Pacific Journal of Mathematics

EVALUATION OF AN E-FUNCTION WHEN THREE OF ITS UPPER PARAMETERS DIFFER BY INTEGRAL VALUES

T. M. MacRobert

Vol. 12, No. 3 March 1962

EVALUATION OF AN E-FUNCTION WHEN THREE OF ITS UPPER PARAMETERS DIFFER BY INTEGRAL VALUES

T. M. MACROBERT.

1. Introduction. If $p \geqq q+1$, [1, p. 353]

$$(1)\qquad E(p;\alpha_r\colon q;\rho_s\colon z)=\sum_{r=1}^{p} z^{\alpha_r}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha_r+n)\prod_{t=1}^{p}{}'\Gamma(\alpha_t-\alpha_r-n)}{n!\prod_{s=1}^{q}\Gamma(\rho_s-\alpha_r-n)}(-z)^n\,,$$

where, if $p=q+1$, $|z|<1$. The dash in the product sign indicates that the factor for which $t=r$ is omitted.

Now, if two or more of the α's are equal or differ by integral values, some of the series on the right cease to exist. For instance, if $\alpha_1=\alpha+l$, $\alpha_2=\alpha$, where l is zero or a positive integer, it has been shown [2, p. 30] that the first two series can be replaced by the expression

$$(2)\qquad \begin{aligned}&(-1)^l z^{\alpha+l}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+l+n)\prod_{t=3}^{p}\Gamma(\alpha_t-\alpha-l-n)}{n!(l+n)!\prod_{s=1}^{q}\Gamma(\rho_s-\alpha-l-n)}\Delta_n z^n\\ &+z^{\alpha}\sum_{n=0}^{l-1}\frac{\Gamma(\alpha+n)(l-n-1)!\prod_{t=3}^{p}\Gamma(\alpha_t-\alpha-n)}{n!\prod_{s=1}^{q}\Gamma(\rho_s-\alpha-n)}(-z)^n\,,\end{aligned}$$

where

$$\begin{aligned}\Delta_n&=\psi(l+n)+\psi(n)-\psi(\alpha+l+n-1)-\log z\\ &+\sum_{t=3}^{p}\psi(\alpha_t-\alpha-l-n-1)-\sum_{s=1}^{q}\psi(\rho_s-\alpha-l-n-1)\,.\end{aligned}$$

Here

$$(3)\qquad \psi(z)=\frac{d}{dz}\log\Gamma(z+1)\,,$$

so that

$$(4)\qquad \frac{d}{dz}\Gamma(z+1)=\Gamma(z+1)\psi(z)\,.$$

It will now be shown that, in the case in which

Received September 28, 1961.

$$\alpha_1 = \alpha,\ \alpha_2 = \alpha + l,\ \alpha_3 = \alpha + l + m\ ,$$

where l and m are zero or positive integers, the first three series can be replaced by the expression

$$\tfrac{1}{2}(-1)^l z^{\alpha+l+m}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+l+m+n)\prod_{t=4}^{p}\Gamma(\alpha_t-\alpha-l-m-n)(-z)^n}{n!(m+n)!(l+m+n)!\prod_{s=1}^{q}\Gamma(\rho_s-\alpha-l-m-n)}\Lambda_n$$

(5) $$-(-1)^l z^{\alpha+l}\sum_{n=0}^{m-1}\frac{\Gamma(\alpha+l+n)(m-n-1)!\prod_{t=4}^{p}\Gamma(\alpha_t-\alpha-l-n)z^n}{n!(l+n)!\prod_{s=1}^{q}\Gamma(\rho_s-\alpha-l-n)}\Theta_n$$

$$+\ z^{\alpha}\sum_{n=0}^{l-1}\frac{\Gamma(\alpha+n)(l-n-1)!(l+m-n-1)!\prod_{t=4}^{p}\Gamma(\alpha_t-\alpha-n)}{n!\prod_{s=1}^{q}\Gamma(\rho_s-\alpha-n)} \times (-z)^n\ ,$$

where

$$\begin{aligned}\Lambda_n = \pi^2 & \\ + & \left\{\begin{array}{l}\log z + \psi(\alpha+l+m+n-1) - \psi(l+m+n) - \psi(m+n) \\ -\ \psi(n) - \sum_{t=4}^{p}\psi(\alpha_t-\alpha-l-m-n-1) \\ +\ \sum_{s=1}^{q}\psi(\rho_s-\alpha-l-m-n-1)\end{array}\right\}^2 \\ & + \chi(\alpha+l+m+n-1) - \chi(l+m+n) - \chi(m+n) - \chi(n) \\ & + \sum_{t=4}^{p}\chi(\alpha_t-\alpha-l-m-n-1) - \sum_{s=1}^{q}\chi(\rho_s-\alpha-l-m-n-1)\ ,\end{aligned}$$

and

$$\begin{aligned}\Theta_n = & \log z + \psi(\alpha+l+n-1) - \psi(l+n) - \psi(m-n-1) - \psi(n) \\ & - \sum_{t=4}^{p}\psi(\alpha_t-\alpha-l-n-1) + \sum_{s=1}^{q}\psi(\rho_s-\alpha-l-n-1)\ .\end{aligned}$$

Here

(6) $$\chi(z) = \psi'(z) = \sum_{n=1}^{\infty}\frac{1}{(z+n)^2}\ ,$$

where $|\operatorname{amp} z| < \pi$.

2. Proof of the formula. If $\alpha_1 = \alpha, \alpha_2 = \alpha + l, \alpha_3 = \alpha + l + m + \varepsilon$, where l and m are zero or positive integers and ε is small, it follows from (2) and (1) that the first 3 terms of (1) are equal to $A + B + C$, where

$$A=(-1)^l z^{\alpha+l}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+l+n)\Gamma(m-n+\varepsilon)\prod_{t=4}^{p}\Gamma(\alpha_t-\alpha-l-n)}{n!(l+n)!\prod_{s=4}^{q}\Gamma(\rho_s-\alpha-l-n)}K_n z^n ,$$

where

$$K_n=\psi(l+n)+\psi(n)+\psi(m-n-1+\varepsilon)-\psi(\alpha+l+n-1)$$
$$-\log z+\sum_{t=4}^{p}\psi(\alpha_t-\alpha-l-n-1)-\sum_{s=1}^{q}\psi(\rho_s-\alpha-l-n-1) ,$$

$$B=z^{\alpha}\sum_{n=0}^{l-1}\frac{\Gamma(\alpha+n)(l-n-1)!\Gamma(l+m-n+\varepsilon)\prod_{t=4}^{p}\Gamma(\alpha_t-\alpha-n)}{n!\prod_{s=1}^{q}\Gamma(\rho_s-\alpha-n)}(-z)^n ;$$

$$C=z^{\alpha+l+m+\varepsilon}$$
$$\times\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+l+m+n+\varepsilon)\Gamma(-l-m-n-\varepsilon)\Gamma(-m-n-\varepsilon)}{n!\prod_{s=1}^{q}\Gamma(\rho_s-\alpha-l-m-n-\varepsilon)}$$
$$\times\prod_{t=4}^{p}\Gamma(\alpha_t-\alpha-l-m-n-\varepsilon)\times(-z)^n .$$

Then $A=D+E$, where

$$D=(-1)^l z^{\alpha+l}\sum_{n=0}^{m-1}\frac{\Gamma(\alpha+l+n)\Gamma(m-n+\varepsilon)\prod\Gamma(\alpha_t-\alpha-l-n)}{n!(l+n)!\prod\Gamma(\rho_s-\alpha-l-n)}K_n z^n ,$$

$$E=(-1)^l z^{\alpha+l+m}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+l+m+n)\Gamma(-n+\varepsilon)\prod\Gamma(\alpha_t-\alpha-l-m-n)}{(m+n)!(l+m+n)!\prod\Gamma(\rho_s-\alpha-l-m-n)}L_n z^n ,$$

where

$$L_n=\psi(l+m+n)+\psi(m+n)+\psi(-n-1+\varepsilon)$$
$$-\psi(\alpha+l+m+n-1)-\log z+\sum\psi(\alpha_t-\alpha-l-m-n-1)$$
$$-\sum\psi(\rho_s-\alpha-l-m-n-1).$$

Note. In these formulae t takes the values from 4 to p.
Here, since [2, p. 31]

(7) $$\psi(-z-1)=\psi(z)+\pi\cot\pi z ,$$

(8) $$\psi(-n-1+\varepsilon)=\psi(n-\varepsilon)-\pi\cot\pi\varepsilon .$$

Hence

$$C+E=\frac{\pi^2}{\sin^2\pi\varepsilon}(-1)^l z^{\alpha+l+m}$$

$$\times\left[\begin{array}{l} z^{\varepsilon}\sum_{n=0}^{\infty}\dfrac{\Gamma(\alpha+l+m+n+\varepsilon)}{n!\Gamma(m+n+1+\varepsilon)} \\ \qquad\dfrac{\prod\Gamma(\alpha_t-\alpha-l-m-n-\varepsilon)(-z)^n}{\Gamma(l+m+n+1+\varepsilon)\times\prod\Gamma(\rho_s-\alpha-l-m-n-\varepsilon)} \\ +\dfrac{\sin\pi\varepsilon}{\pi}\sum_{n=0}^{\infty}\dfrac{\Gamma(\alpha+l+m+n)\prod\Gamma(\alpha_t-\alpha-l-m-n)(-z)^n}{\Gamma(n+1-\varepsilon)(m+n)!(l+m+n)!\prod\Gamma(\rho_s-\alpha-l-m-n)} \\ \quad\times\left[\begin{array}{l}\psi(l+m+n)+\psi(m+n)+\psi(n-\varepsilon)-\psi(\alpha+l+m+n-1) \\ -\log z+\sum\psi(\alpha_t-\alpha-l-m-n-1) \\ \qquad\qquad -\sum\psi(\rho_s-\alpha-l-m-n-1)\end{array}\right] \\ -\cos\pi\varepsilon\sum_{n=0}^{\infty}\dfrac{\Gamma(\alpha+l+m+n)\prod\Gamma(\alpha_t-\alpha-l-m-n)(-z)^n}{\Gamma(n+1-\varepsilon)(m+n)!(l+m+n)!\prod\Gamma(\rho_s-\alpha-l-m-n)}\end{array}\right].$$

The limit of this function when $\varepsilon\to 0$ is obtained by replacing $\pi^2/\sin^2\pi\varepsilon$ by $\frac{1}{2}$, finding the second derivative with regard to ε of the expression in the large bracket, and then making $\varepsilon\to 0$. It is

$$\tfrac{1}{2}(-1)^l z^{\alpha+l+m}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+l+m+n)\prod\Gamma(\alpha_t-\alpha-l-m-n)(-z)^n}{n!(m+n)!(l+m+n)!\prod\Gamma(\rho_s-\alpha-l-m-n)}$$

$$\times\left[\begin{array}{l}\left\{\begin{array}{l}\log z+\psi(\alpha+l+m+n-1)-\psi(m+n)-\psi(l+m+n) \\ -\sum\psi(\alpha_t-\alpha-l-m-n-1)+\sum\psi(\rho_s-\alpha-l-m-n-1)\end{array}\right\}^2 \\ +\chi(\alpha+l+m+n-1)-\chi(m+n)-\chi(l+m+n) \\ +\sum\chi(\alpha_t-\alpha-l-m-n-1)-\sum\chi(\rho_s-\alpha-l-m-n-1) \\ +2\psi(n)\left\{\begin{array}{l}\psi(l+m+n)+\psi(m+n)+\psi(n) \\ \qquad\qquad -\psi(\alpha+l+m+n-1) \\ -\log z+\sum\psi(\alpha_t-\alpha-l-m-n-1) \\ \qquad\qquad -\sum\psi(\rho_s-\alpha-l-m-n-1)\end{array}\right\} \\ -2\chi(n)+\pi^2-\{\psi(n)\}^2+\chi(n)\end{array}\right].$$

From this, with B and D, formula (5) is obtained.

REFERENCES

1. T. M. MacRobert, *Functions of a complex variable*, London, (1954).

2. T. M. MacRobert, *Evaluation of an E-function when two of the upper parameters differ by an integer*, Proc. Glasgow Math. Assoc., **5** (1961).

THE UNIVERSITY,
GLASGOW

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RALPH S. PHILLIPS
Stanford University
Stanford, California

A. L. WHITEMAN
University of Southern California
Los Angeles 7, California

M. G. ARSOVE
University of Washington
Seattle 5, Washington

LOWELL J. PAIGE
University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH
T. M. CHERRY
D. DERRY
M. OHTSUKA
H. L. ROYDEN
E. SPANIER
E. G. STRAUS
F. WOLF

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 12, No. 3 March, 1962