

# Pacific Journal of Mathematics

**EVALUATION OF AN  $E$ -FUNCTION WHEN THREE OF ITS  
UPPER PARAMETERS DIFFER BY INTEGRAL VALUES**

T. M. MACROBERT

# EVALUATION OF AN $E$ -FUNCTION WHEN THREE OF ITS UPPER PARAMETERS DIFFER BY INTEGRAL VALUES

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1. **Introduction.** If  $p \geq q + 1$ , [1, p. 353]

$$(1) \quad E(p; \alpha_r; q; \rho_s; z) = \sum_{r=1}^p z^{\alpha_r} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_r + n) \prod_{t=1}^p \Gamma(\alpha_t - \alpha_r - n)}{n! \prod_{s=1}^q \Gamma(\rho_s - \alpha_r - n)} (-z)^n,$$

where, if  $p = q + 1$ ,  $|z| < 1$ . The dash in the product sign indicates that the factor for which  $t = r$  is omitted.

Now, if two or more of the  $\alpha$ 's are equal or differ by integral values, some of the series on the right cease to exist. For instance, if  $\alpha_1 = \alpha + l$ ,  $\alpha_2 = \alpha$ , where  $l$  is zero or a positive integer, it has been shown [2, p. 30] that the first two series can be replaced by the expression

$$(2) \quad \begin{aligned} & (-1)^l z^{\alpha+l} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + l + n) \prod_{t=3}^p \Gamma(\alpha_t - \alpha - l - n)}{n!(l+n)! \prod_{s=1}^q \Gamma(\rho_s - \alpha - l - n)} A_n z^n \\ & + z^{\alpha} \sum_{n=0}^{l-1} \frac{\Gamma(\alpha + n)(l - n - 1)! \prod_{t=3}^p \Gamma(\alpha_t - \alpha - n)}{n! \prod_{s=1}^q \Gamma(\rho_s - \alpha - n)} (-z)^n, \end{aligned}$$

where

$$\begin{aligned} A_n = & \psi(l+n) + \psi(n) - \psi(\alpha + l + n - 1) - \log z \\ & + \sum_{t=3}^p \psi(\alpha_t - \alpha - l - n - 1) - \sum_{s=1}^q \psi(\rho_s - \alpha - l - n - 1). \end{aligned}$$

Here

$$(3) \quad \psi(z) = \frac{d}{dz} \log \Gamma(z + 1),$$

so that

$$(4) \quad \frac{d}{dz} \Gamma(z + 1) = \Gamma(z + 1) \psi(z).$$

It will now be shown that, in the case in which

$$\alpha_1 = \alpha, \alpha_2 = \alpha + l, \alpha_3 = \alpha + l + m,$$

where  $l$  and  $m$  are zero or positive integers, the first three series can be replaced by the expression

$$(5) \quad \frac{1}{2}(-1)^l z^{\alpha+l+m} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+l+m+n) \prod_{t=4}^p \Gamma(\alpha_t - \alpha - l - m - n) (-z)^n}{n!(m+n)!(l+m+n)! \prod_{s=1}^q \Gamma(\rho_s - \alpha - l - m - n)} A_n$$

$$- (-1)^l z^{\alpha+l} \sum_{n=0}^{m-1} \frac{\Gamma(\alpha+l+n)(m-n-1)! \prod_{t=4}^p \Gamma(\alpha_t - \alpha - l - n) z^n}{n!(l+n)! \prod_{s=1}^q \Gamma(\rho_s - \alpha - l - n)} \theta_n$$

$$+ z^{\alpha} \sum_{n=0}^{l-1} \frac{\Gamma(\alpha+n)(l-n-1)!(l+m-n-1)! \prod_{t=4}^p \Gamma(\alpha_t - \alpha - n)}{n! \prod_{s=1}^q \Gamma(\rho_s - \alpha - n)} \times (-z)^n,$$

where

$$A_n = \pi^2$$

$$+ \left\{ \begin{aligned} & \log z + \psi(\alpha+l+m+n-1) - \psi(l+m+n) - \psi(m+n) \\ & - \psi(n) - \sum_{t=4}^p \psi(\alpha_t - \alpha - l - m - n - 1) \\ & + \sum_{s=1}^q \psi(\rho_s - \alpha - l - m - n - 1) \end{aligned} \right\}^2$$

$$+ \chi(\alpha+l+m+n-1) - \chi(l+m+n) - \chi(m+n) - \chi(n)$$

$$+ \sum_{t=4}^p \chi(\alpha_t - \alpha - l - m - n - 1) - \sum_{s=1}^q \chi(\rho_s - \alpha - l - m - n - 1),$$

and

$$\theta_n = \log z + \psi(\alpha+l+n-1) - \psi(l+n) - \psi(m-n-1) - \psi(n)$$

$$- \sum_{t=4}^p \psi(\alpha_t - \alpha - l - n - 1) + \sum_{s=1}^q \psi(\rho_s - \alpha - l - n - 1).$$

Here

$$(6) \quad \chi(z) = \psi'(z) = \sum_{n=1}^{\infty} \frac{1}{(z+n)^2},$$

where  $|\operatorname{amp} z| < \pi$ .

**2. Proof of the formula.** If  $\alpha_1 = \alpha, \alpha_2 = \alpha + l, \alpha_3 = \alpha + l + m + \varepsilon$ , where  $l$  and  $m$  are zero or positive integers and  $\varepsilon$  is small, it follows from (2) and (1) that the first 3 terms of (1) are equal to  $A + B + C$ , where

$$A = (-1)^l z^{\alpha+l} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+l+n)\Gamma(m-n+\varepsilon) \prod_{t=4}^p \Gamma(\alpha_t - \alpha - l - n)}{n!(l+n)! \prod_{s=1}^q \Gamma(\rho_s - \alpha - l - n)} K_n z^n,$$

where

$$K_n = \psi(l+n) + \psi(n) + \psi(m-n-1+\varepsilon) - \psi(\alpha+l+n-1) - \log z + \sum_{t=4}^p \psi(\alpha_t - \alpha - l - n - 1) - \sum_{s=1}^q \psi(\rho_s - \alpha - l - n - 1),$$

$$B = z^\alpha \sum_{n=0}^{l-1} \frac{\Gamma(\alpha+n)(l-n-1)! \Gamma(l+m-n+\varepsilon) \prod_{t=4}^p \Gamma(\alpha_t - \alpha - n)}{n! \prod_{s=1}^q \Gamma(\rho_s - \alpha - n)}$$

$(-z)^n$  ;

$$C = z^{\alpha+l+m+\varepsilon}$$

$$\begin{aligned} \times \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+l+m+n+\varepsilon)\Gamma(-l-m-n-\varepsilon)\Gamma(-m-n-\varepsilon)}{n! \prod_{s=1}^q \Gamma(\rho_s - \alpha - l - m - n - \varepsilon)} \\ \times \prod_{t=4}^p \Gamma(\alpha_t - \alpha - l - m - n - \varepsilon) \times (-z)^n. \end{aligned}$$

Then  $A = D + E$ , where

$$D = (-1)^l z^{\alpha+l} \sum_{n=0}^{m-1} \frac{\Gamma(\alpha+l+n)\Gamma(m-n+\varepsilon) \prod \Gamma(\alpha_t - \alpha - l - n)}{n!(l+n)! \prod \Gamma(\rho_s - \alpha - l - n)} K_n z^n,$$

$$E = (-1)^l z^{\alpha+l+m} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+l+m+n)\Gamma(-n+\varepsilon) \prod \Gamma(\alpha_t - \alpha - l - m - n)}{(m+n)!(l+m+n)! \prod \Gamma(\rho_s - \alpha - l - m - n)} L_n z^n,$$

where

$$\begin{aligned} L_n = \psi(l+m+n) + \psi(m+n) + \psi(-n-1+\varepsilon) \\ - \psi(\alpha+l+m+n-1) - \log z + \sum \psi(\alpha_t - \alpha - l - m - n - 1) \\ - \sum \psi(\rho_s - \alpha - l - m - n - 1). \end{aligned}$$

*Note.* In these formulae  $t$  takes the values from 4 to  $p$ .

Here, since [2, p. 31]

$$(7) \quad \psi(-z-1) = \psi(z) + \pi \cot \pi z,$$

$$(8) \quad \psi(-n-1+\varepsilon) = \psi(n-\varepsilon) - \pi \cot \pi \varepsilon.$$

Hence

$$C + E = \frac{\pi^2}{\sin^2 \pi \varepsilon} (-1)^l z^{\alpha+l+m}$$

$$\times \left[ z^\varepsilon \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + l + m + n + \varepsilon)}{n! \Gamma(m + n + 1 + \varepsilon)} \frac{\prod \Gamma(\alpha_t - \alpha - l - m - n - \varepsilon)(-z)^n}{\Gamma(l + m + n + 1 + \varepsilon) \times \prod \Gamma(\rho_s - \alpha - l - m - n - \varepsilon)} \right. \\
+ \frac{\sin \pi \varepsilon}{\pi} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + l + m + n) \prod \Gamma(\alpha_t - \alpha - l - m - n)(-z)^n}{\Gamma(n + 1 - \varepsilon)(m + n)!(l + m + n)! \prod \Gamma(\rho_s - \alpha - l - m - n)} \\
\times \left[ \psi(l + m + n) + \psi(m + n) + \psi(n - \varepsilon) - \psi(\alpha + l + m + n - 1) \right] \\
\times \left[ -\log z + \sum \psi(\alpha_t - \alpha - l - m - n - 1) - \sum \psi(\rho_s - \alpha - l - m - n - 1) \right] \\
\left. - \cos \pi \varepsilon \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + l + m + n) \prod \Gamma(\alpha_t - \alpha - l - m - n)(-z)^n}{\Gamma(n + 1 - \varepsilon)(m + n)!(l + m + n)! \prod \Gamma(\rho_s - \alpha - l - m - n)} \right].$$

The limit of this function when  $\varepsilon \rightarrow 0$  is obtained by replacing  $\pi^2/\sin^2 \pi \varepsilon$  by  $\frac{1}{2}$ , finding the second derivative with regard to  $\varepsilon$  of the expression in the large bracket, and then making  $\varepsilon \rightarrow 0$ . It is

$$\frac{1}{2}(-1)^l z^{\alpha+l+m} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + l + m + n) \prod \Gamma(\alpha_t - \alpha - l - m - n)(-z)^n}{n!(m + n)!(l + m + n)! \prod \Gamma(\rho_s - \alpha - l - m - n)} \\
\times \left[ \left( \log z + \psi(\alpha + l + m + n - 1) - \psi(m + n) - \psi(l + m + n) \right)^2 \right. \\
\left. \left( -\sum \psi(\alpha_t - \alpha - l - m - n - 1) + \sum \psi(\rho_s - \alpha - l - m - n - 1) \right) \right] \\
+ \chi(\alpha + l + m + n - 1) - \chi(m + n) - \chi(l + m + n) \\
+ \sum \chi(\alpha_t - \alpha - l - m - n - 1) - \sum \chi(\rho_s - \alpha - l - m - n - 1) \\
+ 2\psi(n) \left\{ \begin{array}{l} \psi(l + m + n) + \psi(m + n) + \psi(n) \\ - \psi(\alpha + l + m + n - 1) \end{array} \right. \\
\left. \left( -\log z + \sum \psi(\alpha_t - \alpha - l - m - n - 1) - \sum \psi(\rho_s - \alpha - l - m - n - 1) \right) \right\} \\
\left. - 2\chi(n) + \pi^2 - \{\psi(n)\}^2 + \chi(n) \right].$$

From this, with  $B$  and  $D$ , formula (5) is obtained.

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The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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