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ON THE RADIAL LIMITS OF BLASCHKE PRODUCTS

GERALD R. MAC LANE AND FRANK BEALL RYAN

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G. R. MACLANE AND F. B. RYAN

1. Introduction. As is well known, a Blaschke product $f(z)$ in $\{|z| < 1\}$ has radial limits $f(e^{i\theta})$ of modulus one almost everywhere on $\{|z| = 1\}$. The object of the present paper is to give a partial answer to the question: how many times does $f(z)$ assume a given radial limit? We shall prove the following theorem.

THEOREM A. *Let E be a given closed set on $\{|w| = 1\}$ and let E' be the complement of E relative to $\{|w| = 1\}$. Then there exists a Blaschke product $f(z)$, all of whose radial limits are of modulus one, and such that the set*

$$L(\beta) = \{\theta \mid f(e^{i\theta}) = e^{i\beta}\}$$

has the power of the continuum for $e^{i\beta} \in E$ and is countable for $e^{i\beta} \in E'$.

Theorem A is a condensed statement of what we shall actually prove; Theorems 1, 2, and 3 contain somewhat more information on $f(z)$. The method of proof is to construct a suitable regularly-branched covering \mathscr{W} of $\{|w| < 1\}$, corresponding to an automorphic function $w = f(z)$, and then use the geometry of \mathscr{W} to obtain our results.

The question naturally arises as to whether one could prove Theorem A directly. That is: could one produce an $f(z)$ with the desired properties by exhibiting its zeros instead of defining $f(z)$ by means of a surface \mathscr{W} ? The answer to this question does not seem to be obvious.

2. The surface \mathscr{W} . Let E be a given *nonvoid* closed subset of $\{|w| = 1\}$ and let $\{a_n\}_1^\infty$ be an infinite sequence of points in $\{|w| < 1\}$ whose derived set is E . Clearly, we may assume that $a_n \neq 0$ and

$$(1) \quad \arg a_m \neq \arg a_n \quad (m \neq n).$$

Let \mathscr{W} be the simply-connected unbordered covering of $\{|w| < 1\}$ which is regularly-branched over the points $\{a_n\}$ with all branch points of multiplicity 2. It is well known [2, 3, 6] that such a covering, with any specified multiplicity or signature for each a_n , exists and is unique. Instead of appealing to the general theory of regularly-branched coverings, we shall construct the surface \mathscr{W} directly, since the details of the construction play a role in the proof of Theorem A.

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Let C_n be the radial segment $\arg w = \arg a_n, |a_n| \leq |w| < 1$. The C_n are disjoint because of (1). We make cuts in $\{|w| < 1\}$ along each C_n and so obtain a slit disc W , copies of which are joined together, according to the following specifications, to form the surface.

0th level. The surface \mathscr{W}_0 consists of just one slit disc W . Note that \mathscr{W}_0 is simply-connected.

1st level. The surface \mathscr{W}_1 is obtained by adjoining an infinite sequence of *distinct* copies of W , namely $W(1), W(2), \dots$, to \mathscr{W}_0 . $W(n_1)$ is joined to \mathscr{W}_0 along C_{n_1} so as to form a first-order branch-point over a_{n_1} . The surface $\mathscr{W}_1 = \mathscr{W}_0 \cup \bigcup_n W(n)$ is simply-connected; for by adjoining the $W(n)$ one at a time we obtain an increasing sequence of simply-connected surfaces which exhaust \mathscr{W}_1 . We denote by $\chi(n_1)$ the curve in \mathscr{W}_1 along which $W(n_1)$ and \mathscr{W}_0 are identified.

2nd level. Along each free slit on the boundary of \mathscr{W}_1 we adjoin a copy of W . More precisely, the sheet $W(n_1, n_2)$ is adjoined to $W(n_1)$ along the cut C_{n_2} in $W(n_1)$. The added sheets correspond one-to-one with all pairs (n_1, n_2) of positive integers such that $n_1 \neq n_2$. Again we see that the surface $\mathscr{W}_2 = \mathscr{W}_1 \cup \bigcup W(n_1, n_2)$ is simply-connected. The curve over C_{n_2} along which $W(n_1)$ and $W(n_1, n_2)$ are joined is denoted by $\chi(n_1, n_2)$.

kth level. Continuing the construction, the surface \mathscr{W}_k consists of \mathscr{W}_{k-1} and copies of W denoted by $W(n_1, n_2, \dots, n_k), n_i \neq n_{i+1}$, which are joined to \mathscr{W}_{k-1} ; $W(n_1, \dots, n_k)$ is adjoined to $W(n_1, \dots, n_{k-1})$ along the cut C_{n_k} in $W(n_1, \dots, n_{k-1})$. Denote the curve along which those two sheets are joined by $\chi(n_1, n_2, \dots, n_k)$. Clearly \mathscr{W}_k is simply-connected.

We take the surface \mathscr{W} to be $\lim_{k \rightarrow \infty} \mathscr{W}_k$ as $k \rightarrow \infty$; it is clear that \mathscr{W} is simply-connected as $\mathscr{W}_k \uparrow \mathscr{W}$. With the natural projection map onto $\{|w| < 1\}$ it is clear that \mathscr{W} is a regularly-branched, unbordered, covering of $\{|w| < 1\}$. All points of \mathscr{W} over the a_n are branch-points of multiplicity 2, and \mathscr{W} has no other branch-points.

3. The function $f(z)$. Since \mathscr{W} is a covering of $\{|w| < 1\}$ it is hyperbolic. Let $w = f(z)$ be the holomorphic function which maps $\{|z| < 1\}$ onto \mathscr{W} , with $f(0) = 0 \in \mathscr{W}_0$ and $f'(0) > 0$. Clearly $|f(z)| < 1$. The radial limits of $f(z)$ are *all* of modulus one, since if this were not the case a boundary point of $\{|z| < 1\}$ would correspond to an interior point of \mathscr{W} which is unbordered. Thus $f(z)$ is of class U [5, p. 32]. Applying Frostman's theorem [5, p. 33] we see that $f(z)$ is a Blaschke product.

Also, $f(z)$ is an automorphic function with respect to a Fuschian group F , since the decktransformations of \mathscr{W} correspond to linear

transformations preserving $\{|z| < 1\}$. It is easily shown that if $E = \{|w| = 1\}$ then F is of the first kind: the limit points of F fill $\{|z| = 1\}$. If $E \neq \{|w| = 1\}$ then F is of the second kind: the set of limit points of F is a perfect nowhere dense subset of $\{|z| = 1\}$.

The sheets $W(n_1, n_2, \dots, n_k)$ of \mathscr{W} correspond to a set of fundamental regions $R(n_1, \dots, n_k)$ of F . These are the fundamental regions which play a role in the proof; since these are defined via the function f it is not clear that they are the same as the fundamental regions obtained by any of the usual constructions in terms of F . Hence we must derive some properties of these regions.

4. Properties of the fundamental regions. For convenience we reduce the notations $W(n_1, \dots, n_k)$, $R(n_1, \dots, n_k)$, and $\chi(n_1, \dots, n_k)$ to W , R , and χ respectively. To each curve χ in \mathscr{W} there corresponds a simple arc X in $\{|z| < 1\}$. It is evident that the fundamental regions R are bounded by the X 's and points of $\{|z| = 1\}$. We proceed with an investigation of the X 's.

First, each X ends at two distinct points of $\{|z| = 1\}$. The two linear pieces of χ correspond to two simple arcs X' and X'' , and $f(z)$ tends to a limit as $|z| \rightarrow 1$ on X' and X'' . Then by Koebe's lemma [1, p. 213] each of X' and X'' must tend to a definite point of $\{|z| = 1\}$. The end points of X' and X'' must be distinct. If not, let D be that part of $\{|z| < 1\}$ bounded by X and a single point b on $\{|z| = 1\}$. Then the part of \mathscr{W} corresponding to D will contain an infinite number of sheets W joined along various χ 's, which correspond to X 's, all ending at b . Thus $f(z)$ would have infinitely many distinct asymptotic values, namely $\exp(i \arg a_n)$, at b ; but this would contradict the theorem of Lindelöf [4, p. 9] to the effect that a bounded holomorphic function can have at most one asymptotic value at a given point.

Thus each X is a crosscut of $\{|z| < 1\}$. A second property is that *no two X 's have a common endpoint*. To see this, suppose X_1 and X_2 are two distinct X 's with a common endpoint b on $\{|z| = 1\}$. Let the corresponding curves χ_1 and χ_2 in \mathscr{W} end at points α_1 and α_2 , respectively, over $\{|w| = 1\}$. If $\alpha_1 \neq \alpha_2$ then we would again have a contradiction of Lindelöf's theorem. Now suppose $\alpha_1 = \alpha_2$. We may construct a sequence of arcs Δ_n in $\{|z| < 1\}$, each joining a point of X_1 to a point of X_2 , such that $\text{diam } \Delta_n \rightarrow 0$. Since by Lindelöf's theorem $f(z) \rightarrow \alpha_1$ uniformly between X_1 and X_2 we may also require $\text{diam } \{f(\Delta_n)\} < 1/n$. But from the structure of \mathscr{W} it is clear that there exists a curve χ on \mathscr{W} , with endpoint $\neq \alpha_1$, such that *any* curve on \mathscr{W} , joining a point of χ_1 to a point of χ_2 , must intersect χ . Since the projection of χ into $\{|w| < 1\}$ and the common projection of χ_1 and χ_2 are a positive distance δ apart, we must have $\text{diam } \{f(\Delta_n)\} \geq \delta$, which is incompatible with $\text{diam } \{f(\Delta_n)\} < 1/n$.

Next, for any $\varepsilon > 0$, the set $S = \{X \mid \text{diam } X > \varepsilon\}$ is finite. For, any disc $\{|z| < 1 - \delta\}$ intersects only a finite number of the X 's. Hence if S were infinite there would exist an infinite sequence $\{X_n\}_1^\infty$ of distinct crosscuts and a nondegenerate arc A on $\{|z| = 1\}$ such that the radius joining $z = 0$ to an arbitrary point of A crosses every X_n . Now any radial limit $f(e^{i\theta}) = e^{i\alpha}$, $e^{i\theta} \in A$, forces the χ_n , corresponding to X_n and ending at $e^{i\alpha_n}$, to satisfy $\alpha_n \rightarrow \alpha$. But then $f(e^{i\theta}) = e^{i\alpha}$ for almost all $e^{i\theta} \in A$, which contradicts the theorem of F. and M. Riesz. The point of this paragraph is that if b is a limit point of F , then any neighborhood of b contains infinitely many complete fundamental regions R . There are at least some examples of Fuchsian groups possessing a set of fundamental regions (connected) whose diameters are bounded away from zero.

5. Properties of $f(z)$ on the boundary.

THEOREM 1. Let b be a limit point of F , U a neighborhood of b , and let $e^{i\alpha} \in E$. Then the set

$$U \cap L(\alpha)$$

has the power of the continuum.

Proof. There exists a cross-cut X , corresponding to the curve χ in \mathscr{W} , which separates $\{|z| < 1\}$ into two domains, one of which, D , is contained in U . The corresponding part, \mathscr{D} , of \mathscr{W} contains infinitely many sheets. In $\{|w| < 1\}$ we may select among the arcs C_n two sequences, $\{C_n(0)\}_1^\infty$, and $\{C_n(1)\}_1^\infty$, which satisfy either the following three conditions

- (2) the lengths of the $C_n(0)$ and $C_n(1)$ tend to zero,
- (3) $\arg C_n(0) \downarrow \alpha$, and $\arg C_n(1) \downarrow \alpha$,
- (4) $\arg C_{n+1}(0) < \arg C_n(1) < \arg C_n(0)$;

or the same conditions with the arrows in (3) and the inequalities in (4) reversed. Such sequences $\{C_n(\varepsilon)\}$, $\varepsilon = 0, 1$ exist because of $e^{i\alpha} \in E$, the initial choice of $\{a_n\}$, and (1).

Now let $\Gamma(\varepsilon) = \Gamma(\varepsilon_1, \varepsilon_2, \dots)$, $\varepsilon_i = 0, 1$, be an arc in \mathscr{D} with the properties:

- (5) $\Gamma(\varepsilon)$ crosses, in order, curves χ in \mathscr{D} over the arcs $C_1(\varepsilon_1), C_2(\varepsilon_2), C_3(\varepsilon_3), \dots$, and meets no other χ 's.
- (6) $\Gamma(\varepsilon)$ tends to a point on the boundary of \mathscr{D} over $e^{i\alpha}$.

This construction of $\Gamma(\varepsilon)$ is possible by (2), (3), (4), and since all the curves χ over $\alpha < \arg w < \alpha + \delta$, $\delta = \delta(\eta)$, are of length $< \eta$. $\Gamma(\varepsilon)$ corresponds to an arc $A(\varepsilon)$ in $\{|z| < 1\}$ which tends to a definite point $b(\varepsilon) \in U \cap \{|z| = 1\}$, since $f(z) \rightarrow e^{i\alpha}$ on $A(\varepsilon)$. By a well-known theorem of Lindelöf [4, p. 10] then the radial limit of $f(z)$ exists at $b(\varepsilon)$ and has

the value of $e^{i\alpha}$.

By associating $b(\varepsilon)$ with the dyadic expansion $0.\varepsilon_1\varepsilon_2\varepsilon_3\dots$, we see that we have found a set of points $b(\varepsilon)$ in $U \cap \{|z| = 1\}$, associated with the radial limit $e^{i\alpha}$, having the power of the continuum, provided that distinct sequences of ε 's correspond to distinct points $b(\varepsilon)$. To show that, let $\{\varepsilon_i\}$ and $\{\varepsilon'_i\}$ be two distinct sequences and let p be the smallest integer for which $\varepsilon_p \neq \varepsilon'_p$. Then $C_p(\varepsilon_p)$ and $C_p(\varepsilon'_p)$ are distinct and the corresponding crosscuts $X_p(\varepsilon_p)$ and $X_p(\varepsilon'_p)$ subtend two *disjoint* (recall the structure of \mathscr{W}) closed arcs A_p and A'_p on $U \cap \{|z| = 1\}$. But $b(\varepsilon) \in A_p$ and $b(\varepsilon') \in A'_p$ and so $b(\varepsilon) \neq b(\varepsilon')$.

THEOREM 2. *Let b be a limit point of F and let U be a neighborhood of b . The set*

$$\{\theta | e^{i\theta} \in U, f(e^{i\theta}) \text{ does not exist}\}$$

has the power of the continuum.

Proof. Select three distinct arcs, $C(0)$, $C(1)$, $C(2)$, from among the arcs C_n . Suppose a curve Γ in \mathscr{W} meets, in succession, curves χ over the arcs in the sequence

$$C(\varepsilon_1), C(\varepsilon_2), C(\varepsilon_3), \dots \quad (\varepsilon_i = 0, 1, 2; \varepsilon_i \neq \varepsilon_{i+1})$$

and crosses no other χ 's. To those curves χ in \mathscr{W} which Γ meets there corresponds a sequence of crosscuts X_1, X_2, X_3, \dots , which subtend arcs A_1, A_2, A_3, \dots on $\{|z| = 1\}$ satisfying the condition $A_{n+1}^- \subset A_n^+$. Also we choose $\varepsilon_1 = 0$ and X_1 fixed, in U , so that the image of Γ lies in U . The sequence $\{\varepsilon_n\}$ then determines a unique point $b(\varepsilon) = \bigcap A_n^+ \in U$. The radius to $b(\varepsilon)$ intersects all X_n ; hence $f(z)$ has no radial limit at $b(\varepsilon)$, for $C(0)$, $C(1)$, $C(2)$ are all distinct and $\varepsilon_i \neq \varepsilon_{i+1}$. Now given the start of the sequence, $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_p$, there are two possible choices for ε_{p+1} and the two possible arcs A_{p+1} are disjoint. Thus distinct sequences $\{\varepsilon_n\}$ yield distinct points $b(\varepsilon)$. The set of sequences $\{\varepsilon_n\}$ has the power of the continuum.

THEOREM 3. *Let $e^{i\alpha} \in E'$. Then the set $L(\alpha)$ is countable.*

Proof. Let U be a neighborhood of $e^{i\alpha}$ containing none of the points a_n . Then \mathscr{W} contains a countable number of schlicht components $\mathscr{U}_1, \mathscr{U}_2, \dots$ over $U \cap \{|w| < 1\}$. Each \mathscr{U}_n maps onto $V_n \subset \{|z| < 1\}$, where V_n is bounded by an arc A_n of $\{|z| = 1\}$ and a crosscut of $\{|z| < 1\}$. The function $f(z)$ is holomorphic on A_n and there is just one radius, ending on A_n , associated with the radial limit $e^{i\alpha}$. Since \mathscr{W} contains only this countable collection of components over U , the result is clear.

We remark that if E is void, then the use of a two-point set $\{a_1, a_2\}$ leads to a Blaschke product satisfying Theorem 3. With a three-point set we can satisfy both Theorem 2 and Theorem 3. Theorem 1 is of course vacuous.

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