OPERATORS OF FINITE RANK IN A REFLEXIVE BANACH SPACE

ADEGOKE OLUBUMMO
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A. OLUBUMMO

1. Let $X$ be a reflexive Banach space and $F(X)$ the Banach algebra of all uniform limits of operators of finite rank, in $X$. Bonsall [1] has characterized $F(X)$ as a simple, $B^*$-annihilator algebra: $F(X)$ contains no proper closed two-sided ideals, every proper, closed right (left) ideal of $F(X)$ has a nonzero left (right) annihilator, and, given any $T \in F(X)$, there exists $T^* \in F(X)$ such that

$$\|T\| \|T^*\| = \|(TT^*)^n\|^{1/n}, \quad n = 1, 2, 3, \ldots.$$ 

In this note, we obtain a new characterization for $F(X)$ (Theorem 3.2): a Banach algebra $A$ is the algebra $F(X)$ of all uniform limits of operators of finite rank in a reflexive Banach space $X$ if and only if $A$ is a simple, weakly compact, $B^*$-algebra with minimal ideals ($A$ is weakly compact if left- and right-multiplications by every $a \in A$ are weakly compact operators). In the process of proving this result, we obtain a characterization of reflexive Banach spaces which seems to be of some independent interest (Theorem 2.2): a Banach space $X$ is reflexive if and only if every operator in $X$ of rank 1 is a weakly compact element of $B(X)$.

2. Let $X$ be a Banach space and $B = B(X)$ the Banach algebra of all bounded operators in $X$ with the uniform topology. For $T \in B$, let $R_T$ denote the operator in $B$ obtained by multiplying elements of $B$ on the right by $T$: $R_T(A) = AT$ for $A \in B$.

Suppose that $T$ is a fixed operator of rank 1 in $X$ with $H = \{x \in X: Tx = 0\}$. Then $H$ is a closed hyperplane in $X$ and if $x_0$ is an element of $X$ such that $Tx_0 \neq 0$, then $X = H \oplus \langle x_0 \rangle$ and we may assume that $\|x_0\| = 1$. Write $B' = \{S \in B: S(H) = (0)\}$. For each $S \in B'$, we define an element $x_s$ of $X$ by setting $x_s = S(x_0)$. The mapping $S \to x_s$ is clearly linear.

**Lemma 2.1.** The linear mapping $S \to x_s$ is a homeomorphism of $B'$ onto $X$.

**Proof.** It is clear that the mapping is one-to-one and, since $\|S(x_0)\| \leq \|S\|$, it is continuous. It is also onto; in fact, let $\varphi \in X^*$ be such that $\varphi(H) = (0)$, $\varphi(x_0) = 1$. Then for given $x \in X$, the operator $S_x$ defined by setting $S_x(y) = \varphi(y)x$, $y \in X$ belongs to $B'$ and is mapped into $x$ by the mapping $S \to S(x_0)$. Hence, by the closed graph theorem, the

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mapping is bicontinuous and the proof is complete.

Let $B_1$ denote the unit ball in $B$, so that $R_T(B_1) = [P T \in B: \|P\| \leq 1]$.

**Lemma 2.2.** $R_T(B_1) = [A \in B': \|A x_0\| \leq \|T x_0\|].$

**Proof.** It is clear that $R_T(B_1) \subset [A \in B': \|A x_0\| \leq \|T x_0\|]$. Now let $A \in B'$ with $\|A x_0\| \leq \|T x_0\|$; we find $P \in B_1$ such that $A = PT$. There exists $\psi \in X^*$ such that $\|\psi\| = 1$ and $\psi(Tx_0) = \|T x_0\|$. We define $P$ by setting $P x = \psi(x) A x_0/\|T x_0\|$. Then $PT x = 0$ if $x \in H$ and $PT x_0 = A x_0$. Thus $P T$ and $A$ coincide in the subspace $(x_0)$ and must therefore coincide everywhere in $X$. Finally $\|P\| = \sup_{\|x\| \leq 1} \|\psi(x) A x_0\|/\|T x_0\| \leq 1$; hence $P \in B_1$ and $R_T(B_1) = [A \in B': \|A x_0\| \leq \|T x_0\|].$

**Lemma 2.3.** Let $F$ be any subset of $B'$. If $F^w$ denotes the closure of $F$ with respect to the weak topology of $B'$ and $F^w$ the closure of $F$ with respect to the weak topology of $B$, then $F^w = F^w$.

**Proof.** Let $P_0 \in F^w$ and let

$$N = N(P_0; \phi_0, \phi_1, \phi_2, \ldots, \phi_n; \varepsilon) = [P \in B: |\phi_k(P - P_0)| < \varepsilon; k = 1, 2, \ldots, n; \phi_k \in B^*]$$

be an arbitrary neighborhood of $P_0$ in $B$. Then the neighborhood $N' = N(P_0; \phi_0, \phi_1, \phi_2, \ldots, \phi_n; \varepsilon)$ of $P_0$ obtained by taking the restriction of $\phi_k$ to $B'$ for each $k$, contains a point $P$ of $F$. Since $P$ must therefore belong to $N$, it follows that $F^w \subseteq F^w$.

Now suppose that $P_0 \in F^w$. Then $P_0 \in B'$ since $B'$ is closed with respect to the weak topology of $B(X)$ (being linear and strongly closed). Let $N' = [P \in B': |\phi_k(P - P_0)| < \varepsilon; k = 1, 2, \ldots, n; \phi_k \in (B')^*]$ be an arbitrary neighborhood of $P_0$ in $B'$. Then again, by considering the neighborhood $N = [P \in B: |\phi_k(P - P_0)| < \varepsilon; k = 1, 2, \ldots, n, \phi_k \in B^*]$ obtained by extending $\phi_k$ to $\phi_k$, for each $k$, on the whole of $B$, we can find $P \in F$ such that $P \in N'$. Hence $F^w \subseteq F^w$. This completes the proof.

**Theorem 2.1.** A Banach space $X$ is reflexive if and only if every operator in $X$ of rank 1 is a right weakly compact element of $B(X)$.

**Proof.** If $X$ is reflexive and $T$ is of rank 1, then by Lemma 2.1, $B'$ is homeomorphic with $X$ under the correspondence $S \mapsto S(x_0)$. Now the image of $B_1$ under $R_T$ is a bounded subset of $B'$ which is therefore contained in a set $U$ which is compact with respect to the weak topology of $B'$ and by Lemma 2.3, with respect to the weak topology of $B(X)$. Thus $R_T$ is a weakly compact operator in $B(X)$ and $T$ is a right weakly compact element of $B(X)$. 
Now suppose that $R_r$ is weakly compact in $B(X)$. Then $R_r(B)$ is contained in a set $V \subset B'$ which is compact with respect to the weak topology of $B(X)$ and hence also with respect to the weak topology of $B'$. Now the ball $Q = \{ A \in B': ||A|| \leq ||Tx||/||x|| \}$ is contained in $R_r(B) \subset V$ and is weakly closed. Hence $Q$ is compact with respect to the weak topology of $B'$ and therefore $B'$ is reflexive. Since $B'$ is homeomorphic with $X$, it follows that $X$ is reflexive and the proof is complete.

**Corollary 2.1.** If $X$ is a reflexive Banach space, then the algebra $F(X)$ of all uniform limits of operators of finite rank in $X$ is a weakly compact algebra.

**Corollary 2.2.** (Ogasawara [2] Theorem 4.) Let $H$ be a Hilbert space and $B(H)$ the Banach algebra of all bounded operators in $H$. If $T$ is a compact operator in $H$, then $T$ is a weakly compact element of $B(H)$.

3. This section is devoted to the study of simple, weakly compact, $B^*$-algebras with minimal ideals.

**Lemma 3.1.** Let $A$ be a simple Banach algebra with minimal ideals. Then every maximal regular left ideal $M$ of $A$ has a nonzero right annihilator.

**Proof.** Since $A$ is a simple Banach algebra, there exists an idempotent $e \in A$ such that $M \cap Ae = (0)$ and $M \oplus Ae = A$. Since $M$ is regular, there is $j \in A$ such that $xj - x \in M$ for every $x \in A$. For some $a_0 \in A$ and $m_0 \in M$, $j = m_0 + a_0 e$, $a_0 e \neq 0$. Suppose now that $m$ is an arbitrary element in $M$. We have $mj - m \in M$ and $mj - ma_0 e = mm_0 \in M$, from which it follows that $m - ma_0 e \in M$. Now, $m \in M$ and hence $ma_0 e \in M$. However, $ma_0 e \in Ae$ since $Ae$ is a left ideal, thus $ma_0 e \in M \cap Ae = (0)$ and since $m$ is arbitrary in $M$, the lemma is proved.

**Lemma 3.2.** Let $A$ be a simple Banach algebra with minimal right ideals. If $j \in A$ and $j$ has no left reverse, then there exists $a \neq 0$ such that $ja = a$.

**Proof.** Let $J = \{ yj - y: y \in A \}$. Then $J$ is a regular left ideal of $A$ which is proper since $j \notin J$. Hence by Lemma 3.1, there exists $a \in A$, $a \neq 0$ such that $Ja = (0)$, i.e. such that $yja - ya = 0$ for all $y \in A$ or $A(ja - a) = (0)$. Since $(A)_r = (0)$, this implies that $ja = a$.}

**Lemma 3.3.** Let $A$ be a simple $B^*$-algebra with minimal right
ideals. If \( |\cdot| \) is any other norm in \( A \) with \( |a| \leq \|a\| \) for each \( a \in A \), then \( |a| = \|a\| \).

\textbf{Proof.} Lemma 3.2 implies that if \( |\cdot| \) is any other norm in \( A \), then
\[
\lim_{n \to \infty} |a^n|^{1/n} = \lim_{n \to \infty} \|a^n\|^{1/n}
\]
for every \( a \in A \) (Cf [4], Lemma 3.1). Then since \( A \) is a \( B^* \)-algebra, we have
\[
|a^*| |a| \geq |a^*a| \geq \lim_{n \to \infty} |(a^*a)^n|^{1/n}
\]
\[
= \lim_{n \to \infty} \|(a^*a)^n\|^{1/n} = \|a^*\| \|a\|,
\]
and since \( |a^*| \leq \|a^*\| \) and \( |a| \leq \|a\| \), the result follows.

\textbf{Theorem 3.1.} A Banach algebra \( A \) is the algebra \( F(X) \) of all uniform limits of operators of finite rank in a reflexive Banach space \( X \) if and only if \( A \) is a simple, weakly compact, \( B^* \)-algebra with minimal right ideals.

\textbf{Proof.} Let \( A \) be a simple, weakly compact, \( B^* \)-algebra with \( e A \) a minimal right ideal, \( e \) a primitive idempotent. We represent \( A \) as an algebra of operators \( \mathcal{A} \) in \( e A \), the latter regarded as a Banach space. Corresponding to each \( a \in A \), we define an operator \( a \in \mathcal{A} \) by \( a : x \to xa \) for \( x \in e A \). The correspondence \( a \to a \) is obviously an isomorphism and if we take \( \|a\| = \sup_{\|x\| \leq 1} \|xa\| \), \( x \in e A \), the correspondence is an isometry in view of Lemma 3.3. Thus \( A \) is isomorphic and isometric to \( \mathcal{A} \) and \( A \) is the uniform closure of \( \mathcal{A} \).

Next we show that \( eA \) is a reflexive Banach space. Now \( e \) has no left reverse in \( A \); hence by Lemma 3.2, there exists \( a \in A \), \( a \neq 0 \) such that \( ea = a \). The set \( P = \{a \in A : ea = a\} \) is a right ideal of \( A \) and since \( P \subseteq eA \), we must have \( P = eA \) since \( eA \) is minimal. If \( e \) is now regarded as a left weakly compact operator on \( A \), then it is clear that the set \( P = eA \) is a reflexive Banach space.

Our next step is to show that in the representation described above, \( \mathcal{A} \) contains all operators of finite rank in \( eA \). Corresponding to each \( a \in Ae \), there exists a continuous linear functional \( \varphi_a \) on \( eA \) satisfying \( \varphi_a(x)e = xa \), \( x \in eA \). Let \( G = [\varphi_a \in (eA)^* : a \in A] \); then \( G \) is a linear subspace of \( (eA)^* \). We show that \( G \) is closed with respect to the usual norm in \( (eA)^* \) defined by \( \|\varphi\| = \sup_{\|x\| \leq 1} |\varphi(x)| \) \( x \in eA \). For \( a \in Ae \), we have \( xa = \varphi_a(x)e \), \( x \in eA \), and since \( \|a\| = \|\varphi\| \) for each \( a \in A \), we have
\[
\|a\| = \|\varphi\| = \sup_{\|x\| \leq 1} \|xa\| \quad a \in Ae
\]
\[
= \sup_{\|x\| \leq 1} \|\varphi_a(x)e\|
\]
\[
= \sup_{\|x\| \leq 1} |\varphi_a(x)| \|e\|\]
Thus $G$ is topologically equivalent to $\mathcal{A}e$ and hence closed. Having proved that $G$ is a closed linear subspace of $(eA)^*$, we now show that $G$ is in fact the whole of $(eA)^*$. Suppose that there exists $\varphi' \in (eA)^*$ such that $\varphi' \notin G$. Since $G$ is closed, there exists $\Phi \in (eA)^{**}$ such that $\Phi(\varphi_a) = 0$ for all $\varphi_a \in G$ and $\Phi(\varphi') = 1$. However, $eA$ is a reflexive Banach space: hence there exists $u_0 \in eA$, $u_0 \neq 0$ such that $\Phi(\varphi) = \varphi(u_0)$ for all $\varphi \in (eA)^*$. In particular, for $\varphi_a \in G$, this implies that $0 = \varphi_a(u_0)e = u_0 a$ for all $a \in A e$, which in turn implies that $u_0 \in (Ae)_i = (0)$ which is absurd. Hence $G = (eA)^*$. From this it follows that $\mathcal{A}$ contains all operators of rank 1 and hence all operators of finite rank in $eA$, since if $T$ is an operator of rank 1 in $eA$, then there exists $\varphi \in (eA)^*$ and $u_0 \in eA$ such that $xT = \varphi(x)u_0$, $x \in eA$. Since $\varphi \in G$, there exist $a \in A e$ and $\varphi_a \in (eA)^*$ such that $\varphi = \varphi_a$ and $xa = \varphi_a(x)e$. Let $u_0 = ea_0$ for some $a_0 \in A$; we have $xT = \varphi_a(x)u_0 = \varphi_a(x)e a_0 = xaa_0$, and since $aa_0 \in A$, the operator $aa_0 = T$ belongs to $\mathcal{A}$.

Finally, the uniform closure of the set of all operators of finite rank in $eA$ is a closed two-sided ideal of $\mathcal{A}$ which must coincide with $\mathcal{A}$ since $\mathcal{A}$ is simple. Thus the "if" part of the theorem is proved.

That $F(X)$ is a simple, weakly compact $B^*$-algebra with minimal ideals follows form corollary 1 and a result due to Bonsall and Goldie [1], Theorem 2. This completes the proof of the theorem.

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REFERENCES

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