}X_{m-1}]$$

from which we obtain

$$\tag{4} X_{i-1}\beta_{j-1} = X_i\beta_j \qquad i, j \leqq m - 1$$

and

$$\tag{5} X_i\beta_{m-1} = 0 \qquad i \leqq m - 2 .$$

To make this relation clear, we write this result in Table 1.

$$
\begin{array}{llllll}
0 = X_1\beta_{m-1} & & & & & \\
0 = X_1\beta_{m-2} = X_2\beta_{m-1} & & & & & \\
\quad \cdots\cdots\cdots\cdots & & & & & \\
0 = X_1\beta_3 & = X_2\beta_4 = & \cdots\cdots\cdots\cdots & = X_{m-3}\beta_{m-1} & & \\
0 = X_1\beta_2 & = X_2\beta_3 = & \cdots\cdots\cdots\cdots & = X_{m-3}\beta_{m-2} & = X_{m-2}\beta_{m-1} & \\
\quad X_1\beta_1 & = X_2\beta_2 = & \cdots\cdots\cdots\cdots & = X_{m-3}\beta_{m-3} & = X_{m-2}\beta_{m-2} & = X_{m-1}\beta_{m-1} \\
 & \quad X_2\beta_1 = & \cdots\cdots\cdots\cdots & = X_{m-3}\beta_{m-4} & = X_{m-2}\beta_{m-3} & = X_{m-1}\beta_{m-2} \\
 & & & \cdots\cdots\cdots\cdots & & \\
 & & & X_{m-3}\beta_1 & = X_{m-2}\beta_2 & = X_{m-1}\beta_3 \\
 & & & & X_{m-2}\beta_1 & = X_{m-1}\beta_2
\end{array}
$$

TABLE 1

Now let us examine $[Y_i, Y_j] = 0$ for $i < j \leqq m - 1$. We see that this is equivalent to the set of equations (6),

$$
(6)\quad \begin{cases}
(\alpha_{m-i}X_1 + \alpha_{m-i+1}X_2 + \cdots + \alpha_{m-1}X_i)\alpha_{m-1} = 0 \\
\quad \cdots\cdots\cdots\cdots\cdots\cdots \\
(\alpha_{m-i}X_1 + \alpha_{m-i+1}X_2 + \cdots + \alpha_{m-1}X_i)\alpha_{m-j+i} = 0 \\
(\alpha_{m-i}X_1 + \alpha_{m-i+1}X_2 + \cdots + \alpha_{m-1}X_i)\alpha_{m-j+i-1} \\
\qquad\qquad - (\alpha_{m-j}X_1 + \alpha_{m-j+1}X_2 + \cdots + \alpha_{m-1}X_j)\alpha_{m-1} = 0 \\
\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
(\alpha_{m-i}X_1 + \alpha_{m-i+1}X_2 + \cdots + \alpha_{m-1}X_i)\alpha_{m-j} \\
\qquad\qquad - (\alpha_{m-j}X_1 + \alpha_{m-j+1}X_2 + \cdots + \alpha_{m-1}X_j)\alpha_{m-i} = 0
\end{cases}
$$

where $i < j \leqq m - 1$. Using $X_1\alpha_{m-1} = X_1\beta_{m-1} = 0$ from Table 1, we see that (6) is equivalent to the following Table 2.

$$
\left.\begin{array}{llllll}
0 = X_1\alpha_{m-1} & & & & & \\
0 = X_1\alpha_{m-2} = X_2\alpha_{m-1} & & & & & \\
\qquad \cdots\cdots\cdots\cdots & & & & & \\
0 = X_1\alpha_3 & = X_2\alpha_4 = & \cdots = & X_{m-3}\alpha_{m-1} & & \\
0 = X_1\alpha_2 & = X_2\alpha_3 = & \cdots = & X_{m-3}\alpha_{m-2} & = X_{m-2}\alpha_{m-1} & \\
\quad X_1\alpha_1 & = X_2\alpha_2 = & \cdots = & X_{m-4}\alpha_{m-3} & = X_{m-2}\alpha_{m-2} & = X_{m-1}\alpha_{m-1}.
\end{array}\right\}\text{(a)}
$$

$$
\left.\begin{array}{lllll}
X_2\alpha_1 = & \cdots = & X_{m-3}\alpha_{m-4} & = X_{m-2}\alpha_{m-3} & = X_{m-1}\alpha_{m-2} \\
 & \cdots\cdots\cdots\cdots & & & \\
 & & X_{m-3}\alpha_1 & = X_{m-2}\alpha_2 & = X_{m-1}\alpha_3 \\
 & & & X_{m-2}\alpha_1 & = X_{m-1}\alpha_2
\end{array}\right\}\text{(b)}
$$

TABLE 2

Next consider $[Y_i, Y_m] = 0$, $i \leqq m - 1$. This is equivalent to the following (7a, b, c),

$$
\text{(7a)}\quad
\begin{cases}
(\alpha_{m-i}X_1 + \alpha_{m-i+1}X_2 + \cdots + \alpha_{m-1}X_i)(\alpha_{m-2} - \beta_{m-2}) = 0 \\
\cdots\cdots\cdots\cdots \\
(\alpha_{m-i}X_1 + \alpha_{m-i+1}X_2 + \cdots + \alpha_{m-1}X_i)(\alpha_i - \beta_i) = 0
\end{cases}
$$

$$
\text{(7b)}\quad
\begin{cases}
(\alpha_{m-i}X_1 + \alpha_{m-i+1}X_2 + \cdots + \alpha_{m-1}X_i)(\alpha_{i-1} - \beta_{i-1}) \\
\qquad - \{\alpha_0X_1 + (\alpha_1 - \beta_1)X_2 + \cdots + (\alpha_{m-2} - \beta_{m-2})X_{m-1} + X_m\}\alpha_{m-1} = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
(\alpha_{m-i}X_1 + \alpha_{m-i+1}X_2 + \cdots + \alpha_{m-1}X_i)(\alpha_1 - \beta_1) \\
\qquad - \{\alpha_0X_1 + (\alpha_1 - \beta_1)X_2 + \cdots + (\alpha_{m-2} - \beta_{m-2})X_{m-1} + X_m\}\alpha_{m-i+1} = 0
\end{cases}
$$

$$
\text{(7c)}\quad
\begin{aligned}
&(\alpha_{m-i}X_1 + \alpha_{m-i+1}X_2 + \cdots + \alpha_{m-1}X_i)\alpha_0 \\
&\qquad - \{\alpha_0X_1 + (\alpha_1 - \beta_1)X_2 + \cdots + (\alpha_{m-2} - \beta_{m-2})X_{m-1} + X_m\}\alpha_{m-i} = 0
\end{aligned}
$$

where $i \leqq m-1$.

Because of Table 1, we see that (7a) is equivalent to part (a) of Table 2. Using part (a) of Table 2, we see that (7b) reduces to a simpler system (7b′),

$$
\text{(7b}'\text{)}\quad
\begin{cases}
(\alpha_{m-1}X_i)(\alpha_{i-1} - \beta_{i-1}) - \{(\alpha_{m-2} - \beta_{m-2})X_{m-1} + X_m\}\alpha_{m-1} = 0 \\
(\alpha_{m-2}X_{i-1} + \alpha_{m-1}X_i)(\alpha_{i-2} - \beta_{i-2}) - \{(\alpha_{m-3} - \beta_{m-3})X_{m-2} \\
\qquad\qquad + (\alpha_{m-2} - \beta_{m-2})X_{m-1} + X_m\}\alpha_{m-2} = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
(\alpha_{m-i+1}X_2 + \cdots + \alpha_{m-1}X_i)(\alpha_1 - \beta_1) \\
\quad - \{(\alpha_{m-i} - \beta_{m-i})X_{m-i+1} + \cdots + (\alpha_{m-2} - \beta_{m-2})X_{m-1} + X_m\}\alpha_{m-i+1} = 0
\end{cases}
$$

Using Table 1 again, we can show that (7b′) is equivalent to part (b) of Table 2 plus the following equations which are obtained from (7b′) by letting $i = m-1$:

$$
\begin{cases}
(\alpha_{m-1}X_{m-1})(\alpha_{m-2} - \beta_{m-2}) - \{(\alpha_{m-2} - \beta_{m-2})X_{m-1} + X_m\}\alpha_{m-1} = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
(\alpha_2X_2 + \cdots + \alpha_{m-1}X_{m-1})(\alpha_1 - \beta_1) \\
\qquad - \{(\alpha_1 - \beta_1)X_2 + \cdots + (\alpha_{m-2} - \beta_{m-2})X_{m-1} + X_m\}\alpha_2 = 0
\end{cases}
$$

Using Table 1 and part (b) of Table 2, these equations can be written as (8),

$$
\text{(8)}\quad (\alpha_{m-1})^2X_{m-1}\frac{\alpha_{m-k} - \beta_{m-k}}{\alpha_{m-1}} + (\alpha_{m-2})^2X_{m-1}\frac{\alpha_{m-k+1} - \beta_{m-k+1}}{\alpha_{m-2}}
$$

$$
+ \cdots + (\alpha_{m-k+1})^2X_{m-1}\frac{\alpha_{m-2} - \beta_{m-2}}{\alpha_{m-k+1}} - X_m\alpha_{m-k+1} = 0\,,^{1)} \quad k = 2, \cdots, m-1\,.
$$

---

1 For simplicity we write $(\alpha_{m-1-j})^2X_{m-1}(\alpha_{m-k+j} - \beta_{m-k+j}/\alpha_{m-1-j})$, $1 \leqq j \leqq k-2$, for $\alpha_{m-1-j}X_{m-1}(\alpha_{m-k+j} - \beta_{m-k+j}) - (X_{m-1}\alpha_{m-1-j})X_{m-1}(\alpha_{m-k+j} - \beta_{m-k+j})$, although at some point α_{m-1-j} might vanish.

We can now obtain $\alpha_{m-2}, \alpha_{m-3}, \cdots, \alpha_1$ succesively by integrating (8) with respect to x^{m-1}; in fact, start from $k=2$, and integrate to get α_{m-2}, then use this α_{m-2} in (8) for $k=3$ and integrate to get α_{m-3}, in general

$$\text{(9)} \qquad \alpha_{m-k} - \beta_{m-k} = \alpha_{m-1} \int \frac{-1}{(\alpha_{m-1})^2} \Big\{ (\alpha_{m-2})^2 X_{m-1} \frac{\alpha_{m-k+1} - \beta_{m-k+1}}{\alpha_{m-2}} + \cdots + (\alpha_{m-k+1})^2 X_{m-1} \frac{\alpha_{m-2} - \beta_{m-2}}{\alpha_{m-k+1}} - X_m \alpha_{m-k+1} \Big\} dx^{m-1} .$$

We still have to show that $\alpha_{m-2}, \alpha_{m-3}, \cdots, \alpha_1$ thus obtained satisfy Table 2. For simplicity let us write (8) in the form

$$(8_k) \qquad (\alpha_{m-1})^2 X_{m-1} \frac{\alpha_{m-k} - \beta_{m-k}}{\alpha_{m-1}} + A_{m-k+1} = 0 .$$

Then (9) becomes

$$(9_k) \qquad \alpha_{m-k} - \beta_{m-k} = \alpha_{m-1} \int \frac{-1}{(\alpha_{m-1})^2} A_{m-k+1} dx^{m-1} .$$

To show that the α's do satisfy Table 2, it suffices to show (10_k),

$$(10_k) \qquad X_{m-q}(\alpha_{m-k} - \beta_{m-k}) = X_{m-q+1}(\alpha_{m-k+1} - \beta_{m-k+1})$$

for $k, q = 2, \cdots, m-1$. We shall prove (10_k) inductively. For $k=2$ it is easy to check. Suppose $(10_2), \cdots, (10_{k-1})$ are true; using this assumption, we differentiate (9_k) and get (11),

$$\text{(11)} \qquad X_{m-q}(\alpha_{m-k} - \beta_{m-}) = a_{m-1} \int \frac{-1}{(\alpha_{m-1})^2} \Big\{ (X_{m-q\,m-2}\alpha)^2 X_{m-1} \frac{\alpha_{m-k+1} - \beta_{m-k+1}}{X_{m-q}\alpha_{m-2}} + X_{m-q+1} A_{m-k+2} + (\alpha_{m-k+1})^2 X_{m-1} \frac{X_{m-q}(\alpha_{m-2} - \beta_{m-2})}{\alpha_{m-k+1}} \Big\} dx^{m-1} .$$

If $q > 2$, then $X_{m-q}\alpha_{m-2} = 0$, so (11) gives us (10_k). If $q=2$, we observe first that differentiating (8_{k+1}) with respect to x^{m-1} gives us (12),

$$\text{(12)} \qquad (X^2_{m-1}(\alpha_{m-k+1} - \beta_{m-k+1}))\alpha_{m-1} - (\alpha_{m-k+1} - \beta_{m-k+1}) X^2_{m-1}\alpha_{m-1} + X_{m-1} A_{m-k+2} = 0.$$

Using (12) and $X_{m-2}(\alpha_{m-2} - \beta_{m-2}) = 0$ in (11) for $q=2$, we obtain

$$X_{m-2}(\alpha_{m-k} - \beta_{m-k}) = \alpha_{m-1} \int \frac{-1}{(\alpha_{m-1})^2} \Big\{ (X_{m-1}(\alpha_{m-k+1} - \beta_{m-k+1})) X_{m-1}\alpha_{m-1} - (X^2_{m-1}(\alpha_{m-k+1} - \beta_{m-k+1}))\alpha_{m-1} \Big\} dx^{m-1} = X_{m-1}(\alpha_{m-k+1} - \beta_{m-k+1})$$

which completes the proof (10_k).

Finally to obtain α_0, we examine (7c), and find that the same type of argument employed to obtain (8) enables us to show that (7c) is equivalent to

$$(13)\quad \begin{cases} X_1\alpha_0 = X_{m-1}(\alpha_{m-2} - \beta_{m-2}) \\ \quad \cdots\cdots\cdots\cdots \\ X_{m-2}\alpha_0 = X_{m-1}(\alpha_1 - \beta_1) \\ (\alpha_1 X_1 + \cdots + \alpha_{m-1}X_{m-1})\alpha_0 - \{\alpha_0 X_1 + (\alpha_1 - \beta_1)X_2 + \\ \qquad\qquad \cdots + (\alpha_{m-2} - \beta_{m-2})X_{m-1} + X_m\}\alpha_{m-1} = 0\,. \end{cases}$$

Using the first $m - 2$ equations of (13) in the last one, gives us (8_k) for $k = m$, where we agree that $\beta_0 = 0$. Hence we obtain α_0 from (9_m). To check that the first $m - 2$ equations in (13) are satisfied by this α_0, we check (10_k) for $k = m$. The same argument in (11) holds for $k = m$, and it is even simpler than before, because in this case the first term in the integrand vanishes.

If h depends on $x^1, \cdots, x^m$ and some parameters jointly in a C^∞-manner, then it is clear that $\alpha_{m-2}, \cdots, \alpha_1, \alpha_0$ obtained above depend on $x^1, \cdots, x^m$ and the parameters in a C^∞-manner, hence we can claim the same for $y^1, \cdots, y^m$.

4. **The complex case.** For Case II in § 2, where $\deg p(\lambda) = 2$, we have $\dim M = m = 2n$. Let the roots of $p(\lambda) = 0$ be $\sigma \pm i\tau$ $(\tau \neq 0)$. Because the semi-simple part s of h is a polynomial in h with constant coefficients, from $[h, h] = 0$, via Lemma 1.1, we get $[s, s] = 0$. The vector 1-form J_s defined by

$$J_s = \frac{1}{\tau}(s - \sigma I)$$

satisfies $\lambda^2 + 1 = 0$, because s satisfies $p(\lambda) = 0$. So we have an almost complex structure J_s on M, and as $[J_s, J_s] = 0$ (because $[s, s] = 0$), this almost complex structure is integrable [7]. Hence we can introduce a new real local coordinate system $\{x^1, \cdots, x^m\}$ such that $z^k = x^{2k-1} + ix^{2k}$ $(k = 1, \cdots, n)$ gives a local complex coordinate system, with which M becomes the underlying C^∞-manifold of complex manifold $\widetilde{M}$. As h is C^∞ with respect to the coordinates on M and the parameters jointly, so is the almost complex structure J_s. Hence the new coordinate functions $x^1, \cdots, x^m$ are also C^∞ with respect to the coordinates on M and the parameters jointly [7].[2] h is now C^∞ with respect to $x^1, \cdots, x^m$ and the parameters jointly. The vector 1-forms on M induce vector 1-forms on $\widetilde{M}$ in a natural way. The vector 1-form $\tilde{s}$ on $\widetilde{M}$ induced by s is equal to $\rho\tilde{I}$, where $\rho = \sigma + i\tau$ and $\tilde{I}$ is the identity vector 1-from on $\widetilde{M}$. We shall show that polynomials in h with constant coefficients induce holomorphic vector 1-forms on M. In particular, the nilpotent part n of h induces the nilpotent holomorphic vector 1-form $\tilde{n}$ on M.

[2] The author wishes to thank Professor L. Nirenberg for communicating the proof of this fact to him. The dependence on parameters is stated without proof in [7].

Let T_C and $T_C^{(p)}$ be the vector bundles over M, which are obtained by complexifying the tangent space T_x and the space of tangential covariant p-vectors $T_x^{(p)}$ respectively at each point x of M. Then any p-form P on M, i.e. any cross-section of $T\otimes T^{(p)}$, extends in a natural way to a cross-section P_C of $T_C\otimes T_C^{(p)}$. If k and l are two vector 1-forms on M, then k_C, l_C and $[k, l]_C$ are defined. If we define the bracket of two cross-sections of T_C in a natural way, and if we define $[k_C, l_C]$ by (2) of § 1, where we replace h, k by k_C, l_C acd u, v by cross-sections of T_C, then we have $[k, l]_C = [k_C, l_C]$.

Denote $\partial/\partial\bar{z}^i$, $\partial/\partial z^i$ by $Z_i, \bar{Z}_i$ for $i = 1, \cdots, n$. $(Z_1)_x, \cdots, (Z_n)_x, (\bar{Z}_1)_x, \cdots, (\bar{Z}_n)_x$ span the complexification of T_x. $(Z_1)_x, \cdots, (Z_n)_x$ span the eigenspaces of eigenvalue ρ. This eigenspace can be identified with the tangent space of $\tilde{M}$ at x. $(\bar{Z}_1)_x, \cdots, (\bar{Z}_n)_x$ span the eigenspace of $(s_C)_x$ of eigenvalue $\bar{\rho}$. If k is a polynomial in h with constant coefficients, by Lemma 1.1 we have $[s, k] = 0$, and hence $[s_C, k_C] = [s, k]_C = 0$. On the other hand we have

$$[s_C, k_C](Z_i, \bar{Z}_j) = (\rho - s)[Z_i, k_C\bar{Z}_j] + (\bar{\rho} - s)[k_CZ_i, \bar{Z}_j] .$$

s_C and k_C are polynomials in h_C with constant coefficients, so s_C and k_C commute; hence k_C leave the eigenspaces of s_C invariant, so using the coordinate expression for k_C, the equation above can be written as

$$[s_C, k_C](Z_i, \bar{Z}_j) = (\rho - \bar{\rho})\sum_{k=1}^{n}\{(Z_i(k_C)_{\bar{k}\bar{j}})\bar{Z}_k + (\bar{Z}_j(k_C)_{ki})Z_k\}$$

from which we get

$$(\partial/\partial\bar{z}^j)(k_C)_{ki} = 0 . \tag{1}$$

$(k_C)_{ki}$ is the matrix form of $\tilde{k}$ on $\tilde{M}$ (induced by k) with respect to the coordinate system $\{z^1, \cdots, z^n\}$, and (1) expresses the fact that $\tilde{k}$ is holomorphic.[3]

(i) If $v = 1$ in Case II of § 2, then $\tilde{h}$ induced by h on $\tilde{M}$, is equal to $\tilde{s} = \rho\tilde{I}$. So in the real coordinate system $\{x^1, \cdots, x^m\}$ h takes the matrix form

$$\begin{pmatrix} A & & & \\ & A & 0 & \\ & & \cdot & \\ & 0 & & \cdot \\ & & & & A \end{pmatrix}$$

where

$$A = \begin{pmatrix} \sigma & \tau \\ -\tau & \sigma \end{pmatrix},$$

[3] The author is indebted to Professor H. C. Wang for this proof.

so that G-structure is integrable.

(ii) If $v = d = n$ in Case II of § 2, then $\tilde{n}$ satisfies $\tilde{n}^n = 0$ but $\tilde{n}^l \neq 0$ for $l < n$ for all points on $\tilde{M}$. As $\tilde{n}$ is holomorphic, it is meaningful to define the Nijenhuis tensor $[\tilde{n}, \tilde{n}]$ of $\tilde{n}$, using (3) of § 1 as the defining formula, where u, v should be holomorphic vector fields on $\tilde{M}$. As $[n_\sigma, n_\sigma] = [n, n]_\sigma = 0$, we have $[\tilde{n}, \tilde{n}] = 0$.

Now following the method in § 3, it is easy to see that we have a complex version of the Proposition in § 3, i.e.

"Let $\tilde{k}$ be a holomorphic nilpotent vector 1-form on an n-dimensional complex manifold, and suppose $\tilde{k}^n = 0$ but $\tilde{k}^l = 0$ for $l < n$, for all points. Then $[\tilde{k}, \tilde{k}] = 0$ implies that the $\tilde{G}$-structure defined by $\tilde{k}$ is integrable. Moreover, if $\tilde{k}$ depends on some complex [real] parameters and is holomorphic [C^∞] with respect to the local coordinates $z^1, \cdots, z^n$ [the real coordinates $x^1, \cdots, x^m$, where $z^k = x^{2k-1} + ix^{2k}$] and the parameters jointly, then the local coordinates $w^1, \cdots, w^n$ associated to the integrable $\tilde{G}$-structure [the real coordinates $y^1, \cdots, y^m$ obtained from $w^k = y^{2k-1} + iy^{2k}$] are holomorphic [C^∞] with respect to $z^1, \cdots, z^n[x^1, \cdots, x^m]$ and the parameters jointly."

Using this complex version, for each point of $\tilde{M}$, we have a neighbourhood with a local complex coordinate system $w^1, \cdots, w^n$, with respect to which $\tilde{h} = \tilde{s} + \tilde{n}$ takes the matrix form

$$\begin{pmatrix} \rho & 1 & & 0 \\ & \rho & 1 & \\ & & \ddots & \ddots \\ 0 & & & 1 \\ & & & \rho \end{pmatrix}$$

Passing back to the real coordinate system $\{y^1, \cdots, y^m\}(w^k = y^{2k-1} + iy^{2k})$, h takes the matrix form

$$\begin{pmatrix} A & B & & \\ & A & B & 0 \\ & & \ddots & \ddots \\ & 0 & & A & B \\ & & & & A \end{pmatrix}$$

where

$$A = \begin{pmatrix} \sigma & \tau \\ -\tau & \sigma \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The G-structure defined by h is thus integrable. The associated local coordinates $y^1, \cdots, y^m$ are C^∞-functions of the coordinates of M and the parameters jointly.

5. An example.[4] Let M be the euclidean space of dimension 4, and

[4] The author is indebted to Professor H. C. Wang for this example.

suppose x, y, z, t are the coordinates. Let

$$X_1 = \partial/\partial x,\ X_2 = \partial/\partial y,\ X_3 = \partial/\partial z,\ X_4 = (\partial/\partial t) + (1 + z)(\partial/\partial x)\ ,$$

and define h by $hX_1 = X_2$, $hX_i = 0$ for $i = 2, 3, 4$. It is easy to check that

(i) $h^2 = 0$,

(ii) $[h, h] = 0$,

and (iii) $[X_3, X_4] = X_1$.

Now, if the G-structure defined by h would be integrable, so would the distributions intrinsically given by h. However, (iii) shows that the distribution given by the kernel of h at each point of M is not integrable, hence we conclude that the G-structure is not integrable.

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