

Pacific Journal of Mathematics

**EVALUATION OF AN E -FUNCTION WHEN THREE OF ITS
UPPER PARAMETERS DIFFER BY INTEGRAL VALUES**

T. M. MACROBERT

EVALUATION OF AN E -FUNCTION WHEN THREE OF ITS UPPER PARAMETERS DIFFER BY INTEGRAL VALUES

T. M. MACROBERT.

1. Introduction. If $p \geq q + 1$, [1, p. 353]

$$(1) \quad E(p; \alpha_r; q; \rho_s; z) = \sum_{r=1}^p z^{\alpha_r} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_r + n) \prod_{t=1}^p \Gamma(\alpha_t - \alpha_r - n)}{n! \prod_{s=1}^q \Gamma(\rho_s - \alpha_r - n)} (-z)^n,$$

where, if $p = q + 1$, $|z| < 1$. The dash in the product sign indicates that the factor for which $t = r$ is omitted.

Now, if two or more of the α 's are equal or differ by integral values, some of the series on the right cease to exist. For instance, if $\alpha_1 = \alpha + l$, $\alpha_2 = \alpha$, where l is zero or a positive integer, it has been shown [2, p. 30] that the first two series can be replaced by the expression

$$(2) \quad \begin{aligned} & (-1)^l z^{\alpha+l} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + l + n) \prod_{t=3}^p \Gamma(\alpha_t - \alpha - l - n)}{n!(l+n)! \prod_{s=1}^q \Gamma(\rho_s - \alpha - l - n)} \Delta_n z^n \\ & + z^{\alpha} \sum_{n=0}^{l-1} \frac{\Gamma(\alpha + n)(l - n - 1)! \prod_{t=3}^p \Gamma(\alpha_t - \alpha - n)}{n! \prod_{s=1}^q \Gamma(\rho_s - \alpha - n)} (-z)^n, \end{aligned}$$

where

$$\begin{aligned} \Delta_n &= \psi(l + n) + \psi(n) - \psi(\alpha + l + n - 1) - \log z \\ &+ \sum_{t=3}^p \psi(\alpha_t - \alpha - l - n - 1) - \sum_{s=1}^q \psi(\rho_s - \alpha - l - n - 1). \end{aligned}$$

Here

$$(3) \quad \psi(z) = \frac{d}{dz} \log \Gamma(z + 1),$$

so that

$$(4) \quad \frac{d}{dz} \Gamma(z + 1) = \Gamma(z + 1) \psi(z).$$

It will now be shown that, in the case in which

Received September 28, 1961.

$$\alpha_1 = \alpha, \alpha_2 = \alpha + l, \alpha_3 = \alpha + l + m,$$

where l and m are zero or positive integers, the first three series can be replaced by the expression

$$\begin{aligned} & \frac{1}{2}(-1)^l z^{\alpha+l+m} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+l+m+n) \prod_{t=4}^p \Gamma(\alpha_t - \alpha - l - m - n) (-z)^n}{n!(m+n)!(l+m+n)! \prod_{s=1}^q \Gamma(\rho_s - \alpha - l - m - n)} A_n \\ (5) \quad & - (-1)^l z^{\alpha+l} \sum_{n=0}^{m-1} \frac{\Gamma(\alpha+l+n)(m-n-1)! \prod_{t=4}^p \Gamma(\alpha_t - \alpha - l - n) z^n}{n!(l+n)! \prod_{s=1}^q \Gamma(\rho_s - \alpha - l - n)} \theta_n \\ & + z^\alpha \sum_{n=0}^{l-1} \frac{\Gamma(\alpha+n)(l-n-1)!(l+m-n-1)! \prod_{t=4}^p \Gamma(\alpha_t - \alpha - n)}{n! \prod_{s=1}^q \Gamma(\rho_s - \alpha - n)} \\ & \qquad \qquad \qquad \times (-z)^n, \end{aligned}$$

where

$$\begin{aligned} A_n = \pi^2 & \left\{ \begin{aligned} & \log z + \psi(\alpha+l+m+n-1) - \psi(l+m+n) - \psi(m+n) \\ & - \psi(n) - \sum_{t=4}^p \psi(\alpha_t - \alpha - l - m - n - 1) \\ & + \sum_{s=1}^q \psi(\rho_s - \alpha - l - m - n - 1) \end{aligned} \right\}^2 \\ & + \chi(\alpha+l+m+n-1) - \chi(l+m+n) - \chi(m+n) - \chi(n) \\ & + \sum_{t=4}^p \chi(\alpha_t - \alpha - l - m - n - 1) - \sum_{s=1}^q \chi(\rho_s - \alpha - l - m - n - 1), \end{aligned}$$

and

$$\begin{aligned} \theta_n = \log z + \psi(\alpha+l+n-1) - \psi(l+n) - \psi(m-n-1) - \psi(n) \\ - \sum_{t=4}^p \psi(\alpha_t - \alpha - l - n - 1) + \sum_{s=1}^q \psi(\rho_s - \alpha - l - n - 1). \end{aligned}$$

Here

$$(6) \quad \chi(z) = \psi'(z) = \sum_{n=1}^{\infty} \frac{1}{(z+n)^2},$$

where $|\text{amp } z| < \pi$.

2. Proof of the formula. If $\alpha_1 = \alpha, \alpha_2 = \alpha + l, \alpha_3 = \alpha + l + m + \varepsilon$, where l and m are zero or positive integers and ε is small, it follows from (2) and (1) that the first 3 terms of (1) are equal to $A + B + C$, where

$$A = (-1)^l z^{\alpha+l} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+l+n)\Gamma(m-n+\varepsilon)\prod_{t=4}^p \Gamma(\alpha_t-\alpha-l-n)}{n!(l+n)!\prod_{s=4}^q \Gamma(\rho_s-\alpha-l-n)} K_n z^n,$$

where

$$K_n = \psi(l+n) + \psi(n) + \psi(m-n-1+\varepsilon) - \psi(\alpha+l+n-1) - \log z + \sum_{t=4}^p \psi(\alpha_t-\alpha-l-n-1) - \sum_{s=1}^q \psi(\rho_s-\alpha-l-n-1),$$

$$B = z^\alpha \sum_{n=0}^{l-1} \frac{\Gamma(\alpha+n)(l-n-1)!\Gamma(l+m-n+\varepsilon)\prod_{t=4}^p \Gamma(\alpha_t-\alpha-n)}{n!\prod_{s=1}^q \Gamma(\rho_s-\alpha-n)}$$

$(-z)^n ;$

$$C = z^{\alpha+l+m+\varepsilon}$$

$$\times \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+l+m+n+\varepsilon)\Gamma(-l-m-n-\varepsilon)\Gamma(-m-n-\varepsilon)}{n!\prod_{s=1}^q \Gamma(\rho_s-\alpha-l-m-n-\varepsilon)} \times \prod_{t=4}^p \Gamma(\alpha_t-\alpha-l-m-n-\varepsilon) \times (-z)^n .$$

Then $A = D + E$, where

$$D = (-1)^l z^{\alpha+l} \sum_{n=0}^{m-1} \frac{\Gamma(\alpha+l+n)\Gamma(m-n+\varepsilon)\prod_{t=4}^p \Gamma(\alpha_t-\alpha-l-n)}{n!(l+n)!\prod_{s=4}^q \Gamma(\rho_s-\alpha-l-n)} K_n z^n,$$

$$E = (-1)^l z^{\alpha+l+m} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+l+m+n)\Gamma(-n+\varepsilon)\prod_{t=4}^p \Gamma(\alpha_t-\alpha-l-m-n)}{(m+n)!(l+m+n)!\prod_{s=4}^q \Gamma(\rho_s-\alpha-l-m-n)} L_n z^n,$$

where

$$L_n = \psi(l+m+n) + \psi(m+n) + \psi(-n-1+\varepsilon) - \psi(\alpha+l+m+n-1) - \log z + \sum \psi(\alpha_t-\alpha-l-m-n-1) - \sum \psi(\rho_s-\alpha-l-m-n-1).$$

Note. In these formulae t takes the values from 4 to p . Here, since [2, p. 31]

$$(7) \quad \psi(-z-1) = \psi(z) + \pi \cot \pi z,$$

$$(8) \quad \psi(-n-1+\varepsilon) = \psi(n-\varepsilon) - \pi \cot \pi \varepsilon.$$

Hence

$$C + E = \frac{\pi^2}{\sin^2 \pi \varepsilon} (-1)^l z^{\alpha+l+m}$$

$$\times \left[\begin{aligned} & z^\varepsilon \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + l + m + n + \varepsilon)}{n! \Gamma(m + n + 1 + \varepsilon)} \\ & \frac{\prod \Gamma(\alpha_i - \alpha - l - m - n - \varepsilon) (-z)^n}{\Gamma(l + m + n + 1 + \varepsilon) \times \prod \Gamma(\rho_s - \alpha - l - m - n - \varepsilon)} \\ & + \frac{\sin \pi \varepsilon}{\pi} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + l + m + n) \prod \Gamma(\alpha_i - \alpha - l - m - n) (-z)^n}{\Gamma(n + 1 - \varepsilon) (m + n)! (l + m + n)! \prod \Gamma(\rho_s - \alpha - l - m - n)} \\ & \times \left[\begin{aligned} & \psi(l + m + n) + \psi(m + n) + \psi(n - \varepsilon) - \psi(\alpha + l + m + n - 1) \\ & - \log z + \sum \psi(\alpha_i - \alpha - l - m - n - 1) \\ & \qquad \qquad \qquad - \sum \psi(\rho_s - \alpha - l - m - n - 1) \end{aligned} \right] \\ & - \cos \pi \varepsilon \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + l + m + n) \prod \Gamma(\alpha_i - \alpha - l - m - n) (-z)^n}{\Gamma(n + 1 - \varepsilon) (m + n)! (l + m + n)! \prod \Gamma(\rho_s - \alpha - l - m - n)} \end{aligned} \right].$$

The limit of this function when $\varepsilon \rightarrow 0$ is obtained by replacing $\pi^2/\sin^2 \pi \varepsilon$ by $\frac{1}{2}$, finding the second derivative with regard to ε of the expression in the large bracket, and then making $\varepsilon \rightarrow 0$. It is

$$\frac{1}{2} (-1)^l z^{\alpha+l+m} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + l + m + n) \prod \Gamma(\alpha_i - \alpha - l - m - n) (-z)^n}{n! (m + n)! (l + m + n)! \prod \Gamma(\rho_s - \alpha - l - m - n)} \times \left[\begin{aligned} & \left\{ \begin{aligned} & (\log z + \psi(\alpha + l + m + n - 1) - \psi(m + n) - \psi(l + m + n)) \\ & - \sum \psi(\alpha_i - \alpha - l - m - n - 1) + \sum \psi(\rho_s - \alpha - l - m - n - 1) \end{aligned} \right\}^2 \\ & + \chi(\alpha + l + m + n - 1) - \chi(m + n) - \chi(l + m + n) \\ & + \sum \chi(\alpha_i - \alpha - l - m - n - 1) - \sum \chi(\rho_s - \alpha - l - m - n - 1) \\ & + 2\psi(n) \left\{ \begin{aligned} & \psi(l + m + n) + \psi(m + n) + \psi(n) \\ & \qquad \qquad \qquad - \psi(\alpha + l + m + n - 1) \end{aligned} \right\} \\ & - \log z + \sum \psi(\alpha_i - \alpha - l - m - n - 1) \\ & \qquad \qquad \qquad - \sum \psi(\rho_s - \alpha - l - m - n - 1) \end{aligned} \right] \\ & - 2\chi(n) + \pi^2 - \{\psi(n)\}^2 + \chi(n) \end{aligned} \right].$$

From this, with B and D , formula (5) is obtained.

REFERENCES

1. T. M. MacRobert, *Functions of a complex variable*, London, (1954).
2. T. M. MacRobert, *Evaluation of an E-function when two of the upper parameters differ by an integer*, Proc. Glasgow Math. Assoc., **5** (1961).

THE UNIVERSITY,
GLASGOW

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RALPH S. PHILLIPS

Stanford University
Stanford, California

M. G. ARSOVE

University of Washington
Seattle 5, Washington

A. L. WHITEMAN

University of Southern California
Los Angeles 7, California

LOWELL J. PAIGE

University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH

T. M. CHERRY

D. DERRY

M. OHTSUKA

H. L. ROYDEN

E. SPANIER

E. G. STRAUS

F. WOLF

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 12, No. 3

March, 1962

Alfred Aeppli, <i>Some exact sequences in cohomology theory for Kähler manifolds</i>	791
Paul Richard Beesack, <i>On the Green's function of an N-point boundary value problem</i>	801
James Robert Boen, <i>On p-automorphic p-groups</i>	813
James Robert Boen, Oscar S. Rothaus and John Griggs Thompson, <i>Further results on p-automorphic p-groups</i>	817
James Henry Bramble and Lawrence Edward Payne, <i>Bounds in the Neumann problem for second order uniformly elliptic operators</i>	823
Chen Chung Chang and H. Jerome (Howard) Keisler, <i>Applications of ultraproducts of pairs of cardinals to the theory of models</i>	835
Stephen Urban Chase, <i>On direct sums and products of modules</i>	847
Paul Civin, <i>Annihilators in the second conjugate algebra of a group algebra</i>	855
J. H. Curtiss, <i>Polynomial interpolation in points equidistributed on the unit circle</i>	863
Marion K. Fort, Jr., <i>Homogeneity of infinite products of manifolds with boundary</i>	879
James G. Glimm, <i>Families of induced representations</i>	885
Daniel E. Gorenstein, Reuben Sandler and William H. Mills, <i>On almost-commuting permutations</i>	913
Vincent C. Harris and M. V. Subba Rao, <i>Congruence properties of $\sigma_r(N)$</i>	925
Harry Hochstadt, <i>Fourier series with linearly dependent coefficients</i>	929
Kenneth Myron Hoffman and John Wermer, <i>A characterization of $C(X)$</i>	941
Robert Weldon Hunt, <i>The behavior of solutions of ordinary, self-adjoint differential equations of arbitrary even order</i>	945
Edward Takashi Kobayashi, <i>A remark on the Nijenhuis tensor</i>	963
David London, <i>On the zeros of the solutions of $w''(z) + p(z)w(z) = 0$</i>	979
Gerald R. Mac Lane and Frank Beall Ryan, <i>On the radial limits of Blaschke products</i>	993
T. M. MacRobert, <i>Evaluation of an E-function when three of its upper parameters differ by integral values</i>	999
Robert W. McKelvey, <i>The spectra of minimal self-adjoint extensions of a symmetric operator</i>	1003
Adegoke Olubummo, <i>Operators of finite rank in a reflexive Banach space</i>	1023
David Alexander Pope, <i>On the approximation of function spaces in the calculus of variations</i>	1029
Bernard W. Roos and Ward C. Sangren, <i>Three spectral theorems for a pair of singular first-order differential equations</i>	1047
Arthur Argyle Sagle, <i>Simple Malcev algebras over fields of characteristic zero</i>	1057
Leo Sario, <i>Meromorphic functions and conformal metrics on Riemann surfaces</i>	1079
Richard Gordon Swan, <i>Factorization of polynomials over finite fields</i>	1099
S. C. Tang, <i>Some theorems on the ratio of empirical distribution to the theoretical distribution</i>	1107
Robert Charles Thompson, <i>Normal matrices and the normal basis in abelian number fields</i>	1115
Howard Gregory Tucker, <i>Absolute continuity of infinitely divisible distributions</i>	1125
Elliot Carl Weinberg, <i>Completely distributed lattice-ordered groups</i>	1131
James Howard Wells, <i>A note on the primes in a Banach algebra of measures</i>	1139
Horace C. Wisner, <i>Decomposition and homogeneity of continua on a 2-manifold</i>	1145