ON THE RADIAL LIMITS OF BLASCHKE PRODUCTS

GERALD R. MAC LANE AND FRANK BEALL RYAN
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G. R. MacLane and F. B. Ryan

1. Introduction. As is well known, a Blaschke product \( f(z) \) in \( \{ |z| < 1 \} \) has radial limits \( f(e^{i\theta}) \) of modulus one almost everywhere on \( \{ |z| = 1 \} \). The object of the present paper is to give a partial answer to the question: how many times does \( f(z) \) assume a given radial limit? We shall prove the following theorem.

**Theorem A.** Let \( E \) be a given closed set on \( \{ |w| = 1 \} \) and let \( E' \) be the complement of \( E \) relative to \( \{ |w| = 1 \} \). Then there exists a Blaschke product \( f(z) \), all of whose radial limits are of modulus one, and such that the set

\[
L(\beta) = \{ \theta | f(e^{i\theta}) = e^{i\beta} \}
\]

has the power of the continuum for \( e^{i\beta} \in E \) and is countable for \( e^{i\beta} \in E' \).

Theorem A is a condensed statement of what we shall actually prove; Theorems 1, 2, and 3 contain somewhat more information on \( f(z) \). The method of proof is to construct a suitable regularly-branched covering \( \mathcal{W} \) of \( \{ |w| < 1 \} \), corresponding to an automorphic function \( w = f(z) \), and then use the geometry of \( \mathcal{W} \) to obtain our results.

The question naturally arises as to whether one could prove Theorem A directly. That is: could one produce an \( f(z) \) with the desired properties by exhibiting its zeros instead of defining \( f(z) \) by means of a surface \( \mathcal{W} \)? The answer to this question does not seem to be obvious.

2. The surface \( \mathcal{W} \). Let \( E \) be a given nonvoid closed subset of \( \{ |w| = 1 \} \) and let \( \{ a_n \}_n \) be an infinite sequence of points in \( \{ |w| < 1 \} \) whose derived set is \( E \). Clearly, we may assume that \( a_n \neq 0 \) and

\[
(1) \quad \arg a_m \neq \arg a_n \quad (m \neq n).
\]

Let \( \mathcal{W} \) be the simply-connected unbordered covering of \( \{ |w| < 1 \} \) which is regularly-branched over the points \( \{ a_n \} \) with all branch points of multiplicity 2. It is well known [2, 3, 6] that such a covering, with any specified multiplicity or signature for each \( a_n \), exists and is unique. Instead of appealing to the general theory of regularly-branched coverings, we shall construct the surface \( \mathcal{W} \) directly, since the details of the construction play a role in the proof of Theorem A.

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Let \( C_n \) be the radial segment \( \arg w = \arg \alpha w, \) \( |a_n| \leq |w| < 1. \) The \( C_n \) are disjoint because of (1). We make cuts in \( \{|w| < 1\} \) along each \( C_n \) and so obtain a slit disc \( W, \) copies of which are joined together, according to the following specifications, to form the surface.

**0\(^{th}\) level.** The surface \( \mathcal{W}_0 \) consists of just one slit disc \( W. \) Note that \( \mathcal{W}_0 \) is simply-connected.

**1\(^{st}\) level.** The surface \( \mathcal{W}_1 \) is obtained by adjoining an infinite sequence of distinct copies of \( W, \) namely \( W(1), W(2), \ldots, \) to \( \mathcal{W}_0. \) \( W(n_i) \) is joined to \( \mathcal{W}_0 \) along \( C_{n_i} \) so as to form a first-order branch-point over \( a_{n_i}. \) The surface \( \mathcal{W}_1 = \mathcal{W}_0 \cup \bigcup_n W(n_i) \) is simply-connected; for by adjoining the \( W(n_i) \) one at a time we obtain an increasing sequence of simply-connected surfaces which exhaust \( \mathcal{W}_1. \) We denote by \( \chi(n_i) \) the curve in \( \mathcal{W}_1 \) along which \( W(n_i) \) and \( W(n_i) \) are joined.

**2\(^{nd}\) level.** Along each free slit on the boundary of \( \mathcal{W}_1, \) we adjoin a copy of \( W. \) More precisely, the sheet \( W(n_1, n_2) \) is adjoined to \( W(n_i) \) along the cut \( C_{n_2} \) in \( W(n_i). \) The added sheets correspond one-to-one with all pairs \( (n_1, n_2) \) of positive integers such that \( n_1 \neq n_2. \) Again we see that the surface \( \mathcal{W}_2 = \mathcal{W}_1 \cup \bigcup W(n_1, n_2) \) is simply-connected. The curve over \( C_{n_2} \) along which \( W(n_1) \) and \( W(n_1, n_2) \) are joined is denoted by \( \chi(n_1, n_2). \)

**\( k \)^{th} level.** Continuing the construction, the surface \( \mathcal{W}_k \) consists of \( \mathcal{W}_{k-1} \) and copies of \( W \) denoted by \( W(n_1, n_2, \ldots, n_k), \) \( n_i \neq n_{i+1}, \) which are joined to \( \mathcal{W}_{k-1}; \) \( W(n_1, \ldots, n_k) \) is adjoined to \( W(n_1, \ldots, n_{k-1}) \) along the cut \( C_{n_k} \) in \( W(n_1, \ldots, n_{k-1}). \) Denote the curve along which those two sheets are joined by \( \chi(n_1, n_2, \ldots, n_k). \) Clearly \( \mathcal{W}_k \) is simply-connected.

We take the surface \( \mathcal{W} \) to be \( \lim \mathcal{W}_k \) as \( k \to \infty; \) it is clear that \( \mathcal{W} \) is simply-connected as \( \mathcal{W}_k \uparrow \mathcal{W}. \) With the natural projection map onto \( \{|w| < 1\} \) it is clear that \( \mathcal{W} \) is a regularly-branched, unbordered, covering of \( \{|w| < 1\}. \) All points of \( \mathcal{W} \) over the \( a_n \) are branch-points of multiplicity 2, and \( \mathcal{W} \) has no other branch-points.

**3. The function \( f(z). \)** Since \( \mathcal{W} \) is a covering of \( \{|w| < 1\} \) it is hyperbolic. Let \( w = f(z) \) be the holomorphic function which maps \( \{|z| < 1\} \) onto \( \mathcal{W}, \) with \( f(0) = 0 \in \mathcal{W}_0 \) and \( f'(0) > 0. \) Clearly \( |f(z)| < 1. \) The radial limits of \( f(z) \) are all of modulus one, since if this were not the case a boundary point of \( \{|z| < 1\} \) would correspond to an interior point of \( \mathcal{W} \) which is unbordered. Thus \( f(z) \) is of class \( U \) [5, p. 32]. Applying Frostman's theorem [5, p. 33] we see that \( f(z) \) is a Blaschke product.

Also, \( f(z) \) is an automorphic function with respect to a Fuschian group \( F, \) since the decktransformations of \( \mathcal{W} \) correspond to linear
transformations preserving \(|z| < 1\). It is easily shown that if \(E = \{|w| = 1\}\) then \(F\) is of the first kind: the limit points of \(F\) fill \(|z| = 1\). If \(E \neq \{|w| = 1\}\), then \(F\) is of the second kind: the set of limit points of \(F\) is a perfect nowhere dense subset of \(|z| = 1\).

The sheets \(W(n_1, n_2, \ldots, n_k)\) of \(\mathbb{W}\) correspond to a set of fundamental regions \(R(n_1, \ldots, n_k)\) of \(F\). These are the fundamental regions which play a role in the proof; since these are defined via the function \(f\) it is not clear that they are the same as the fundamental regions obtained by any of the usual constructions in terms of \(F\). Hence we must derive some properties of these regions.

4. Properties of the fundamental regions. For convenience we reduce the notations \(W(n_1, \ldots, n_k), R(n_1, \ldots, n_k), \) and \(\chi(n_1, \ldots, n_k)\) to \(W, R, \) and \(\chi\) respectively. To each curve \(\chi\) in \(\mathbb{W}\) there corresponds a simple arc \(X\) in \(|z| < 1\). It is evident that the fundamental regions \(R\) are bounded by the \(X\)'s and points of \(|z| = 1\). We proceed with an investigation of the \(X\)'s.

First, each \(X\) ends at two distinct points of \(|z| = 1\). The two linear pieces of \(\chi\) correspond to two simple arcs \(X'\) and \(X''\), and \(f(z)\) tends to a limit as \(|z| \to 1\) on \(X'\) and \(X''\). Then by Koebe's lemma [1, p. 213] each of \(X'\) and \(X''\) must tend to a definite point of \(|z| = 1\). The end points of \(X'\) and \(X''\) must be distinct. If not, let \(D\) be that part of \(|z| < 1\) bounded by \(X\) and a single point \(b\) on \(|z| = 1\). Then the part of \(\mathbb{W}\) corresponding to \(D\) will contain an infinite number of sheets \(W\) joined along various \(\chi\)'s, which correspond to \(X\)'s, all ending at \(b\). Thus \(f(z)\) would have infinitely many distinct asymptotic values, namely \(\exp(i \arg a_n)\), at \(b\); but this would contradict the theorem of Lindelöf [4, p. 9] to the effect that a bounded holomorphic function can have at most one asymptotic value at a given point.

Thus each \(X\) is a crosscut of \(|z| < 1\). A second property is that no two \(X\)'s have a common endpoint. To see this, suppose \(X_1\) and \(X_2\) are two distinct \(X\)'s with a common endpoint \(b\) on \(|z| = 1\). Let the corresponding curves \(\chi_1\) and \(\chi_2\) in \(\mathbb{W}\) end at points \(\alpha_1\) and \(\alpha_2\), respectively, over \(|w| = 1\). If \(\alpha_1 \neq \alpha_2\), then we would again have a contradiction of Lindelöf's theorem. Now suppose \(\alpha_1 = \alpha_2\). We may construct a sequence of arcs \(A_n\) in \(|z| < 1\), each joining a point of \(X_1\) to a point of \(X_2\), such that \(\text{diam } A_n \to 0\). Since by Lindelöf's theorem \(f(z) \to \alpha_1\) uniformly between \(X_1\) and \(X_2\), we may also require \(\text{diam } \{f(A_n)\} < 1/n\). But from the structure of \(\mathbb{W}\) it is clear that there exists a curve \(\chi\) on \(\mathbb{W}\), with endpoint \(\neq \alpha_n\), such that any curve on \(\mathbb{W}\), joining a point of \(\chi_1\) to a point of \(\chi_2\), must intersect \(\chi\). Since the projection of \(\chi\) into \(|w| < 1\) and the common projection of \(\chi_1\) and \(\chi_2\) are a positive distance \(\delta\) apart, we must have \(\text{diam } \{f(A_n)\} \geq \delta\), which is incompatible with \(\text{diam } \{f(A_n)\} < 1/n\).
Next, for any \( \varepsilon > 0 \), the set \( S = \{ X \, | \, \text{diam} \, X > \varepsilon \} \) is finite. For, any disc \( \{|z| < 1 - \delta\} \) intersects only a finite number of the \( X \)'s. Hence if \( S \) were infinite there would exist an infinite sequence \( \{X_n\}_n \) of distinct crosscuts and a nondegenerate arc \( A \) on \( \{z\, | \, \text{diam} \, X > \varepsilon \} \) such that the radius joining \( z = 0 \) to an arbitrary point of \( A \) crosses every \( X_n \). Now any radial limit \( f(e^{i\theta}) = e^{ia} \), \( e^{ia} \in A \), forces the \( \chi_n \), corresponding to \( X_n \) and ending at \( e^{ia_n} \), to satisfy \( a_n \to a \). But then \( f(e^{i\theta}) = e^{ia} \) for almost all \( e^{ia} \in A \), which contradicts the theorem of F. and M. Riesz. The point of this paragraph is that if \( b \) is a limit point of \( F \), then any neighborhood of \( b \) contains infinitely many complete fundamental regions \( R \). There are at least some examples of Fuchsian groups possessing a set of fundamental regions (connected) whose diameters are bounded away from zero.

5. Properties of \( f(z) \) on the boundary.

THEOREM 1. Let \( b \) be a limit point of \( F \), \( U \) a neighborhood of \( b \), and let \( e^{ia} \in E \). Then the set

\[
U \cap L(\alpha)
\]

has the power of the continuum.

Proof. There exists a cross-cut \( X \), corresponding to the curve \( \chi \) in \( \mathcal{W} \), which separates \( \{z\, | \, |z| < 1\} \) into two domains, one of which, \( D \), is contained in \( U \). The corresponding part, \( \mathcal{D} \), of \( \mathcal{W} \) contains infinitely many sheets. In \( \{ |w| < 1 \} \) we may select among the arcs \( C_n \) two sequences, \( \{C_n(0)\}_n \) and \( \{C_n(1)\}_n \), which satisfy either the following three conditions

(2) the lengths of the \( C_n(0) \) and \( C_n(1) \) tend to zero,
(3) \( \arg C_n(0) \downarrow \alpha \), and \( \arg C_n(1) \uparrow \alpha \),
(4) \( \arg C_{n+1}(0) < \arg C_n(1) < \arg C_n(0) \);

or the same conditions with the arrows in (3) and the inequalities in (4) reversed. Such sequences \( \{C_n(\varepsilon)\} \), \( \varepsilon = 0, 1 \) exist because of \( e^{ia} \in E \), the initial choice of \( \{a_n\} \), and (1).

Now let \( \Gamma(\varepsilon) = \Gamma(\varepsilon_1, \varepsilon_2, \ldots) \), \( \varepsilon_i = 0, 1 \), be an arc in \( \mathcal{D} \) with the properties:

(5) \( \Gamma(\varepsilon) \) crosses, in order, curves \( \chi \) in \( \mathcal{D} \) over the arcs \( C_1(\varepsilon_1), C_2(\varepsilon_2), C_3(\varepsilon_3), \ldots \), and meets no other \( \chi \)'s.
(6) \( \Gamma(\varepsilon) \) tends to a point on the boundary of \( \mathcal{D} \) over \( e^{ia} \).

This construction of \( \Gamma(\varepsilon) \) is possible by (2), (3), (4), and since all the curves \( \chi \) over \( \alpha < \arg w < \alpha + \delta, \delta = \delta(\eta) \), are of length \( < \eta \). \( \Gamma(\varepsilon) \) corresponds to an arc \( \mathcal{A}(\varepsilon) \) in \( \{|z| < 1\} \) which tends to a definite point \( b(\varepsilon) \in U \cap \{|z| = 1\} \), since \( f(z) \to e^{ia} \) on \( \mathcal{A}(\varepsilon) \). By a well-known theorem of Lindelöf [4, p. 10] then the radial limit of \( f(z) \) exists at \( b(\varepsilon) \) and has
the value of $e^{ia}$.

By associating $b(\varepsilon)$ with the dyadic expansion $0 \cdot \varepsilon_1 \varepsilon_2 \varepsilon_3 \cdots$, we see that we have found a set of points $b(\varepsilon)$ in $U \cap \{|z| = 1\}$, associated with the radial limit $e^{ia}$, having the power of the continuum, provided that distinct sequences of $\varepsilon$'s correspond to distinct points $b(\varepsilon)$. To show that, let $\{\varepsilon_i\}$ and $\{\varepsilon'_i\}$ be two distinct sequences and let $p$ be the smallest integer for which $\varepsilon_p \neq \varepsilon'_p$. Then $C_p(\varepsilon_p)$ and $C_p(\varepsilon'_p)$ are distinct and the corresponding crosscuts $X_p(\varepsilon_p)$ and $X_p(\varepsilon'_p)$ subtend two disjoint (recall the structure of $\mathcal{W}$) closed arcs $A_p$ and $A'_p$ on $U \cap \{|z| = 1\}$. But $b(\varepsilon) \in A_p$ and $b(\varepsilon') \in A'_p$ and so $b(\varepsilon) \not= b(\varepsilon')$.

**Theorem 2.** Let $b$ be a limit point of $F$ and let $U$ be a neighborhood of $b$. The set

$$\{\theta \mid e^{i\theta} \in U, f(e^{i\theta}) \text{ does not exist}\}$$

has the power of the continuum.

**Proof.** Select three distinct arcs, $C(0)$, $C(1)$, $C(2)$, from among the arcs $C_n$. Suppose a curve $I'$ in $\mathcal{W}$ meets, in succession, curves $\chi$ over the arcs in the sequence

$$C(\varepsilon_i), C(\varepsilon_2), C(\varepsilon_3), \cdots \quad (\varepsilon_i = 0, 1, 2; \; \varepsilon_i \not= \varepsilon_{i+1})$$

and crosses no other $\chi$'s. To those curves $\chi$ in $\mathcal{W}$ which $I'$ meets there corresponds a sequence of crosscuts $X_1, X_2, X_3, \cdots$, which subtend arcs $A_1, A_2, A_3, \cdots$ on $\{|z| = 1\}$ satisfying the condition $A_{n+1} \subset A_n$. Also we choose $\varepsilon_1 = 0$ and $X_1$ fixed, in $U$, so that the image of $I'$ lies in $U$. The sequence $\{\varepsilon_i\}$ then determines a unique point $b(\varepsilon) = \bigcap A_n \in U$. The radius to $b(\varepsilon)$ intersects all $X_n$; hence $f(z)$ has no radial limit at $b(\varepsilon)$, for $C(0), C(1), C(2)$ are all distinct and $\varepsilon_i \not= \varepsilon_{i+1}$. Now given the start of the sequence, $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_p$, there are two possible choices for $\varepsilon_{p+1}$ and the two possible arcs $A_{p+1}$ are disjoint. Thus distinct sequences $\{\varepsilon_i\}$ yield distinct points $b(\varepsilon)$. The set of sequences $\{\varepsilon_i\}$ has the power of the continuum.

**Theorem 3.** Let $e^{ia} \in E'$. Then the set $L(\alpha)$ is countable.

**Proof.** Let $U$ be a neighborhood of $e^{ia}$ containing none of the points $a_n$. Then $\mathcal{W}$ contains a countable number of schlicht components $\mathcal{W}_1, \mathcal{W}_2, \cdots$ over $U \cap \{|w| < 1\}$. Each $\mathcal{W}_n$ maps onto $V_n \subset \{|z| < 1\}$, where $V_n$ is bounded by an arc $A_n$ of $\{|z| = 1\}$ and a crosscut of $\{|z| < 1\}$. The function $f(z)$ is holomorphic on $A_n$ and there is just one radius, ending on $A_n$, associated with the radial limit $e^{ia}$. Since $\mathcal{W}$ contains only this countable collection of components over $U$, the result is clear.
We remark that if $E$ is void, then the use of a two-point set $\{a_1, a_2\}$ leads to a Blaschke product satisfying Theorem 3. With a three-point set we can satisfy both Theorem 2 and Theorem 3. Theorem 1 is of course vacuous.

REFERENCES


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