SIMPLE MALCEV ALGEBRAS OVER FIELDS OF CHARACTERISTIC ZERO

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1. Introduction. Malcev algebras are a natural generalization of Lie algebras suggested by introducing the commutator of two elements as a new multiplicative operation in an alternative algebra [3]. The defining identities obtained in this way for a Malcev algebra $A$ are

\begin{align}
\text{(1.1)} & \quad xy = -yx \\
\text{(1.2)} & \quad xy \cdot xz = (xy \cdot z)x + (yz \cdot x)x + (zx \cdot x)y
\end{align}

for all $x, y, z \in A$. Since Albert [1] has shown that every simple alternative ring which contains an idempotent not its unity quantity is either associative or the split Cayley-Dickson algebra $C$, it is natural to see if a simple Malcev algebra can be obtained from $C$. In [3] a seven dimensional simple non-Lie Malcev algebra $A^*$ is obtained from $C$ and is discussed in detail. In this paper we shall prove the following

**Theorem.** Let $A$ be a finite dimensional simple non-Lie Malcev algebra over an algebraically closed field of characteristic zero. Furthermore assume $A$ contains an element $u$ such that the right multiplication by $u$, $R_u$, is not a nilpotent linear transformation. Then $A$ is isomorphic to $A^*$.

The necessary identities and notation from [3] for any algebra $A$ are repeated here for convenience:

\begin{align}
\text{(1.3)} & \quad \text{Commutator, } (x, y) = [x, y] = xy - yx \\
\text{(1.4)} & \quad \text{Associator, } (x, y, z) = xy \cdot z - x \cdot yz \\
\text{(1.5)} & \quad \text{Jacobian, } J(x, y, z) = xy \cdot z + yz \cdot x + zx \cdot y
\end{align}

for $x, y, z \in A$. If $h(x_1, \ldots, x_n)$ is a function of $n$ indeterminates such that for any $n$ subsets $B_i$ of $A$ and $b_i \in B_i$, the elements $h(b_1, \ldots, b_n)$ are in $A$, then $h(B_1, \ldots, B_n)$ will denote the linear subspace of $A$ spanned by all of the elements $h(b_1, \ldots, b_n)$.

For a Malcev algebra $A$ of characteristic not 2 or 3, we shall use the following identities and theorems from [3]:

\begin{align}
\text{(1.6)} & \quad J(x, y, xz) = J(x, y, z)x
\end{align}
\( J(x, y, wz) + J(w, y, xz) = J(x, y, z)w + J(w, y, z)x \)

\( 2wJ(x, y, z) = J(w, x, yz) + J(w, y, zx) + J(w, z, xy) \)

\( J(wx, y, z) = wJ(x, y, z) + J(w, y, z)x - 2J(yz, w, x) \)

\( xy \cdot zw = x(wy \cdot z) + w(yz \cdot x) + y(zx \cdot w) + z(xw \cdot y) \)

for all \( w, x, y, z \in A \). If \( N = \{ x \in A: J(x, A, A) = 0 \} \), then it is shown in [3] that \( N \) is an ideal of \( A \) which is a Lie subalgebra and furthermore for \( a, b \in A \)

\( J(a, b, A) = 0 \) implies \( ab \in N \).

It is also shown in [3] that \( J(A, A, A) \) is an ideal of \( A \). Thus if \( A \) is a simple non-Lie Malcev algebra we have

\( N = 0 \) and \( A = J(A, A, A) \).

We shall assume throughout this paper that \( A \) is a finite dimensional simple non-Lie Malcev algebra over an algebraically closed field \( F \) of characteristic not 2 or 3 containing an element \( u \) such that \( R_u \) is not a nilpotent linear transformation. In § 2 the basic multiplicative identities are derived using methods analogous to those of Lie algebras. Decomposing \( A = A_0 \oplus A_{a} \oplus \cdots \oplus A_{\gamma} \) into weight spaces relative to \( R_u \) [2; page 132] we prove the block multiplication identities \( A_{\alpha}A_{\beta} \subset A_{\alpha+\beta} \) if \( \alpha \neq \beta, A_{a} \subset A_{-a} \) and \( A_{0}^3 = 0 \). Further identities are derived in § 3 which lead to the important result that there exists a nonzero weight \( \alpha \) such that \( A = A_0 \oplus A_{a} \oplus A_{-a} \), where \( A_0 = A_{\alpha}A_{-\alpha} \).

In § 4 we show that \( R(A_0) \), the set of right multiplications \( R_{x_0} \) by elements \( x_0 \in A_0 \), is a set of commuting linear transformations on the subspaces \( A_0, A_{a} \) and \( A_{-a} \). Analogous to Lie algebras we decompose \( A = A_0 \oplus A_{a} \oplus A_{-a} \) into weight spaces relative to \( R(A_0) \) [2; page 133] and thus find a basis of \( A \) which simultaneously triangulates the matrices of \( R(A_0) \). We now introduce the trace form, \( (x, y) = \text{trace } R_xR_y \), in § 5 and assume for the remainder of the paper that the algebraically closed field is of characteristic zero. With this and the results of § 4 we easily show that \( (x, y) \) is a nondegenerate invariant form on \( A = A_0 \oplus A_{a} \oplus A_{-a} \) and \( A_0 = uF \).

In § 6 we show that \( R_u \) has a diagonal matrix of the form

\[
\begin{bmatrix}
0 & 0 \\
\alpha I & 0 \\
0 & -\alpha I
\end{bmatrix}
\]

Using this and a few more identities we show in § 7 that the simple Malcev algebra \( A = A_0 \oplus A_{a} \oplus A_{-a} \) is isomorphic to the seven dimen-
sional algebra \( A^* \).

2. Basic multiplication identities. Let \( R_u \) (\( u \in A \)) be a fixed non-nilpotent linear transformation and decompose the simple Malcev algebra \( A \) into the weight space direct sum \( A = A_0 \oplus A_1 \oplus \cdots \oplus A_\gamma \) relative to \( R_u \) where the weight space of \( R_u \),

\[
A_a = \{ x \in A : x(\alpha I - R_u)^k = 0 \text{ for some integer } k > 0 \},
\]

is a nonzero \( R_u \)-invariant subspace of \( A \) corresponding to the weight \( a \) of \( R_u \). Let \( x_\alpha \in A_\alpha \), \( x_\beta \in A_\beta \), then using (1.6)

\[
J(u, x_\alpha, x_\beta)R_u = J(u, x_\alpha, x_\beta)u = J(u, x_\alpha, ux_\beta) = -J(u, x_\alpha, x_\beta R_u)
\]

and therefore

\[
J(u, x_\alpha, x_\beta)(\beta I + R_u) = J(u, x_\alpha, x_\beta(\beta I - R_u)).
\]

Now letting \( y_\beta = x_\beta(\beta I - R_u) \in A_\beta \) we have

\[
J(u, x_\alpha, x_\beta(\beta I - R_u)^n) = J(u, x_\alpha, y_\beta(\beta I - R_u^n)) = J(u, x_\alpha, y_\beta(\beta I + R_u)) = J(u, x_\alpha, x_\beta(\beta I - R_u^n)(\beta I + R_u)) = (u, x_\alpha, y_\beta(\beta I + R_u^n)).
\]

Continuing by induction we obtain

\[
(2.1) \quad J(u, x_\alpha, x_\beta)(\beta I + R_u)^n = J(u, x_\alpha, x_\beta(\beta I - R_u)^n)
\]

for every integer \( n \). Since \( x_\beta \in A_\beta \) there exists an integer \( N \) such that \( 0 = J(u, x_\alpha, x_\beta(\beta I - R_u)^N) = J(u, x_\alpha, x_\beta(\beta I + R_u)^N) \) and this shows \( J(u, x_\alpha, x_\beta) \in A_{-\beta} \). Now interchanging the roles of \( x_\beta \) and \( x_\alpha \) in (2.1) we also obtain \( J(u, x_\alpha, x_\beta) \in A_{-\beta} \) and thus

\[
(2.2) \quad J(u, A_\alpha, A_\beta) \subset A_{-\alpha} \cap A_{-\beta}.
\]

From (2.2) we have the following relations

\[
(2.3) \quad J(u, A_\alpha, A_\alpha) \subset A_{-\alpha}
\]

\[
(2.4) \quad J(u, A_\alpha, A_\beta) = 0 \quad \text{if } \alpha \neq \beta.
\]

We shall now prove

\[
(2.5) \quad A_\alpha A_\beta \subset A_{\alpha + \beta} \quad \text{if } \alpha \neq \beta.
\]

For if \( \alpha \neq \beta \) and \( x_\alpha \in A_\alpha \), \( x_\beta \in A_\beta \) we have by (2.4),

\[
0 = J(u, x_\alpha, x_\beta) = (x_\alpha x_\beta)R_u - x_\alpha R_u \cdot x_\beta - x_\alpha \cdot x_\beta R_u;
\]

that is, \( (x_\alpha x_\beta)R_u = x_\alpha R_u \cdot x_\beta + x_\alpha \cdot x_\beta R_u \) and so \( R_u \) is a derivation of
$A_\alpha A_\beta$ into $A_\alpha A_\beta$. This yields

$$(x_\alpha x_\beta)(R_u - (\alpha + \beta)I) = x_\alpha(R_u - \alpha I) \cdot x_\beta + x_\alpha \cdot x_\beta(R_u - \beta I)$$

and in the usual way we prove the Leibnitz rule for derivations which then yields that for some integer $N$, $(x_\alpha x_\beta)(R_u - (\alpha + \beta)I)^N = 0$ and therefore $x_\alpha x_\beta \in A_{\alpha + \beta}$. In particular we have

$$\tag{2.6} A_0 A_\alpha \subset A_\alpha \text{ if } \alpha \neq 0.$$ 

We shall now investigate $A_0$ more closely. Let $x_\alpha \in A_\alpha$, $x_\beta \in A_\beta$ and $x_0 \in A_0$, then by (1.7) $J(x_0, x_\beta, ux_\alpha) + J(u, x_\beta, x_0x_\alpha) = J(x_0, x_\beta, x_\alpha)u + J(u, x_\beta, x_0)\alpha x_\alpha$. Therefore if $0 \neq \alpha \neq \beta$ we have by (2.4) $J(x_0, x_\beta, ux_\alpha) = J(x_0, x_\beta, x_\alpha)u$. This yields $J(x_0, x_\beta, x_\alpha(\alpha I - R_u)) = J(x_0, x_\beta, x_\alpha)(\alpha I + R_u)$ and as in the proof of (2.4) we obtain

$$\tag{2.7} J(A_0, A_\alpha, A_\beta) = 0 \text{ if } 0 \neq \alpha \neq \beta \neq 0.$$ 

Next let $x_0, y_0 \in A_0$ and $x_\alpha \in A_\alpha$ where $\alpha \neq 0$, then using (1.9), (2.4) and (2.6) we have

$$J(x_0u, y_0, x_\alpha) = x_0J(u, y_0, x_\alpha) + J(x_0, y_0, x_\alpha)u - 2J(y_0x_\alpha, x_\alpha)u$$

and in general we have $J(x_\alpha R_\alpha^u, y_0, x_\alpha) = J(x_0, y_0, x_\alpha)R_\alpha^u$ which implies $J(x_0, y_0, x_\alpha) \in A_0$. Now by (1.7), $J(x_0, y_0, ux_\alpha) + J(u, y_0, x_\alpha)x_\alpha = J(x_0, y_0, x_\alpha)u + J(u, y_0, x_\alpha)x_0$; and using (2.4) and (2.6) we obtain $J(x_0, y_0, x_\alpha R_u) = -J(x_0, y_0, x_\alpha)R_u$ which implies $J(x_0, y_0, x_\alpha(\alpha I - R_u)) = -J(x_0, y_0, x_\alpha)(R_u + \alpha I)$. Thus, as usual, we have $J(x_0, y_0, x_\alpha) \in A_{\alpha}$ and therefore $J(x_0, y_0, x_\alpha) \in A_0 \cap A_{\alpha}$ which proves

$$\tag{2.8} J(A_0, A_\alpha, A_\beta) = 0 \text{ if } \alpha \neq 0.$$ 

We shall now show $A_0^a \subset A_0$. From our basic decomposition $A = A_0 \oplus A_\alpha \oplus \cdots \oplus A_\gamma$ relative to $R_u$ we can find a basis $\{x_1(\tau), \cdots, x_m(\tau)\}$ ($m = m_\tau$) of $A_\tau$ such that

$$\tag{2.9} x_i(\tau)R_u = \sum_{j=1}^{i-1} a_{ij}x_j(\tau) + \tau x_i(\tau)$$

where $\tau, a_{ij} \in F$ and $i = 1, \cdots, m$. In particular let $\{x_1(0), \cdots, x_n(0)\} = \{x_1, \cdots, x_n\}$ be the above type for $A_0$. Then $x_i R_u = 0$ and

$$x_i R_u = \sum_{k=1}^{i-1} a_{ik}x_k \quad (i = 2, \cdots, n).$$

Furthermore,

$$J(u, x_i, x_j) = (x_i x_j)R_u + x_j R_u \cdot x_i + x_j \cdot x_i R_u$$

$$= (x_i x_j)R_u + \sum_{k=1}^{i-1} a_{jk}x_k x_i + \sum_{k=1}^{i-1} a_{ik}x_j x_k$$
with the understanding that $a_{10} = 0$.

Using (1.6) and operating on both sides of the previous equation with $R_n^u$, we obtain

$$(-1)^n J(u, x_i, x_j R_n^u) = J(u, x_i, x_j) R_n^u$$

$$= (x_i x_j) R_n^{n+1} + \sum_{k=1}^{j-1} a_{jk} (x_k x_i) R_n^u$$

$$+ \sum_{k=1}^{i-1} a_{ik} (x_j x_k) R_n^u .$$

Now by assuming $i < j$ and choosing $n$ large enough, a simple inductive argument yields $x_i x_j \in A_0$ for all $i$ and $j$. Thus $A_0^2 \subset A_0$.

Using (1.8), $A_0^2 \subset A_0$ and (2.8) we have

$$A_\alpha J(A_0, A_0, A_0) \subset J(A_\alpha, A_0, A_0) \subset J(A_\alpha, A_0, A_0) = 0 \quad \text{for } \alpha \neq 0 .$$

Thus, $A J(A_0, A_0, A_0) \subset \sum_{\alpha} A_\alpha J(A_0, A_0, A_0) = A_0 J(A_0, A_0, A_0) \subset J(A_0, A_0, A_0)$, or $J(A_0, A_0, A_0)$ is an ideal of $A$. But since $J(A_0, A_0, A_0) \subset A_0 \neq A$ and $A$ is simple we have

$$J(A_0, A_0, A_0) = 0 .$$

Now using (2.8) and (2.10) we have $J(A_0, A_0, A) = \sum_{\alpha} J(A_0, A_0, A_\alpha) = 0$ and by (1.11) and (1.12),

$$A_0^2 \subset N = 0 .$$

In particular this means the kernel of $R_u$ is $A_0$.

We shall now show $A_\alpha \subset A_{-\alpha}$. Let $x_\alpha, y_\alpha \in A_\alpha$ for $\alpha \neq 0$, then by

$$J(u, x_\alpha, y_\alpha) = (x_\alpha y_\alpha) R_u + y_\alpha R_u \cdot x_\alpha + y_\alpha \cdot x_\alpha R_u = w_{-\alpha} \in A_{-\alpha}.$$ 

Therefore $(x_\alpha y_\alpha) R_u = x_\alpha R_u \cdot y_\alpha + y_\alpha \cdot y_\alpha R_u + w_{-\alpha}$ which yields

$$(x_\alpha y_\alpha)(R_u - 2\alpha I) = x_\alpha (R_u - \alpha I) \cdot y_\alpha + x_\alpha \cdot y_\alpha (R_u - \alpha I) + w^{(1)}_{-\alpha} .$$

By induction we obtain

$$(x_\alpha y_\alpha)(R_u - 2\alpha I)^n = w^{(n)}_{-\alpha} + \sum_{r=0}^n C_{\alpha, r} x_\alpha (R_u - \alpha I)^{n-r} \cdot y_\alpha (R_u - \alpha I)^r ,$$

where $w^{(n)}_{-\alpha} \in A_{-\alpha}$. Therefore for large enough $N$, $(x_\alpha y_\alpha)(R_u - 2\alpha I)^N \in A_{-\alpha}$.

Now let $x_\alpha y_\alpha = \sum_{\gamma} z_\gamma$ where $z_\gamma \in A_\gamma$, then $(x_\alpha y_\alpha)(R_u - 2\alpha I)^N = \sum_{\gamma} z_\gamma (R_u - 2\alpha I)^N \in A_{-\alpha}$. Therefore by the $R_u$-invariance of the $A_\gamma$ and the uniqueness of the decomposition $A = A_0 \oplus A_\alpha \oplus \cdots \oplus A_\lambda$, $z_{\gamma}(R_u - 2\alpha I)^N \times 0$ if $\gamma \neq -\alpha$. Thus if $\gamma \neq -\alpha, z_\gamma \in A_{2\alpha}$. Therefore $x_\alpha y_\alpha = z_{2\alpha} + z_{-\alpha}$ which proves

$$A_\alpha^2 \subset A_{2\alpha} \oplus A_{-\alpha} .$$
**Lemma 2.13.** \( J(u, A^2_a, A^2_\alpha) = 0. \)

**Proof.** Using (2.12), (2.7) and (2.3) we have
\[
J(u, A^2_a, A^2_\alpha) \subseteq J(u, A_{-\alpha}, A^2_\alpha) + J(u, A^2_\alpha, A^2_\alpha) \subseteq J(u, A^2_\alpha, A^2_\alpha) \subseteq A_{-\alpha}.
\]
Now for any \( x, y \in A_\alpha, z \in A_{2\alpha} \) we have by (1.7) \( J(z, u, xy) + J(x, u, zy) = J(z, u, y)x + J(x, u, y)z \) and using (2.4), (2.5) and (2.3) this yields
\[
J(z, u, xy) = J(x, u, y)z \in A_{-\alpha} \cdot A_{2\alpha} \subseteq A_\alpha.
\]
Combining these results we have \( J(u, A^2_a, A^2_\alpha) \subseteq A_\alpha \cap A_{-\alpha} = 0. \)

Now let \( w \in A_{2\alpha}, x, y \in A_\alpha \) and \( xy = z_{2\alpha} + z_{-\alpha} \) where \( z_{2\alpha} \in A_{2\alpha}, z_{-\alpha} \in A_{-\alpha} \), then using Lemma 2.13 and the fact \( J(u, A_{-\alpha}, A^2_{\alpha}) = 0 \) we have
\[
0 = J(u, xy, w) = J(u, z_{2\alpha}, w) + J(u, z_{-\alpha}, w) = J(u, z_{2\alpha}, w);
\]
that is,
\[
J(u, z_{2\alpha}, A^2_{2\alpha}) = 0.
\]
Now since \( z_{2\alpha} \in A_{2\alpha} \) we also have by (2.4) \( J(u, z_{2\alpha}, A_{\beta}) = 0 \) if \( \beta \neq 2\alpha \).

Combining these results, \( J(u, z_{2\alpha}, A) = \sum_{\beta} J(u, z_{2\alpha}, A_{\beta}) = 0 \) and therefore \( z_{2\alpha}u \in N = 0 \) by (1.11) and (1.12). Thus \( 0 = z_{2\alpha}R_u \) and therefore \( z_{2\alpha} \in A_0 \cap A_{2\alpha} = 0 \) and this proves
\[
(2.14) \quad A^2_\alpha \subseteq A_{-\alpha}.
\]
Also note that we now have
\[
(2.15) \quad J(A_0, A_\alpha, A_\alpha) \subseteq A_{-\alpha}.
\]

3. **More identities.** Let \( A = A_0 \oplus A_\alpha \oplus \cdots \oplus A_\gamma \) be the decomposition of \( A \) into a weight space direct sum relative to \( R_u \) and suppose that for weights \( \alpha, \beta, \gamma \) of \( R_u, \beta \neq \gamma \) and \( \beta + \gamma \neq \alpha \). Then for \( x \in A_\alpha, y \in A_\beta \) and \( z \in A_\gamma \) we have by (1.9) and (2.4)
\[
J(xu, y, z) = xuJ(u, y, z) + J(x, y, z)u - 2J(yz, x, u) = J(x, y, z)u
\]
and therefore \( J(x(R_u - \alpha I), y, z) = J(x, y, z)(R_u - \alpha I). \) By induction we have \( J(x(R_u - \alpha I)^n, y, z) = J(x, y, z)(R_u - \alpha I)^n \) and hence
\[
(3.1) \quad J(A_\alpha, A_\beta, A_\gamma) \subseteq A_\alpha \quad \text{if} \quad \beta \neq \gamma \quad \text{and} \quad \beta + \gamma \neq \alpha.
\]
By the symmetry of the \( \alpha, \beta \) and \( \gamma \) we may also conclude
\[
(3.2) \quad J(A_\beta, A_\gamma, A_\alpha) \subseteq A_\beta \quad \text{if} \quad \gamma \neq \alpha \quad \text{and} \quad \gamma + \alpha \neq \beta
\]
\[
(3.3) \quad J(A_\gamma, A_\alpha, A_\beta) \subseteq A_\gamma \quad \text{if} \quad \alpha \neq \beta \quad \text{and} \quad \alpha + \beta \neq \gamma.
\]
Now assume \( \alpha \neq \beta \neq \gamma \neq \alpha \). Suppose \( \beta + \gamma = \alpha \). If \( \gamma + \alpha = \beta \),
then \( \gamma = 0 \) and therefore \( \alpha = \beta \), a contradiction. Therefore \( \gamma + \alpha = \beta \) and by (3.2) \( J(A_\beta, A_\gamma, A_\alpha) \subseteq A_\beta \). Similarly if \( \alpha + \beta = \gamma \), then \( \beta = 0 \) and \( \alpha = \gamma \), a contradiction. Therefore \( \alpha + \beta = \gamma \) and by (3.3) \( J(A_\gamma, A_\alpha, A_\beta) \subseteq A_\gamma \). Thus we have \( J(A_\alpha, A_\beta, A_\gamma) \subseteq A_\gamma \cap A_\beta = 0 \) if \( \alpha \neq \beta = \gamma = \alpha \) and \( \beta + \gamma = \alpha \).

With the assumption \( \alpha \neq \beta \neq \gamma \neq \alpha \), suppose now that \( \beta + \gamma = \alpha \). Then by (3.1), \( J(A_\alpha, A_\beta, A_\gamma) \subseteq A_\alpha \). We next note that it is impossible to have \( \gamma + \alpha = \beta \) and \( \alpha + \beta = \gamma \). So using (3.2) or (3.3) together with \( J(A_\alpha, A_\beta, A_\gamma) \subseteq A_\alpha \) we conclude \( J(A_\alpha, A_\beta, A_\gamma) = 0 \). Thus we can conclude, using the preceding paragraph,

\[
J(A_\alpha, A_\beta, A_\gamma) = 0 \quad \text{if} \quad \alpha \neq \beta \neq \gamma \neq \alpha .
\]

Now assume two weights are equal, that is, \( \alpha = \beta \). Suppose \( \gamma \neq 0, \alpha, -\alpha \) or \( 2\alpha \), then

\[
J(A_\alpha, A_\alpha, A_\gamma) \subseteq A_\alpha^2 A_\gamma + A_\alpha A_\gamma A_\alpha + A_\gamma A_\alpha A_\alpha
\]

\[
\subseteq A_{-\alpha} A_\gamma + A_{\alpha+\gamma} A_\alpha
\]

\[
\subseteq A_{-\alpha + \gamma} \oplus A_{\gamma + 2\alpha}.
\]

However using (3.1) \( J(A_\alpha, A_\alpha, A_\gamma) \subseteq A_\alpha \) and therefore \( J(A_\alpha, A_\alpha, A_\gamma) \subseteq A_\alpha \cap (A_{-\alpha + \gamma} \oplus A_{\gamma + 2\alpha}) = 0 \). This proves

\[
J(A_\alpha, A_\alpha, A_\gamma) = 0 \quad \text{if} \quad \gamma \neq 0, \alpha, \text{ or } -\alpha 2\alpha .
\]

For the “exceptional” cases we have

\[
J(A_\alpha, A_\alpha, A_\alpha) \subseteq A_\alpha^2 A_\alpha \subseteq A_{-\alpha} A_\alpha A_\alpha \subseteq A_\alpha .
\]

\[
J(A_\alpha, A_\alpha, A_\alpha) \subseteq A_\alpha^2 A_\alpha \subseteq A_{-\alpha} A_\alpha A_\alpha \subseteq A_{-\alpha}.
\]

\[
J(A_\alpha, A_\alpha, A_\alpha) \subseteq A_\alpha^2 A_\alpha \subseteq A_{-\alpha} A_\alpha A_\alpha \subseteq A_{-\alpha}.
\]

\[
J(A_\alpha, A_\alpha, A_\alpha) = 0 .
\]

To prove (3.9) let \( x, y \in A_\alpha, z \in A_{2\alpha} \), then by (1.9), (2.5) and (2.4)

\[
J(xu, y, z) = xJ(u, y, z) + J(x, y, z)u - 2J(yz, x, u)
\]

\[
= J(x, y, z)u
\]

and as usual we have \( J(x(R_u - \alpha I)^n, y, z) = J(x, y, z)(R_u - \alpha I)^n \). Therefore \( J(x, y, z) \in A_\alpha \). However by (1.7) \( J(x, y, uz) + J(u, y, xz) = J(x, y, z)u + J(u, y, z)x \) and using (2.4) we obtain \( J(x, y, uz) = J(x, y, z)u \). This yields \( J(x, y, z(2\alpha I - R_u)^n) = J(x, y, z)(2\alpha I + R_u)^n \) and therefore \( J(x, y, z) \in A_{-2\alpha} \). Combining the above results we have \( J(x, y, z) \in A_\alpha \cap A_{-2\alpha} = 0 \) if \( \alpha \neq 0 \).

We shall now show \( A_\alpha A_\beta = 0 \) if \( \alpha \neq 0 \) and \( \beta \neq 0, \pm \alpha \). Let \( \alpha \) and \( \beta \) be fixed weights of \( R_u \) and assume \( \beta \neq k\alpha, k = 0, \pm 1, \pm 2, \cdots \), with
\( \alpha \neq 0 \). Then for any other weight \( \gamma \) we have by (3.4) \( J(A_{\beta}, A_{\alpha}, A_{\gamma}) = 0 \) if \( \beta \neq \alpha \neq \gamma \neq \beta \). However \( \alpha \neq \beta \) and therefore \( J(A_{\beta}, A_{\alpha}, A_{\gamma}) = 0 \) if \( \alpha \neq \gamma \neq \beta \). Suppose \( \gamma = \alpha \), then by (3.5) and the choice of \( \beta \), \( J(A_{\beta}, A_{\alpha}, A_{\alpha}) = 0 \). Suppose \( \gamma = \beta \), then \( J(A_{\beta}, A_{\alpha}, A_{\beta}) = J(A_{\beta}, A_{\beta}, A_{\alpha}) = 0 \) if \( \alpha \neq 0, \beta, -\beta \) or \( 2\beta \). We know \( \alpha \neq 0, \beta \) or \( -\beta \) so if \( \alpha = 2\beta \), then by (3.9) \( J(A_{\beta}, A_{\beta}, A_{\alpha}) = 0 \). Combining all these cases we have shown \( J(A_{\beta}, A_{\alpha}, A_{\gamma}) = 0 \) for any weight \( \gamma \) and therefore \( J(A_{\beta}, A_{\alpha}, A_{\gamma}) = \sum_{\gamma} J(A_{\beta}, A_{\alpha}, A_{\gamma}) = 0 \). By (1.11) and (1.12) \( A_{\alpha}A_{\beta} \subset N = 0 \). This proves

\[ (3.10) \quad A_{\alpha}A_{\beta} = 0 \quad \text{if} \quad \alpha \neq 0 \quad \text{and} \quad \beta \neq k\alpha, k = 0, \pm 1, \pm 2, \ldots . \]

We now assume \( \alpha \neq 0 \) and \( \beta = k\alpha \) for \( k \neq 0, \pm 1 \), then \( J(A_{\alpha}, A_{\beta}, A_{\gamma}) = J(A_{\alpha}, A_{k\alpha}, A_{\gamma}) = 0 \) if \( \alpha \neq k\alpha \neq \gamma \neq \alpha \), by (3.4). But since \( k \neq 1 \) we have \( J(A_{\alpha}, A_{k\alpha}, A_{\gamma}) = 0 \) if \( \alpha \neq \gamma \neq k\alpha \). Suppose \( \gamma = \alpha \), then using (3.5)

\[
\begin{align*}
J(A_{\alpha}, A_{\beta}, A_{\gamma}) &= J(A_{\alpha}, A_{k\alpha}, A_{\gamma}) \\
&= J(A_{\alpha}, A_{k\alpha}, A_{\alpha}) \\
&= 0
\end{align*}
\]

if \( k\alpha \neq 0, \alpha, -\alpha \) or \( 2\alpha \). But by the choice of \( k \) we need only consider \( k\alpha = 2\alpha \) and in this case \( J(A_{\alpha}, A_{\alpha}, A_{k\alpha}) = 0 \) by (3.9). Now suppose \( \gamma = k\alpha \), then

\[
\begin{align*}
J(A_{\alpha}, A_{\beta}, A_{\gamma}) &= J(A_{\alpha}, A_{k\alpha}, A_{\gamma}) \\
&= J(A_{\alpha}, A_{k\alpha}, A_{k\alpha}) \\
&= 0
\end{align*}
\]

if \( \alpha \neq 0, k\alpha, -k\alpha \) or \( 2k\alpha \), by (3.5). Again by the choice of \( k \) and \( \alpha \) we need only consider \( \alpha = 2k\alpha \). In this case \( k = 1/2 \) and therefore \( \gamma = \beta = k\alpha = 1/2\alpha \). This yields \( J(A_{\alpha}, A_{\beta}, A_{\gamma}) = J(A_{\beta}, A_{\beta}, A_{2\alpha}) = 0 \) by (3.9). Combining all of these cases we have for any weight \( \gamma \), \( J(A_{\alpha}, A_{k\alpha}, A_{\gamma}) = 0 \) if \( \alpha \neq 0, k \neq 0, \pm 1 \) and as before this gives

\[ (3.11) \quad A_{\alpha}A_{k\alpha} = 0 \quad \text{if} \quad \alpha \neq 0, k \neq 0, \pm 1 . \]

(3.10) and (3.11) yield

\[ (3.12) \quad A_{\alpha}A_{\beta} = 0 \quad \text{if} \quad \alpha \neq 0, \beta \neq 0, \pm \alpha . \]

Since \( R_u \) is not nilpotent, there exists a weight \( \alpha \neq 0 \). We shall now show that \( -\alpha \) is also a weight of \( R_u \). For suppose \( -\alpha \) is not a weight, then by the usual convention \( A_{-\alpha} = 0 \) and noting that none of the previously derived identities use the fact that \( A_{-\alpha} \neq 0 \) we have for \( \beta \neq 0 \) or \( \alpha \), that \( A_{\alpha}A_{\beta} = 0 \) by (3.12). For \( \beta = 0 \), \( A_{\alpha}A_{\beta} \subset A_{\alpha} \) and for
\[ \beta = \alpha, \ A_\beta A_\beta \subset A_{-\alpha} = 0 \text{ using (2.14)}. \] Therefore \( A_\alpha \) is a nonzero ideal of \( A \) and so \( A = A_\alpha \). But \( u \in A \) and \( u \not\in A_\alpha = A \), a contradiction. Therefore \(-\alpha\) is a weight if \( \alpha \) is a weight.

Now set \( \mathcal{N}_\alpha = A_\alpha A_{-\alpha} \oplus A_\alpha \oplus A_{-\alpha} \) where \( \alpha \) is a nonzero weight. Then \( \mathcal{N}_\alpha \neq 0 \) and for \( \beta = 0, \pm \alpha \) we have \( \mathcal{N}_\alpha A_\beta \subset \mathcal{N}_\alpha \). For \( \beta \neq 0, \pm \alpha \) we have \( A_\alpha A_\beta = A_{-\alpha} A_\beta = 0 \) by (3.12). Now by (3.4) and (3.12) we have for \( x \in A_\alpha, y \in A_{-\alpha}, z \in A_\beta \) that \( 0 = J(x, y, z) = xy \cdot z + yz \cdot x + zx \cdot y = xy \cdot z \) and so \( 0 = A_\alpha A_{-\alpha} A_\beta \). Thus in all cases \( \mathcal{N}_\alpha A_\beta \subset \mathcal{N}_\alpha \) and therefore \( \mathcal{N}_\alpha \) is a nonzero ideal of \( A \) and we have \( A = \mathcal{N}_\alpha \). This proves

**Proposition 3.13.** If \( A \) is a finite dimensional simple non-Lie Malcev algebra over an algebraically closed field of characteristic not 2 or 3 and \( A \) contains an element \( u \) such that \( R_u \) is not a nilpotent linear transformation, then there exists an \( \alpha \neq 0 \) such that \( A = A_0 \oplus A_\alpha \oplus A_{-\alpha} \) where \( A_\alpha = \{ x \in A : x(aI - R_u)^k = 0 \text{ for some } k > 0 \} \) and \( A_0 = A_\alpha A_{-\alpha} \).

4. A decomposition of \( A \) relative to \( A_0 \). Let us consider the decomposition of \( A \) as given Proposition 3.13; that is,

\[ A = A_0 \oplus A_\alpha \oplus A_{-\alpha}. \]

For any \( y_0, z_0 \in A_0 \) and \( x \in A_\alpha (a = 0, \pm \alpha) \), we use (2.8) and (2.11) to see that

\[ 0 = J(x, y_0, z_0) = x(R_{y_0} R_{z_0} - R_{z_0} R_{y_0}) \cdot \]

Therefore,

\[ R(A_0) = \{ R_{z_0} : x_0 \in A_0 \} \]

is a commuting set of linear transformations acting on \( A_\alpha \). We can find \( R(A_0) \)-invariant subspaces \( M_\lambda (\alpha) \) [2; Chapter 4] such that

\[ A_\alpha = \sum_\lambda \bigoplus M_\lambda (\alpha) \quad (a = 0, \pm \alpha), \]

where on each \( M_\lambda (\alpha) \) the transformation \( R_{z_0} \), for any \( x_0 \in A_0 \), has a matrix of the form

\[
\begin{bmatrix}
\lambda(x_0) & 0 \\
* & \lambda(x_0)
\end{bmatrix};
\]

that is, \( M_\lambda (\alpha) \) has a basis \( \{ x_1, x_2, \ldots, x_m \} (m = m(\lambda, \alpha)) \) such that for any \( x_0 \in A_0 \), there exists \( a_{i,j}(x_0) \in F \) for which

\[ x_i R_{z_0} = \sum_{j=1}^{i-1} a_{i,j}(x_0)x_j + \lambda(x_0)x_i, \]

where \( \lambda(x_0) \in F \) and, of course, \( i = 1, 2, \ldots, m. \)
Using the usual terminology we call the function \( \lambda \) defined by \( \lambda: x_0 \rightarrow \lambda(x_0) \) a weight of \( A_0 \) in \( A_a \) or just a weight and the corresponding \( M_\lambda(a) \) a weight space of \( A_a \) corresponding to \( \lambda \) or just a weight space of \( A_a \). It is easily seen [2] that \( A_a \) has finitely many weights and the weights are linear functionals on \( A_0 \) to \( F \). Also

\[
M_\lambda(a) = \{ x \in A_a : \text{for all } x_0 \in A_0, x(R_{x_0} - \lambda(x_0)I)^k = 0 \text{ for some integer } k > 0 \}
\]

and for this weight \( \lambda \) we have \( \lambda(u) = a \). For suppose \( \lambda(u) = b \), then there exists an \( x \neq 0 \) in \( M_\lambda(a) \) such that \( bx = xR_u \). But \( M_\lambda(a) \subset A_a = \{ x \in A : x(R_u - aI)^n = 0 \} \); therefore \( (b - a)x = x(R_u - aI)^n \) so for some integer \( N \), \( (b - a)^nx = x(R_u - aI)^n = 0 \) and thus \( a = b = \lambda(u) \). We now combine the weight space decompositions of the \( A_a \) to form a weight space decomposition of \( A \) in

**Proposition 4.2.** Let \( A = A_0 \oplus A_a \oplus A_{-a} \) be a simple Malcev algebra as determined by Proposition 3.13, then we can write \( A = A_0 \oplus \Sigma_\lambda \oplus M_\lambda(\alpha) \oplus \Sigma_\mu \oplus M_\mu(-\alpha) \) where all weights are distinct and any nonzero weight \( \rho \) of \( A_0 \) in \( A \) is a weight of \( A_0 \) in \( A_a \) or \( A_{-a} \) but not both.

**Proof.** The first part is clear noting that in the original weight space decomposition \( A_a = \sum_\gamma \oplus M_\gamma(\alpha) \) the weights of \( A_0 \) in \( A_a \) can be taken to be distinct. Also if \( \lambda \) is a weight of \( A_0 \) in \( A_a \) and \( \mu \) a weight of \( A_0 \) in \( A_{-a} \), then \( \lambda(u) = \alpha = -\alpha = \mu(u) \) and therefore \( \lambda \neq \mu \). Now let \( \rho \neq 0 \) be any weight of \( A_0 \) in \( A \) with weight space \( M_\rho = \{ x \in A : x(R_{x_0} - \rho(x_0)I)^k = 0 \} \) and let \( y = y_0 + y_a + y_{-a} \in M_\rho \) where \( y_a \in A_a \) with \( a = 0, \pm \alpha \). Then for some integer \( N > 0 \),

\[
0 = y(R_{x_0} - \rho(x_0)I)^N = y_0(R_{x_0} - \rho(x_0)I)^N + y_a(R_{x_0} - \rho(x_0)I)^N + y_{-a}(R_{x_0} - \rho(x_0)I)^N
\]

and by the uniqueness of the decomposition \( A = A_0 \oplus A_a \oplus A_{-a} \) we have \( y_a(R_{x_0} - \rho(x_0)I)^N = 0 \) for \( a = 0, \pm \alpha \). Now by using the binomial theorem and \( A_0^2 = 0 \) we have \( 0 = y_a(R_{x_0} - \rho(x_0)I)^N = y_0\rho(x_0)^N \) and since \( \rho \neq 0, y_0 = 0 \). Thus we have \( y_a(R_{x_0} - \rho(x_0)I)^N = 0, a = \pm \alpha, \) for some integer \( N \) and so \( \rho \) is a weight of \( A_0 \) in \( A_a \) and \( A_{-a} \). Now suppose \( y_a \) and \( y_{-a} \) are both nonzero, then since \( \rho \) is a weight of \( A_0 \) in \( A_{-a}, \rho(u) = \alpha \) and since \( \rho \) is a weight of \( A_0 \) in \( A_{-a}, \rho(u) = -\alpha \), a contradiction. Thus \( \rho \) is a weight of \( A_0 \) in either \( A_a \) or \( A_{-a} \) but not both.

We shall use the usual convention that if \( \rho \) is not a weight of \( A_0 \) in \( A \), then \( M_\rho = 0 \). Let \( M_\lambda(a) \) and \( M_\mu(a) \) be weight spaces of \( A_0 \) in \( A_a \).
and let \( x_0, y_0 \in A_0 \) and \( x \in M_\lambda(a), y \in M_\mu(a) \), then using (2.8) and (1.7) we have
\[
J(x, x_0, y_0 y) = J(y_0, x_0, xy) + J(x, x_0, y_0 y) \\
= J(y_0, x_0, y)x + J(x, x_0, y)y_0 \\
= J(x, x_0, y)y_0.
\]
Thus \( J(x_0, x, y(R_y - \mu(y_0)I)) = -J(x_0, x, y)(R_y + \mu(y_0)I) \) and by induction
\[
J(x_0, x, y(R_y - \mu(y_0)I)^n) = (-1)^nJ(x_0, x, y)(R_y + \mu(y_0)I)^n.
\]
From this we obtain \( J(x_0, x, y) \in M_{-\mu}(-a) \) and interchanging the roles of \( x \) and \( y \) we see \( J(x_0, x, y) \in M_{-\lambda}(-a) \); this proves
\[
(4.3) \quad J(A_0, M_\lambda(a), M_\mu(a)) \subset M_{-\lambda}(-a) \cap M_{-\mu}(-a).
\]
From (4.3) we obtain
\[
(4.4) \quad J(A_0, M_\lambda(a), M_\lambda(a)) \subset M_{-\lambda}(-a)
\]
\[
(4.5) \quad J(A_0, M_\lambda(a), M_\mu(a)) = 0 \quad \text{if} \quad \lambda \neq \mu.
\]
We shall next show
\[
M_\lambda(a)M_\mu(a) = 0 \quad \text{if} \quad \lambda \neq \mu.
\]
For let \( x_0 \in A_0, x \in M_\lambda(a) \) and \( y \in M_\mu(a) \), then by (4.5) \( 0 = J(x, y, x_0) \) and therefore \( xyR_{x_0} = xR_{x_0} \cdot y + x \cdot yR_{x_0} \) and hence \( xy(R_{x_0} - (\mu(x_0) + \lambda(x_0))I) = x(R_{x_0} - \lambda(x_0)I) \cdot y + x \cdot y(R_{x_0} - \mu(x_0)I) \). In the usual way we can prove there exists an integer \( N \) such that \( xy(R_{x_0} - (\mu(x_0) + \lambda(x_0))I)^N = 0 \) and since we know \( xy \in A_{-a} \) this shows \( xy \in M_{\lambda+\mu}(-a) \) if \( \lambda + \mu \) (defined by \( (\lambda + \mu)(x_0) = \lambda(x_0) + \mu(x_0) \)) is a weight of \( A_0 \) in \( A_{-a} \), or \( xy = 0 \). If \( xy \neq 0 \), then \( \lambda + \mu \) is a weight of \( A_0 \) in \( A_{-a} \) where \( \lambda \) and \( \mu \) are weights of \( A_0 \) in \( A_a \) and therefore \( -a = (\lambda + \mu)(u) = \lambda(u) + \mu(u) = a + a \), a contradiction.

Next we have for any weight \( \lambda \) of \( A_0 \) in \( A_a \)
\[
M_\lambda(a)M_\lambda(a) \subset M_{-\lambda}(-a)
\]
if \( -\lambda \) is a weight of \( A_0 \) in \( A_{-a} \). For let \( x_0 \in A_0 \) and \( \lambda \equiv \lambda(x_0) \in F \) and let \( M_\lambda(a) \) have basis \( \{x_1, \cdots, x_m\} \) as in (4.1). Then using (1.2) we obtain
\[
\lambda^2x_1x_2 = \lambda x_1(\lambda x_2 + a_2x_1) \\
= x_1R_{x_0} \cdot x_1R_{x_0} \\
= (x_1x_1 \cdot x_2)x_0 + (x_1x_2 \cdot x_0)x_0 + (x_2x_0 \cdot x_0)x_1 \\
= -\lambda x_1x_2R_{x_0} + x_1x_2R_{x_0}^2 + \lambda^2x_2x_1
\]
and thus
\[
0 = x_1x_2(R_{x_0}^2 - \lambda R_{x_0} - 2\lambda^2I) = x_1x_2(R_{x_0} + \lambda I)(R_{x_0} - 2\lambda I).
\]
Now since \( \lambda \) is a weight of \( A_0 \) in \( A_a \), \(-2\lambda \) is not a weight of \( A_0 \) in \( A_{-a} \): \(-a = (2\lambda)(u) = 2\lambda(u) = 2a \). Thus the above equation implies 
\[ x_i x_2 (R_{x_0} + \lambda I) = 0 \]
and therefore 
\[ x_i x_2 \in M_{-\lambda}(-a) \]. Next 
\[ x_i x_0 \cdot x_3 x_0 = \lambda x_1 (\lambda x_3 + a_3 x_3) = \lambda^3 x_3 + s \]
where \( s \in M_{-\lambda}(-a) \) and 
\[ (x_0 x_1 \cdot x_3) x_0 + (x_0 x_2 \cdot x_3) x_0 + (x_0 x_0 \cdot x_3) x_1 = -\lambda x_3 x_2 R_{x_0} + x_3 x_3 R_{x_0} + \lambda^3 x_3 x_3 + t \]
where \( t \in M_{-\lambda}(-a) \). Therefore using (1.2) we obtain 
\[ 0 = x_i x_3 (R_{x_0} + \lambda I) (R_{x_0} - 2\lambda I) + w \]
where \( w \in M_{-\lambda}(-a) \) and actually 
\[ w = 3x_3 x_2 x_0 \]. Therefore 
\[ 0 = x_i x_3 (R_{x_0} + \lambda I) (R_{x_0} - 2\lambda I) + w \]
and as before 
\[ x_i x_3 \in M_{-\lambda}(-a) \]. Continuing this process we obtain 
\[ x_i x_k \in M_{-\lambda}(-a) \] for \( k = 1, 2, \ldots, m \). Next consider the product 
\[ x_i x_3 \]
where \( s \in M_{-\lambda}(-a) \) and 
\[ (x_0 x_2 \cdot x_0) x_0 + (x_0 x_3 \cdot x_0) x_0 + (x_0 x_0 \cdot x_0) x_2 = x_2 x_3 (R_{x_0}^2 - \lambda R_{x_0} - \lambda^2 I) + w \]
where \( w \in M_{-\lambda}(-a) \), therefore 
\[ 0 = x_i x_3 (R_{x_0} + \lambda I)(R_{x_0} - 2\lambda I) + w \]
where \( w \in M_{-\lambda}(-a) \). Therefore for some integer \( k > 0 \) such that \( w(R_{x_0} + \lambda I)^k = 0 \) we have 
\[ 0 = x_i x_3 (R_{x_0} + \lambda I)^{k+1} (R_{x_0} - 2\lambda I) \]
and as before 
\[ x_i x_3 \in M_{-\lambda}(-a) \]. We continue this process showing 
\[ x_i x_k \in M_{-\lambda}(-a) \] and in general 
\[ x_i x_j \in M_{-\lambda}(-a) \] for \( i, j = 1, \ldots, m \). This completes the proof of (4.7).

We now show

\[ M_\lambda(a) \cdot M_\mu(-a) = 0 \quad \text{if} \quad \lambda + \mu \neq 0 . \]

By (2.7) we have for \( x \in M_\lambda(a) \), \( y \in M_\mu(-a) \) and \( x_0 \in A_0 \) that 
\[ 0 = J(x, y, x_0) \]
and as usual we obtain 
\[ xy(R_{x_0} - (\lambda(x_0) + \mu(x_0)) I)^N = 0 \]
for some integer \( N > 0 \). Now 
\[ z = xy \in A_0 \]
and suppose \( z \neq 0 \), then, since \( \lambda + \mu \neq 0 \), \( \lambda + \mu \) is a nonzero weight of \( A_0 \) in \( A_0 \), a contradiction to Proposition 4.2.

Let \( x \in M_\lambda(a) \), \( y \in M_\lambda(a) \) and \( z \in M_\mu(-a) \), then using (1.9), (2.7) and (2.8) we have

\[ J(xx_0, y, z) = xx_0 J(x, y, z) + J(x, y, z)x_0 - 2J(yz, x, x_0) \]

and therefore 
\[ J(x(R_{x_0} - \rho(x_0) I), y, z) = J(x, y, z)(R_{x_0} - \rho(x_0) I) \]
and as usual we obtain 
\[ J(x, y, z) \in M_\lambda(a) \] and therefore 
\[ J(x, y, z) \in M_\lambda(a) \cap M_\mu(a) = 0 \] if \( \lambda \neq \rho \). Now assume \( \lambda = \rho \) and assume \( \mu = -\lambda \) is a weight of \( A_0 \) in \( A_{-a} \), then

\[ 0 = J(x, y, z) = xy \cdot z + yz \cdot x + zx \cdot y = yx \cdot x \]

using (4.6) and (4.8). This proves
if $\lambda \neq \rho$ are weights of $A_o$ in $A_\alpha$ such that $-\lambda$ is a weight of $A_o$ in $A_{-\alpha}$.

We shall now show if $\lambda$ is a nonzero weight of $A_o$ in $A_\alpha$ with weight space $M_\lambda(a)$, then $-\lambda$ is a nonzero weight of $A_o$ in $A_{-\alpha}$ with weight space $M_{-\lambda}(-a)$. The proof is similar to that following (3.12): Suppose $-\lambda$ is not a weight of $A_o$ in $A_{-\alpha}$, then $M_{-\lambda}(-a) = 0$; $M_\lambda(a)M_{-\lambda}(-a) = 0$ if $\rho \neq \lambda$; $A_oM_\lambda(a) \subset M_\lambda(a)$ and $M_\lambda(a)M_{-\lambda}(-a) = 0$ if $\mu + \lambda \neq 0$. Thus $M_\lambda(a)$ is a proper ideal of $A$, a contradiction.

Set $M_\lambda = M_\lambda(\alpha)M_{-\lambda}(-\alpha) + M_\lambda(\alpha) \oplus M_{-\lambda}(-\alpha)$ for some nonzero weight $\lambda$ of $A_o$ in $A_\alpha$. Then analogous to Proposition 3.13, $M_\lambda$ can be shown to be a nonzero ideal of $A$ and we have

**Proposition 4.10.** If $A = A_o \oplus A_\alpha \oplus A_{-\alpha}$ is a simple Malcev algebra as determined by Proposition 3.13, then there exists a nonzero weight $\lambda$ of $A_o$ in $A$ with weight space $M_\lambda(\alpha) = A_\alpha$ and such that $-\lambda$ is a weight of $A_o$ in $A$ with weight space $M_{-\lambda}(-\alpha) = A_{-\alpha}$.

We shall identify $\alpha$ with $\lambda$ as a weight, that is, use the notation $\alpha(x_o)$ for $\lambda(x_o)$ and also identify $M_\lambda(\alpha) = A_\alpha$, $M_{-\lambda}(-\alpha) = A_{-\alpha}$. Note that Proposition 4.10 implies there exists a basis for $A$ so that for every $x \in A_o$, $R_x$ has a matrix of the form

\[
\begin{bmatrix}
0 & 0 & 0 \\
[\alpha(x) & 0 & 0] \\
[\ast & 0 & 0] \\
0 & 0 & 0 \\
\end{bmatrix}.
\]

5. The trace form. Set $(x, y) = \text{trace } R_xR_y$, then it is shown [3] that this is actually an invariant form; that is $(x, y)$ is a bilinear form on $A$ such that for all $x, y, z \in A$, $(xy, z) = (x, yz)$. Also a bilinear form $(x, y)$ is nondegenerate on $A$ if $(x, y) = 0$ for all $y \in A$ implies $x = 0$.

**Theorem 5.1.** If $A = A_o \oplus A_\alpha \oplus A_{-\alpha}$ is a finite dimensional simple non-Lie Malcev algebra over an algebraically closed field of characteristic zero and if $A$ contains an element $u$ such that $R_u$ is not nilpotent, then $(x, y) = \text{trace } R_xR_y$ is a nondegenerate invariant form on $A$ and dimension $A_\alpha = \text{dimension } A_{-\alpha}$.

**Proof.** On $A = A_o \oplus A_\alpha \oplus A_{-\alpha}$ $R_u$ has the matrix
and since \( u \in A = J(A, A, A) \) (by 1.12) we have by [3; 2.12] that \( 0 = \text{trace } R_u = \alpha(n_\alpha - n_{-\alpha}) \) where \( n_\alpha = \text{dimension } A_\alpha, \alpha = \pm \alpha \).

Now to show \((x, y)\) is nondegenerate, let \( T = \{ x \in A : (x, A) = 0 \} \) where for subsets \( B, C \) of \( A \) we set \((B, C) = \{(b, c) : b \in B, c \in C\}\) and for \( x \in A, (x, C) = \{(x, c) : c \in C\}\). Since \((x, y)\) is an invariant form on \( A, T \) is an ideal of \( A \) and since \( A \) is simple, \( T = 0 \) or \( T = A \). If \( T = A \), then \((A, A) = 0\) and from the matrix of \( R_u \) we see that

\[
0 = (u, u) = \text{trace } R_u^2 = 2n\alpha^2
\]

where \( n = \text{dimension } A_\alpha \). Since \( F \) is of characteristic zero, \( \alpha = 0 \), a contradiction. Thus \( T = 0 \) which implies \((x, y)\) is nondegenerate on \( A \).

**Corollary 5.2.** If \( A = A_0 \oplus A_\alpha \oplus A_{-\alpha} \) is a simple Malcev algebra as above then

\[
(A_0, A_\alpha) = (A_0, A_{-\alpha}) = (A_\alpha, A_\alpha) = (A_{-\alpha}, A_{-\alpha}) = 0.
\]

**Proof.** Since \( R_u \) is nonsingular on \( A_\alpha, \alpha \neq 0, A_\alpha = A_\alpha R_u \). Therefore \((A_\alpha, A_\alpha) = (A_\alpha, A_\alpha R_u) = (A_\alpha R_u, A_\alpha) = 0\), the second equality uses \((x, y)\) is an invariant form and the third uses (2.11). Also \((A_\alpha, A_\alpha) = (u A_\alpha, A_\alpha) = (u, A_\alpha A_\alpha) \subset (u, A_{-\alpha}) = 0 \).

**Corollary 5.3.** If \( A_0^* \) is the dual space of \( A_0 \) consisting of linear functionals on \( A_0 \) and \( f \in A_0^* \), then \( f = c\alpha \) for some \( c \in F \).

**Proof.** First, \((x, y)\) is nondegenerate on \( A_0 \). For if \( x_0 \in A_0 \) is such that \((x_0, A_0) = 0\), then

\[
(x_0, A) = (x_0, A_0 \oplus A_\alpha \oplus A_{-\alpha}) \subset (x_0, A_0) + (x_0, A_\alpha) + (x_0, A_{-\alpha}) = 0
\]

by the preceding corollary and therefore \( x_0 = 0 \) by Theorem 5.1. Now if \( f \in A_0^* \), then there exists a unique element [2, page 141] \( a_r \in A_0 \)
such that for all \( x \in A_0, f(x) = (x, a_f) = \text{trace} R_x R_{a_f} = \)

\[
\begin{pmatrix}
0 & 0 & 0 \\
\alpha(x) & 0 & 0 \\
0 & \cdots & \alpha(x)
\end{pmatrix}
\]

\[
\begin{pmatrix}
\alpha(a_f) & 0 & 0 \\
0 & \cdots & \alpha(a_f)
\end{pmatrix}
\]

\[
= 2n\alpha(a_f)\alpha(x); \text{ using the remarks at the end of } \S 4 \text{ to obtain the form of the matrices of } R_x \text{ and } R_{a_f}. \text{ Thus } f = c\alpha \text{ where } c = 2n\alpha(a_f) \in F.
\]

**Corollary 5.4.** The dimension of \( A_0 \) is one.

**Proof.** 0 < dimension \( A_0 = \) dimension \( A_0^* = \) dimension \( uF = 1. \)

We shall frequently refer to a Malcev algebra \( A \) that satisfies Theorem 5.1 as a "usual simple non-Lie Malcev algebra" and for the remainder of this paper we shall assume the algebraically closed field \( F \) is of characteristic zero.

6. The diagonalization of \( R_u \). Using Proposition 4.10 and Corollary 5.4 we are able to decompose \( A \) relative to \( R(A_0) \) into the form

\[ A = A_0 \oplus A_a \oplus A_{-a} \]

where \( A_0 = uF \). From this the matrix of \( R_u \) on \( A_a, a = \pm \alpha \), has the form

\[
\begin{pmatrix}
a & \cdots & 0 \\
\cdots & \cdots & \alpha \end{pmatrix}
\]

We shall show in this section that \( R_u \) can be diagonalized. Put \( R_u \) into its Jordan canonical form on \( A_a \), that is, find \( R_u \)-invariant subspaces \( U_i(a) \) of \( A_a \) such that \( A_a = U_1(a) \oplus \cdots \oplus U_m(a) \) and each \( U_i(a) \) has a basis \( \{x_{i1}, \cdots, x_{im}\} \) so that the action of \( R_u \) is given by

\[
x_{i1}R_u = ax_{i1}
\]

\[
x_{ij}R_u = ax_{ij} + x_{i(j-1)}
\]

\( j = 2, \cdots, m_i \).

Thus on \( U_i(a) \), \( R_u \) has an \( m \times m \) matrix of the form

\[
\begin{pmatrix}
a & 0 \\
1 & a \\
\vdots & \ddots & \ddots \\
0 & \cdots & 1 & a
\end{pmatrix}
\]
where \( m = \text{dimension } U_i(a) \). We shall now investigate the multiplicative relations between the \( U \)'s and show that the dimension of all the \( U_i(a) \) is one and therefore \( R_u \) will have a diagonal matrix.

**Lemma 6.2.** \( U_i(a) U_j(a) = 0 \).

**Proof.** Let \( U_i(a) \) have basis \( \{x_1, \ldots, x_m\} \) as given by (6.1). If \( m = 1 \), we are finished. Suppose \( m > 1 \), then using (1.6)

\[
0 = -J(u, x_2, x_2) R_u = J(u, x_2, x_2 R_u) = aJ(u, x_2, x_2) + J(u, x_2, x_1) = J(u, x_2, x_1)
\]

\[
= x_2 x_1 \cdot u + x_1 u \cdot x_2 + u x_2 \cdot x_1 = x_2 x_1 (R_u - 2aI).
\]

But we know \( A_{2a} = 0 \), therefore \( x_1 x_2 = 0 \). Now using (1.6) we have, in general, for any \( i = 1, \ldots, m \),

\[
0 = J(u, x_i, x_i R_u) = J(u, x_i, x_i) + aJ(u, x_i, x_i) = J(u, x_i, x_i)
\]

and again using (1.6),

\[
0 = J(u, x_i, x_{i-1} R_u) = J(u, x_i, x_{i-1}) + aJ(u, x_i, x_{i-1}) = J(u, x_i, x_{i-1}).
\]

Continuing this process we have

\[
J(u, x_i, x_k) = 0
\]

for all \( k \leq i \). Now if \( i < k \), then by the preceding sentence

\[
0 = J(u, x_k, x_i) = J(u, x_i, x_k).
\]

Thus

\[
J(u, x_i, x_k) = 0 \quad \text{for all } i, k = 1, \ldots, m.
\]

By linearity this implies

\[
J(u, x, y) = 0 \quad \text{for all } x, y \in U_i(a).
\]

Thus

\[
xy R_u = x R_u \cdot y + y R_u
\]
and
\[ xy(R_u - 2aI) = x(R_u - aI) \cdot y + x \cdot y(R_u - aI) \]
As usual we can find an \( N \) large enough so that \( xy(R_u - 2aI)^N = 0 \).
But we know \( A_{2n} = 0 \), therefore \( xy = 0 \).

**Lemma 6.3.** Let \( x \in A_a \) be such that \( xR_u = ax \) and let \( U_i(-a) = \{y_1, \ldots, y_m\} \), then \( xy_i = 0 \) for \( i = 1, \ldots, m-1 \) and \( xy_m = \lambda u \) where \( \lambda = -(y_m, x)/2na \).

**Proof.** Using the invariant form \((x, y)\) we have \((y_m, u, u) = (y_m, xu) = a(y_m, x)\). Since \( xy_m \in A_o = uF \) we may write \( xy_m = \lambda u \), then \((y_m, x, u) = (-\lambda u, u) = -\lambda(u, u) = -\lambda 2na^2(a = \pm \alpha) \). Thus \( \lambda = -(y_m, x)/2na \).

Now since \( x \in A_a \) and \( U_i(-a) \subset A_{-a} \), we have by (2.4) and (2.11) that
\[ 0 = J(x, y, u) = xy_2 \cdot u + y_2 \cdot x + ux \cdot y_2 = (\lambda u x) - axy_2 = y_2 x. \]
Again \( 0 = J(x, y, u) = xy_2 \cdot u + y_2 \cdot x + ux \cdot y_2 = (\lambda u x) - axy_2 = y_2 x \).
Continuing this process we eventually obtain \( 0 = J(x, y, u) = xy_m \cdot u + y_m u \cdot x + ux \cdot y_m = y_m x. \)

**Theorem 6.4.** Let \( x \in A_a \) be such that \( xR_u = ax \) and let \( U_i(-a) \) be such that \( xU_i(-a) \neq 0 \), then dimension \( U_i(-a) = 1 \).

**Proof.** Let \( B = uF \oplus xF \oplus U_i(-a) \), then using the preceding lemmas and their notation we see that \( B \) is a subalgebra of \( A \) and \( xy_m = \lambda u \) where \( \lambda \neq 0 \). Now by (2.4) we have \( J(u, x, y_m) = 0 \), therefore by [3; Corollary 4.4] we see that \( u, x \) and \( y_m \) are contained in a Lie subalgebra, \( L \), of \( A \). However this implies \( y_m u = -ay_m + y_{m-1} \in L \) and therefore \( y_{m-1} \in L \); again \( y_{m-1} u = -ay_{m-1} + y_{m-2} \in L \) and therefore \( y_{m-2} \in L \). Continuing this process we obtain \( B \subset L \) and so \( B \) is a Lie subalgebra of \( A \). Thus for any \( z \in B \),
\[ 0 = J(z, x, y_m) \]
\[ = z(R_z R_{y_m} - R_{y_m} R_z - R_{xy_m}) \]
\[ = z([R_z, R_{y_m}] - \lambda R_u). \]
Thus on \( B \) we have \( \lambda R_u = [R_z, R_{y_m}] \) and therefore the trace of \( R_u \) on \( B \) is zero. But calculating the trace of \( R_u \) from its matrix on \( B \), we obtain that the trace is \( 0 + a - am \). Thus \( m = 1 \).

**Corollary 6.5.** The dimensional of all the \( U_i(-a), a = \pm \alpha, \) is one.

**Proof.** Suppose there exists \( U_j(-a) = \{y_1, \ldots, y_m\} \) of dimension \( m > 1 \). Then for every \( U_i(a), y_i U_i(a) = 0 \). For if there exists some
$U_i(a)$ such that $y_i U_i(a) \neq 0$, then by Theorem 6.4, dimension $U_i(a) = 1$. But this means there exists $x \in A_a$ such that $x R_u = ax$ and $0 \neq xy_i \in x U_i(-a)$; so again by Theorem 6.4, dimension $U_i(-a) = 1$, a contradiction. Thus $y_i U_i(a) = 0$ for all $i$ and this implies $y_i A_a = y_i (U_i(a) \oplus \cdots \oplus U_m(a)) = 0$. Now from Corollary 5.2 we have, since $y_i \in A_{-a}$, $(A_a, y_i) = (A_{-a}, y_i) = 0$ and using the preceding sentence

$$(A_a, y_i) = (A_a, y_i u) = (A_y y_i, u) = 0.$$ 

Thus $(A, y_i) = 0$ and since $(x, y)$ is nondegenerate on $A$, $y_1 = 0$, a contradiction.

7. Proof of the theorem. Let $A = A_0 \oplus A_a \oplus A_{-a}$ be the usual simple non-Lie Malcev algebra, then we have just seen that $A_a$ is the null space of $R_u - a I$, $a = 0, \pm \alpha$. The choice of $\alpha \neq 0$ is fixed but arbitrary. In particular we want to consider the case $\alpha = -2$, then all we must do is consider $u' = (-2/\alpha)u$ and decompose $A$ relative to $R_u'$ (which is also not nilpotent) to obtain $A = A_0 \oplus A_{-2} \oplus A_2$. However we shall work with a fixed $\alpha$ and normalize when necessary.

Let $a, b \in F$ be any characteristic roots (weights) of $R_u$, that is, $a, b = 0, \pm \alpha$ with characteristic vectors $x, y \in A$; that is, $a x = x R_u$, by $y \in A_a$, $y \in A_b$, then we have

$$(7.1) \quad J(x, y, u) = xy \cdot u - (a + b)xy \quad \text{where } x \in A_a, y \in A_b.$$

Using (2.4) and (7.1) we also have

$$(7.2) \quad xy \cdot u = (a + b)xy \quad \text{where } y \in A_a, y \in A_b \text{ and } a \neq b.$$

Since $xy \in A_{-a}$ if $x, y \in A_a$, we have

$$(7.3) \quad xy \cdot u = -axy \quad \text{where } x, y \in A_a.$$

Combining (7.3) and (7.1) yields

$$(7.4) \quad J(x, y, u) = -3axy \quad \text{where } x, y \in A_a.$$

Let $x, y, z \in A_a$, then using (2.14), (2.4), (1.9) and (7.4) we have

$$0 = J(xy, z, u)$$

$$= x J(y, z, u) + J(x, z, u)y - 2 J(z u, x, y)$$

$$= x (-3ayz) + (-3axz)y - 2a J(z, x, y).$$

Therefore

$$2J(x, y, z) = -3(x \cdot yz + xz \cdot y)$$

$$= 3(xy \cdot z + yz \cdot x + xz \cdot y) - 3xy \cdot z$$

and thus
Now \( J(x, z, y) = -x \cdot yz \) and adding this to (7.5) yields \( 0 = xy \cdot z + xz \cdot y \) and with a slight change of notation we have

\[
xy \cdot z = -x \cdot yz \quad \text{where } x, y, z \in A_a.
\]

From (7.6) with \( z = x \) we obtain

\[
xy \cdot x = 0 \quad \text{where } x, y \in A_a.
\]

Now let \( x, y \in A_a, z \in A_{-a} \), then

\[
-xaJ(x, y, z) = J(x, y, zu) - aJ(z, y, x) = -aJ(x, y, z).
\]

So

\[
-2aJ(x, y, z) = J(z, y, xu) + J(x, y, zu) = J(z, y, u)x + J(x, y, u)z = J(x, y, u)z,
\]

using (1.7) for the second equality, (2.4) for the third. Thus we have

\[
-2aJ(x, y, z) = J(x, y, u)z = (-3axy)z \quad \text{using (7.4)}
\]

and hence

\[
2J(x, y, z) = 3xy \cdot z \quad \text{where } x, y \in A_a, z \in A_{-a}.
\]

This yields \( 3xy \cdot z = 2(xy \cdot z + yz \cdot x + zx \cdot y) \) or

\[
xy \cdot z = -2(xz \cdot y + x \cdot yz) \quad \text{where } x, y \in A_a, z \in A_{-a}.
\]

We now use (7.9) to prove the important identity (7.10). Thus let \( w, x, y, z \) be elements of \( A_a \) and set \( v = J(x, y, z), 2x' = yz, -2y' = xz \) and \( 2z' = xy \). Then

\[
vw = 6(x'w \cdot x + y'w \cdot y + z'w \cdot z).
\]

To prove this note that \( x', y', z' \in A_{-a} \) and using (7.9) we have

\[
2x' \cdot w = xw \cdot x' - 2wx' \cdot x, 2y' \cdot w = yw \cdot y' - 2wy' \cdot y, 2z' \cdot w = zw \cdot z' - 2wz' \cdot z.
\]

Adding these equations and multiplying by 2 yield

\[
2vw = 2(xw \cdot x' + yw \cdot y' + zw \cdot z') + 4(x'w \cdot x + y'w \cdot y + z'w \cdot z).
\]

Now using (1.10),

\[
2(xw \cdot x' + yw \cdot y' + zw \cdot z') = xw \cdot yz + yw \cdot zx + zw \cdot xy
\]

noting some cancellation to obtain the third equality. Thus \( 2vw = vw + 2(x'w \cdot x + y'w \cdot y + z'w \cdot z) + 4(x'w \cdot x + y'w \cdot y + z'w \cdot z) \) and this proves (7.10).
Since A is simple non-Lie Malcev algebra, we shall use the facts $A^2 = A$ and $A = J(A, A, A)$ to obtain more identities for A. First we have

$$A_0 \oplus A_\alpha \oplus A_{-\alpha} = A = J(A, A, A)$$
$$\subset J(A_0, A, A) + J(A_\alpha, A, A) + J(A_{-\alpha}, A, A)$$
$$\subset J(A_0, A_\alpha, A_\alpha) + J(A_\alpha, A_{-\alpha}, A_{-\alpha}) + J(A_\alpha, A_\alpha, A_\alpha)$$
$$+ J(A_{-\alpha}, A_{-\alpha}, A_{-\alpha}) + J(A_\alpha, A_\alpha, A_{-\alpha}) + J(A_\alpha, A_{-\alpha}, A_{-\alpha})$$
$$\subset A_0 \oplus A_\alpha \oplus A_{-\alpha}$$

and therefore

$$A_0 = J(A_\alpha, A_\alpha, A_\alpha) + J(A_{-\alpha}, A_{-\alpha}, A_{-\alpha}) ,$$
$$A_\alpha = J(A_0, A_{-\alpha}, A_{-\alpha}) + J(A_\alpha, A_\alpha, A_\alpha) ,$$
$$A_{-\alpha} = J(A_0, A_\alpha, A_\alpha) + J(A_\alpha, A_\alpha, A_{-\alpha}) .$$

We now use $A = A^2$ to obtain

$$A_0 \oplus A_\alpha \oplus A_{-\alpha} = A = A^2$$
$$= A_0 A_\alpha + A_\alpha A_{-\alpha} + A_{-\alpha} A_\alpha + A_\alpha^2$$

and therefore

$$A_0 = A_\alpha A_{-\alpha} ,$$
$$A_\alpha = A_\alpha A_\alpha + A_{-\alpha}^2 ,$$
$$A_{-\alpha} = A_\alpha A_\alpha + A_\alpha^2 .$$

Since $A_0 = uF$ we have $A_0 A_\alpha = A_\alpha (\alpha = \pm \alpha)$. Also

$$J(A_0, A_{-\alpha}, A_{-\alpha}) \subset A_\alpha = A_0 A_{-\alpha}$$
$$\subset A_0 J(A_\alpha, A_{-\alpha}, A_{-\alpha}) + A_\alpha J(A_\alpha, A_\alpha, A_{-\alpha})$$
$$\subset J(A_0, A_\alpha, A_{-\alpha}^2) + J(A_\alpha, A_{-\alpha}, A_{-\alpha} A_{-\alpha}) + J(A_\alpha, A_{-\alpha}, A_{-\alpha} A_{-\alpha})$$
$$+ J(A_\alpha, A_\alpha, A_{-\alpha} A_{-\alpha}) + J(A_\alpha, A_{-\alpha}, A_{-\alpha} A_{-\alpha}) + J(A_\alpha, A_{-\alpha}, A_\alpha^2)$$
$$\subset J(A_0, A_{-\alpha}, A_{-\alpha}) ,$$

obtaining the second inclusion from $A_\alpha = J(A_0, A_{-\alpha}, A_{-\alpha}) + J(A_\alpha, A_{-\alpha}, A_{-\alpha})$ and the third inclusion from (1.8). Thus we have

$$A_\alpha = J(A_0, A_{-\alpha}, A_{-\alpha}) , \quad \alpha \neq 0 .$$

From this and remembering $A_0 = uF$ we obtain

$$A_\alpha = A_{-\alpha} A_{-\alpha} , \quad \alpha \neq 0 .$$

For $A_{-\alpha} A_{-\alpha} \subset A_\alpha = J(A_0, A_{-\alpha}, A_{-\alpha}) \subset A_{-\alpha} A_{-\alpha}$. Also

$$A_0 = J(A_\alpha, A_\alpha, A_\alpha) , \quad \alpha = \pm \alpha .$$

For
\[ J(A_0, A_0, A_0) \subset A_0 = A_{a}A_{-a} \]
\[ = A_{a}J(A_0, A_0, A_0) \]
\[ \subset J(A_0, A_0, A_0^2) + J(A_0, A_0, A_0A_0) + J(A_0, A_0, A_0A_0) \]
\[ \subset J(A_0, A_0, A_0) . \]

We summarize these identities in

**Proposition 7.11.** Let \( A = A_0 \oplus A_0 \oplus A_{-a} \) be the usual simple non-Lie Malcev algebra, then we have for \( a = \pm \alpha \),

\[ A_a = A_aA_a = A_{-a}A_{-a} \]

and

\[ A_0 = A_aA_{-a} = J(A_0, A_0, A_0) . \]

**Theorem 7.12.** Let \( A = A_0 \oplus A_0 \oplus A_{-a} \) be the usual simple non-Lie Malcev algebra, then \( A \) is isomorphic to the simple seven dimensional Malcev algebra \( A* \) discussed in the introduction.

**Proof.** Since \( uF = A_0 = A_{a}A_{-a} = A_a \cdot A_aA_a \), there exists \( x, y, z \in A_a \) such that \( x \cdot yz = 2u \). Define \( 2x' = yz, -2y' = xz \) and \( 2z' = xy \) and form the subspace \( B \) generated by \( \{u, x, y, z, x', y', z'\} \). First the \( x, y \) and \( z \) are linearly independent over \( F \). For if \( ax + by + cz = 0 \) with \( a, b, c \in F \) and, for example, \( a \neq 0 \), then write \( x = b'y + c'z \) and therefore using (7.7) \( 2u = x \cdot yz = b'y \cdot yz + c'z \cdot yz = 0 \), a contradiction. Similarly noting \( u = xx' \) and assuming a relation of the type \( x' = b'y' + c'z' \) and using the definitions of \( x', y' \) and \( z' \) we see that the \( x', y' \) and \( z' \) are also linearly independent. Since \( A = A_0 \oplus A_0 \oplus A_{-a} \), \( \{u, x, y, z, x', y', z'\} \) is a linearly independent set of vectors over \( F \). Using identities (1.2), (7.6) and (7.7) we obtain the following multiplication table for \( B \).

<table>
<thead>
<tr>
<th></th>
<th>( u )</th>
<th>( x )</th>
<th>( y )</th>
<th>( z )</th>
<th>( x' )</th>
<th>( y' )</th>
<th>( z' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u )</td>
<td>0</td>
<td>(-\alpha x)</td>
<td>(-\alpha y)</td>
<td>(-\alpha z)</td>
<td>(\alpha x')</td>
<td>(\alpha y')</td>
<td>(\alpha z')</td>
</tr>
<tr>
<td>( x )</td>
<td>(\alpha x)</td>
<td>0</td>
<td>(2z')</td>
<td>(-2y')</td>
<td>(u)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( y )</td>
<td>(\alpha y)</td>
<td>(-2z')</td>
<td>0</td>
<td>(2x')</td>
<td>0</td>
<td>(u)</td>
<td>0</td>
</tr>
<tr>
<td>( z )</td>
<td>(\alpha z)</td>
<td>(2y')</td>
<td>(-2x')</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(u)</td>
</tr>
<tr>
<td>( x' )</td>
<td>(-\alpha x')</td>
<td>(-u)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(\alpha z)</td>
<td>(-\alpha y)</td>
</tr>
<tr>
<td>( y' )</td>
<td>(-\alpha y')</td>
<td>0</td>
<td>(-u)</td>
<td>0</td>
<td>(-\alpha z)</td>
<td>0</td>
<td>(\alpha x)</td>
</tr>
<tr>
<td>( z' )</td>
<td>(-\alpha z')</td>
<td>0</td>
<td>0</td>
<td>(-u)</td>
<td>(\alpha y)</td>
<td>(-\alpha x)</td>
<td>0</td>
</tr>
</tbody>
</table>

By the remarks at the beginning of this section we can choose \( \alpha = -2 \)
and consequently obtain that $B$ is isomorphic to $A^*$. It remains to show the dimension of $A$ over $F$ is seven. For this it suffices to show dimension $A_\alpha = 3$, since dimension $A_\alpha = \dim A_{-\alpha}$. Let $0 \neq w \in A_\alpha$, then by (7.5)

$$6u = 3x \cdot yz = -J(x, y, z)$$

and therefore by (7.10),

$$6\alpha w = 6wu = x_0 x + y_0 y + z_0 z$$

where $x_0, y_0, z_0 \in A_0 = uF$. But by the action of $u$ on $x, y$ and $z$ we have $6\alpha w = a_0 x + b_0 y + c_0 z$ where $a_0, b_0, c_0 \in F$. Thus the dimension of $A_\alpha$ is three.

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