

Pacific Journal of Mathematics

**OPERATORS OF FINITE RANK IN A REFLEXIVE BANACH
SPACE**

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OPERATORS OF FINITE RANK IN A REFLEXIVE BANACH SPACE

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1. Let X be a reflexive Banach space and $F(X)$ the Banach algebra of all uniform limits of operators of finite rank, in X . Bonsall [1] has characterized $F(X)$ as a simple, B^* -annihilator algebra: $F(X)$ contains no proper closed two-sided ideals, every proper, closed right (left) ideal of $F(X)$ has a nonzero left (right) annihilator, and, given any $T \in F(X)$, there exists $T^* \in F(X)$ such that

$$\|T\| \|T^*\| = \|(TT^*)^n\|^{1/n}, \quad n = 1, 2, 3, \dots$$

In this note, we obtain a new characterization for $F(X)$ (Theorem 3.2): a Banach algebra A is the algebra $F(X)$ of all uniform limits of operators of finite rank in a reflexive Banach space X if and only if A is a simple, weakly compact, B^* -algebra with minimal ideals (A is weakly compact if left- and right-multiplications by every $a \in A$ are weakly compact operators). In the process of proving this result, we obtain a characterization of reflexive Banach spaces which seems to be of some independent interest (Theorem 2.2): a Banach space X is reflexive if and only if every operator in X of rank 1 is a weakly compact element of $B(X)$.

2. Let X be a Banach space and $B = B(X)$ the Banach algebra of all bounded operators in X with the uniform topology. For $T \in B$, let R_T denote the operator in B obtained by multiplying elements of B on the right by T : $R_T(A) = AT$ for $A \in B$.

Suppose that T is a fixed operator of rank 1 in X with $H = [x \in X: Tx = 0]$. Then H is a closed hyperplane in X and if x_0 is an element of X such that $Tx_0 \neq 0$, then $X = H \oplus (x_0)$ and we may assume that $\|x_0\| = 1$. Write $B' = [S \in B: S(H) = (0)]$. For each $S \in B'$, we define an element x_S of X by setting $x_S = S(x_0)$. The mapping $S \rightarrow x_S$ is clearly linear.

LEMMA 2.1. *The linear mapping $S \rightarrow x_S$ is a homeomorphism of B' onto X .*

Proof. It is clear that the mapping is one-to-one and, since $\|S(x_0)\| \leq \|S\|$, it is continuous. It is also onto; in fact, let $\varphi \in X^*$ be such that $\varphi(H) = (0)$, $\varphi(x_0) = 1$. Then for given $x \in X$, the operator S_x defined by setting $S_x(y) = \varphi(y)x$, $y \in X$ belongs to B' and is mapped into x by the mapping $S \rightarrow S(x_0)$. Hence, by the closed graph theorem, the

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mapping is bicontinuous and the proof is complete.

Let B_1 denote the unit ball in B , so that $R_T(B_1) = [PT \in B: \|P\| \leq 1]$.

LEMMA 2.2. $R_T(B_1) = [A \in B': \|Ax_0\| \leq \|Tx_0\|]$.

Proof. It is clear that $R_T(B_1) \subset [A \in B': \|Ax_0\| \leq \|Tx_0\|]$. Now let $A \in B'$ with $\|Ax_0\| \leq \|Tx_0\|$; we find $P \in B_1$ such that $A = PT$. There exists $\psi \in X^*$ such that $\|\psi\| = 1$ and $\psi(Tx_0) = \|Tx_0\|$. We define P by setting $Px = \psi(x)Ax_0/\|Tx_0\|$. Then $PTx = 0$ if $x \in H$ and $PTx_0 = Ax_0$. Thus PT and A coincide in the subspace (x_0) and must therefore coincide everywhere in X . Finally $\|P\| = \sup_{\|x\| \leq 1} \|\psi(x)Ax_0\|/\|Tx_0\| \leq 1$; hence $P \in B_1$ and $R_T(B_1) = [A \in B': \|Ax_0\| \leq \|Tx_0\|]$.

LEMMA 2.3. *Let F be any subset of B' . If $F^{B'}$ denotes the closure of F with respect to the weak topology of B' and F^B the closure of F with respect to the weak topology of B , then $F^{B'} = F^B$.*

Proof. Let $P_0 \in F^{B'}$ and let

$$N = N(P_0; \Phi_1, \Phi_2, \dots, \Phi_n; \varepsilon) \\ = [P \in B: |\Phi_k(P - P_0)| < \varepsilon; k = 1, 2, \dots, n; \Phi_k \in B^*]$$

be an arbitrary neighborhood of P_0 in B . Then the neighborhood $N' = N(P_0; \Phi'_1, \Phi'_2, \dots, \Phi'_n; \varepsilon)$ of P_0 obtained by taking the restriction of Φ_k to B' for each k , contains a point P of F . Since P must therefore belong to N , it follows that $F^{B'} \subseteq F^B$.

Now suppose that $P_0 \in F^B$. Then $P_0 \in B'$ since B' is closed with respect to the weak topology of $B(X)$ (being linear and strongly closed). Let $N' = [P \in B': |\varphi_k(P - P_0)| < \varepsilon, k = 1, 2, \dots, n; \varphi_k \in (B')^*]$ be an arbitrary neighborhood of P_0 in B' . Then again, by considering the neighborhood $N = [P \in B: |\Phi_k(P - P_0)| < \varepsilon, k = 1, 2, \dots, n, \Phi_k \in B^*]$ obtained by extending φ_k to Φ_k , for each k , on the whole of B , we can find $P \in F$ such that $P \in N'$. Hence $F^B \subseteq F^{B'}$. This completes the proof.

THEOREM 2.1. *A Banach space X is reflexive if and only if every operator in X of rank 1 is a right weakly compact element of $B(X)$.*

Proof. If X is reflexive and T is of rank 1, then by Lemma 2.1, B' is homeomorphic with X under the correspondence $S \mapsto S(x_0)$. Now the image of B_1 under R_T is a bounded subset of B' which is therefore contained in a set U which is compact with respect to the weak topology of B' and by Lemma 2.3, with respect to the weak topology of $B(X)$. Thus R_T is a weakly compact operator in $B(X)$ and T is a right weakly compact element of $B(X)$.

Now suppose that R_T is weakly compact in $B(X)$. Then $R_T(B_1)$ is contained in a set $V \subset B'$ which is compact with respect to the weak topology of $B(X)$ and hence also with respect to the weak topology of B' . Now the ball $Q = [A \in B': \|A\| \leq \|Tx_0\|/\|x_0\|]$ is contained in $R_T(B_1) \subset V$ and is weakly closed. Hence Q is compact with respect to the weak topology of B' and therefore B' is reflexive. Since B' is homeomorphic with X , it follows that X is reflexive and the proof is complete.

COROLLARY 2.1. *If X is a reflexive Banach space, then the algebra $F(X)$ of all uniform limits of operators of finite rank in X is a weakly compact algebra.*

COROLLARY 2.2. (Ogasawara [2] Theorem 4.) *Let H be a Hilbert space and $B(H)$ the Banach algebra of all bounded operators in H . If T is a compact operator in H , then T is a weakly compact element of $B(H)$.*

3. This section is devoted to the study of simple, weakly compact, B^* -algebras with minimal ideals.

LEMMA 3.1. *Let A be a simple Banach algebra with minimal ideals. Then every maximal regular left ideal M of A has a nonzero right annihilator.*

Proof. Since A is a simple Banach algebra, there exists an idempotent $e \in A$ such that $M \cap Ae = (0)$ and $M \oplus Ae = A$. Since M is regular, there is $j \in A$ such that $xj - x \in M$ for every $x \in A$. For some $a_0 \in A$ and $m_0 \in M$, $j = m_0 + a_0e$, $a_0e \neq 0$. Suppose now that m is an arbitrary element in M . We have $mj - m \in M$ and $mj - ma_0e = mm_0 \in M$, from which it follows that $m - ma_0e \in M$. Now, $m \in M$ and hence $ma_0e \in M$. However, $ma_0e \in Ae$ since Ae is a left ideal, thus $ma_0e \in M \cap Ae = (0)$ and since m is arbitrary in M , the lemma is proved.

LEMMA 3.2. *Let A be a simple Banach algebra with minimal right ideals. If $j \in A$ and j has no left reverse, then there exists $a \neq 0$ such that $ja = a$.*

Proof. Let $J = [yj - y: y \in A]$. Then J is a regular left ideal of A which is proper since $j \notin J$. Hence by Lemma 3.1, there exists $a \in A$, $a \neq 0$ such that $Ja = (0)$, i.e. such that $yja - ya = 0$ for all $y \in A$ or $A(ja - a) = (0)$. Since $(A)_r = (0)$, this implies that $ja = a$.

LEMMA 3.3. *Let A be a simple B^* -algebra with minimal right*

ideals. If $\|\cdot\|$ is any other norm in A with $\|a\| \leq \|a\|$ for each $a \in A$, then $\|\cdot\| = \|\cdot\|$.

Proof. Lemma 3.2 implies that if $\|\cdot\|$ is any other norm in A , then $\lim_{n \rightarrow \infty} \|a^n\|^{1/n} = \lim_{n \rightarrow \infty} \|a^n\|^{1/n}$ for every $a \in A$ (Cf [4], Lemma 3.1). Then since A is a B^* -algebra, we have

$$\begin{aligned} \|a^*\| \|a\| &\geq \|a^*a\| \geq \lim_{n \rightarrow \infty} \|(a^*a)^n\|^{1/n} \\ &= \lim_{n \rightarrow \infty} \|(a^*a)^n\|^{1/n} = \|a\| \|a\|, \end{aligned}$$

and since $\|a^*\| \leq \|a^*\|$ and $\|a\| \leq \|a\|$, the result follows.

THEOREM 3.1. *A Banach algebra A is the algebra $F(X)$ of all uniform limits of operators of finite rank in a reflexive Banach space X if and only if A is a simple, weakly compact, B^* -algebra with minimal right ideals.*

Proof. Let A be a simple, weakly compact, B^* -algebra with eA a minimal right ideal, e a primitive idempotent. We represent A as an algebra of operators \mathcal{A} in eA , the latter regarded as a Banach space. Corresponding to each $a \in A$, we define an operator $\bar{a} \in \mathcal{A}$ by $\bar{a}: x \rightarrow xa$ for $x \in eA$. The correspondence $a \rightarrow \bar{a}$ is obviously an isomorphism and if we take $\|\bar{a}\| = \sup_{\|x\| \leq 1} \|xa\|$, $x \in eA$, the correspondence is an isometry in view of Lemma 3.3. Thus A is isomorphic and isometric to \mathcal{A} and A is the uniform closure of \mathcal{A} .

Next we show that eA is a reflexive Banach space. Now e has no left reverse in A ; hence by Lemma 3.2, there exists $a \in A$, $a \neq 0$ such that $ea = a$. The set $P = [a \in A: ea = a]$ is a right ideal of A and since $P \subseteq eA$, we must have $P = eA$ since eA is minimal. If e is now regarded as a left weakly compact operator on A , then it is clear that the set $P = eA$ is a reflexive Banach space.

Our next step is to show that in the representation described above, \mathcal{A} contains all operators of finite rank in eA . Corresponding to each $a \in Ae$, there exists a continuous linear functional φ_a on eA satisfying $\varphi_a(x)e = xa$, $x \in eA$. Let $G = [\varphi_a \in (eA)^*: a \in Ae]$; then G is a linear subspace of $(eA)^*$. We show that G is closed with respect to the usual norm in $(eA)^*$ defined by $\|\varphi\| = \sup_{\|x\| \leq 1} |\varphi(x)|$, $x \in eA$. For $a \in Ae$, we have $xa = \varphi_a(x)e$, $x \in eA$, and since $\|a\| = \|\bar{a}\|$ for each $a \in A$, we have

$$\begin{aligned} \|a\| = \|\bar{a}\| &= \sup_{\|x\| \leq 1} \|xa\| && a \in Ae \\ &= \sup_{\|x\| \leq 1} \|\varphi_a(x)e\| \\ &= \sup_{\|x\| \leq 1} |\varphi_a(x)| \|e\| \end{aligned}$$

$$= \|\varphi_a\| \cdot \|e\| .$$

Thus G is topologically equivalent to Ae and hence closed. Having proved that G is a closed linear subspace of $(eA)^*$, we now show that G is in fact the whole of $(eA)^*$. Suppose that there exists $\varphi' \in (eA)^*$ such that $\varphi' \notin G$. Since G is closed, there exists $\phi \in (eA)^{**}$ such that $\phi(\varphi_a) = 0$ for all $\varphi_a \in G$ and $\phi(\varphi') = 1$. However, eA is a reflexive Banach space: hence there exists $u_0 \in eA$, $u_0 \neq 0$ such that $\phi(\varphi) = \varphi(u_0)$ for all $\varphi \in (eA)^*$. In particular, for $\varphi_a \in G$, this implies that $0 = \varphi_a(u_0)e = u_0a$ for all $a \in Ae$, which in turn implies that $u_0 \in (Ae)_l = (0)$ which is absurd. Hence $G = (eA)^*$. From this it follows that \mathcal{A} contains all operators of rank 1 and hence all operators of finite rank in eA , since if T is an operator of rank 1 in eA , then there exists $\varphi \in (eA)^*$ and $u_0 \in eA$ such that $xT = \varphi(x)u_0$, $x \in eA$. Since $\varphi \in G$, there exist $a \in Ae$ and $\varphi_a \in (eA)^*$ such that $\varphi = \varphi_a$ and $xa = \varphi_a(x)e$. Let $u_0 = ea_0$ for some $a_0 \in A$; we have $xT = \varphi_a(x)u_0 = \varphi_a(x)ea_0 = xaa_0$, and since $aa_0 \in A$, the operator $aa_0 = T$ belongs to \mathcal{A} .

Finally, the uniform closure of the set of all operators of finite rank in eA is a closed two-sided ideal of \mathcal{A} which must coincide with \mathcal{A} since \mathcal{A} is simple. Thus the ‘‘if’’ part of the theorem is proved.

That $F(X)$ is a simple, weakly compact B^* -algebra with minimal ideals follows from corollary 1 and a result due to Bonsall and Goldie [1], Theorem 2. This completes the proof of the theorem.

REMARKS. 1. The problems discussed here were suggested by reading Ogasawara and Yoshinaga [2,3] and Bonsall [1].

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