NORMAL MATRICES AND THE NORMAL BASIS IN ABELIAN NUMBER FIELDS

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1. Introduction. Throughout this note $F$ denotes a normal field of algebraic numbers of finite degree $n$ over the rational number field. Let $G_1, G_2, \ldots, G_n$ denote the elements of the Galois group $G$ of $F$. It is known [2] that $F$ may possess a "normal" basis for the integers consisting of the conjugates $\alpha^{g_1}, \alpha^{g_2}, \ldots, \alpha^{g_n}$ of an integer $\alpha$. In [4] the question of the uniqueness of the normal basis was answered when $G$ is cyclic. (See also [1, 6].) If $\beta_1, \beta_2, \ldots, \beta_n$ is any integral basis of $F$ then the matrix $(\beta_i^j), 1 \leq i, j \leq n$, is called a discriminant matrix. It was shown in [4] that if $G$ is abelian then the discriminant matrix of the normal basis $\beta_i = \alpha^{g_1}, \ldots, \beta_n = \alpha^{g_n}$ is a normal matrix and, if $G$ is cyclic and $F$ has a normal basis, then any integral basis $\beta_1, \ldots, \beta_n$ for which the discriminant matrix is normal is of the form $\beta_{\sigma(i)} = \pm \alpha^{g_1}, \ldots, \beta_{\sigma(n)} = \pm \alpha^{g_n}$ for a suitable choice of the $\pm$ signs, where $\sigma$ is a permutation of $1, 2, \ldots, n$.

It is the purpose of this note to use the methods of [4] to extend these results for cyclic fields to abelian fields. In particular, in Theorem 1, we shall give a new proof of a result obtained by G. Higman in [1]. The author wishes to thank Dr. O. Taussky-Todd for drawing the problems considered here to his attention.

2. Preliminary material. We suppose throughout that

\[ G = (S_1) \times (S_2) \times \cdots \times (S_k) \]

is the direct product of $k$ cyclic groups $(S_i)$ of order $n_i$. Of course, each $n_i > 1$ and $n = n_1 n_2 \cdots n_k$. If $X$ and $Y = (y_{i,j})$ are two matrices with elements in a group or a ring then we define $X \times Y = (X y_{i,j})$. $X \times Y$ is the Kronecker product [3] of $X$ and $Y$. Henceforth, in this paper, the symbol $\times$ will always be used to denote the Kronecker product of vectors or matrices. A matrix $A$ is said to be a circulant of type $(n_i)$ if

\[ A = [a_1, a_2, \ldots, a_{n_i}]_{n_i} = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{n_i} \\ a_{n_i} & a_1 & a_2 & \cdots & a_{n_i-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{bmatrix}. \]

Here $a_1, a_2, \ldots, a_{n_i}$ may lie in a group or a ring. For $i > 1$ we define

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by induction \([A_1, A_2, \cdots, A_n]_{k}\) to be a circulant of type \((n_1, n_2, \cdots, n_k)\) if each of \(A_1, A_2, \cdots, A_n\) is a circulant of type \((n_1, n_2, \cdots, n_{i-1})\). For \(1 \leq i \leq k\) let \(H_i = (1, S_i, S_i^2, \cdots, S_i^{n_i-1})\) and \(D_i = [1, S_i^{-1}, S_i^{n_i-2}, \cdots, S_i]\). Henceforth we shall always let \(G_1, G_2, \cdots, G_n\) denote the elements of \(G\) in the order implied by the vector equality

\[
(G_1, G_2, \cdots, G_n) = H_1 \times H_2 \times \cdots \times H_k.
\]

Let \(y(G_1), y(G_2), \cdots, y(G_n)\) be commuting indeterminants and define the matrix \(Y = (y(G_i G_j^{-1})), 1 \leq i, j \leq n\). Then it can be proved by induction on \(k\) that \(D_1 \times D_2 \times \cdots \times D_k = (G_i G_j^{-1}), 1 \leq i, j \leq n\), and hence that \(Y\) is a circulant of type \((n_1, n_2, \cdots, n_k)\). Since any circulant of type \((n_1, n_2, \cdots, n_k)\) is determined by its first row, it follows that any circulant of type \((n_1, n_2, \cdots, n_k)\) may be obtained by assigning particular values to the indeterminants \(y(G_1), \cdots, y(G_n)\) in \(Y\).

**Lemma 1.** Circulants of type \((n_1, n_2, \cdots, n_k)\) with coefficients in a field \(K\) form a commutative matrix algebra containing the inverse of each of its invertible elements. For fixed \(m\), all matrices \(X = (X_{i,j}), 1 \leq i, j \leq m\), in which each \(X_{i,j}\) is a circulant of type \((n_1, n_2, \cdots, n_k)\) with coefficients in \(K\), form a matrix algebra containing the inverse of each of its invertible elements.

**Proof.** Let \(W = (W_{i,j} G_i G_j^{-1}), 1 \leq i, j \leq m\). Then \(W + Y\) and \(aW\) for \(a \in K\) are clearly circulants of type \((n_1, n_2, \cdots, n_k)\). The \((i, j)\) element of \(WY\) is

\[
\sum_{i=1}^{m} W_{i,j} (G_i G_i^{-1}) y(G_i G_j^{-1}) = \sum_{i=1}^{m} W_{i,j} (G_i G_i^{-1} G_i G_j^{-1}) y((G_i^{-1} G_j^{-1}) G_j^{-1})
\]

\[
= \sum_{i=1}^{m} W_{i,j} y(G_i^{-1}) w(G_j^{-1}).
\]

But this is the \((i, j)\) element of \(YW\). Hence \(WY = YW\). Define

\[
z(G_i G_j^{-1}) = \sum_{i=1}^{m} W_{i,j} y(G_i G_j^{-1}).
\]

Then a straightforward calculation shows that \(z(G_i G_j^{-1}) = z(G_j G_i^{-1})\) if \(G_i G_i^{-1} = G_j G_j^{-1}\). Hence the variables \(z(G_i G_j^{-1}), 1 \leq i, j \leq n\), are unambiguously defined, so that \(WY\) is a circulant of type \((n_1, n_2, \cdots, n_k)\). This proves the first half of the first assertion of the lemma. The rest of the first assertion follows from the fact that the inverse of a matrix is a polynomial in the matrix. The other assertion of the lemma is now clear.

We let \(B'\) and \(B^*\) denote, respectively, the transpose and the complex conjugate transpose of the matrix \(B\). The diagonal matrix
whose diagonal entries are \( \lambda_1, \lambda_2, \cdots, \lambda_n \) is denoted by \( \text{diag} (\lambda_1, \lambda_2, \cdots, \lambda_n) \). The zero and identity matrices with \( s \) rows and columns are denoted by \( 0_s \) and \( I_s \), respectively, and for \( i = 1, 2, \cdots, k \), the companion matrix of the polynomial \( x^n - 1 \) is denoted by \( F_i = [0, 1, 0, \cdots, 0]_{s_k} \).

Let \( \zeta_{nu} \) be a primitive root of unity of order \( nu \) for \( 1 \leq i \leq k \). Set \( \Omega_u = (\zeta_{u(i-1)(i-j)}^{-1}) \), \( 1 \leq i, j \leq nu \), and set \( \Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_k \). Define \( T_u = n_u^{-1/2} \Omega_u \) and \( T = n^{-1/2} \Omega \). It can be shown by direct computation that \( T_u \) is a unitary matrix. Hence, using the basic properties \( (Z \times Y) = ZXYW \) and \( (X \times Y)^* = X^* \times Y^* \) of the Kronecker product, it follows immediately that \( T \) is a unitary matrix.

**Lemma 2.** If \( A \) is a circulant of type \( (n_1, n_2, \cdots, n_k) \) with first row \( a = (a_1, a_2, \cdots, a_n) \) and complex coefficients, then \( T^*AT = \text{diag} (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) \) where the vector \( \varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) \) is linked to the vector \( a \) by \( \varepsilon' = \Omega a' \).

**Proof.** The proof is by induction on \( k \). For \( k = 1 \) it is well known (and straightforward to check) that \( AT_1 = T_1 \text{diag} (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) \). Suppose the result known for \( k - 1 \). If

\[
A = [A_1, A_2, \cdots, A_{n_k}]_{s_k} = \sum_{i=1}^{n_k} A_i \times F_i^{i-1}
\]

and if we set \( d = n_1n_2\cdots n_{k-1} \) and define \( (\gamma_{(i-1)d+1}, \gamma_{(i-1)d+2}, \cdots, \gamma_{id}) \) by

\[
\gamma_i = (\gamma_{(i-1)d+1}, \gamma_{(i-1)d+2}, \cdots, \gamma_{id})',
\]

then, by the induction assumption,

\[
(T_1 \times \cdots \times T_{k-1})^*A_i(T_1 \times \cdots \times T_{k-1}) = \text{diag} (\gamma_{(i-1)d+1}, \gamma_{(i-1)d+2}, \cdots, \gamma_{id}),
\]

Then

\[
T^*AT = \sum_{i=1}^{n_k} (T_1 \times \cdots \times T_{k-1} \times T_k)^*(A_i \times F_k^{i-1})(T_1 \times \cdots \times T_{k-1} \times T_k)
\]

\[
= \sum_{i=1}^{n_k} \{(T_1 \times \cdots \times T_{k-1})^*A_i(T_1 \times \cdots \times T_{k-1})\} \times \{T_k^*F_kT_k\}^{i-1}
\]

\[
= \sum_{i=1}^{n_k} \{(\text{diag} (\gamma_{(i-1)d+1}, \gamma_{(i-1)d+2}, \cdots, \gamma_{id})\}
\]

\[
\times \{\text{diag} (1, \zeta_k^{i-1}, \zeta_k^{i+1-1}, \cdots, \zeta_k^{(n_i-1)(i-1)})\}.
\]

Thus \( T^*AT \) is diagonal. If \( r = (b - 1)d + c \) where \( 1 \leq c \leq d \) and \( 1 \leq b \leq n_k \), then the \((r, r)\) diagonal element of \( T^*AT \) is

\[
\varepsilon_r = \sum_{i=1}^{n_k} \gamma_{(i-1)d+c}(x_k^{b-1}(i-1)), \quad 1 \leq r \leq n.
\]
Setting \( \varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \) and \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \), equations (3) are the same as the matrix equation \( \varepsilon' = (I \times \Omega_n) \gamma' \) and equations (2) are the same as \( ((\Omega_1 \times \cdots \times \Omega_{k-1}) \times I_n) a' = \gamma' \). Combining these two facts, we obtain \( \varepsilon' = \Omega a' \), as required.

3. The uniqueness of the normal basis. If \( \beta_1, \ldots, \beta_n \) is another normal basis of \( F \) then \( (\beta_1, \ldots, \beta_n)' = (a_{i,j})(\alpha^{\alpha_i \alpha_j^{-1}}) \), where \( (\alpha^{\alpha_i \alpha_j^{-1}}) \) and \( (\alpha^{\alpha_i \alpha_j^{-1}}) \) are both circulants of type \( (n_1, n_2, \ldots, n_k) \) and \( (a_{i,j}) \) is a unimodular matrix of rational integers. By Lemma 1, \( (a_{i,j}) = (\beta_i \beta_j^{-1})^{-1} \) is also a circulant of type \( (n_1, n_2, \ldots, n_k) \). Conversely, if \( \beta_1, \ldots, \beta_n \) is an integral basis such that \( (\beta_1, \ldots, \beta_n)' = (a_{i,j})(\alpha^{\alpha_i \alpha_j^{-1}}) \), where \( (a_{i,j}) \) is a unimodular circulant of rational integers of type \( (n_1, n_2, \ldots, n_k) \), then \( (\alpha^{\alpha_i \alpha_j^{-1}}) \) so that, by Lemma 1, \( (\beta_i \beta_j^{-1})^{-1} \) is also a circulant. Then, in \( (\beta_i \beta_j^{-1})^{-1} \), the elements in the first column are a permutation on those in the first row. Hence \( \beta_1, \ldots, \beta_n \) is a permutation of a normal basis. Following [4], we call a circulant trivial if it has just a single nonzero entry in each row. Thus \( \beta_1, \ldots, \beta_n \) is necessarily a permutation of \( \alpha^{\alpha_1}, \ldots, \alpha^{\alpha_n} \) or of \( -\alpha^{\alpha_1}, \ldots, -\alpha^{\alpha_n} \). Precisely when all unimodular circulants of rational integers of type \( (n_1, n_2, \ldots, n_k) \) are trivial.

If \( G \) has a cyclic direct factor of order other than 2, 3, 4, or 6, we may choose the notation so that \( (S_i) \) is this cyclic direct factor. By [4] there exists a nontrivial unimodular circulant \( B \) of rational integers of type \( (n_i) \). Then \( B \times I_{n_1 \times \cdots \times n_k} \) is a nontrivial unimodular integral circulant of type \( (n_1, n_2, \ldots, n_k) \) and so the normal basis is not unique. Hence only the following two cases remain to be considered:

(i) each \( n_i \) = 4 or 2;
(ii) each \( n_i \) = 3 or 2; \( 1 \leq i \leq k \).

In either case (i) or case (ii) let \( A \) be a unimodular circulant of rational integers of type \( (n_1, n_2, \ldots, n_k) \). Then, by Lemma 2, the determinant of \( A \) is \( \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n \) where each \( \varepsilon_i \) is an integer and hence a unit in the field \( K \) generated by \( \xi_1, \ldots, \xi_k \). \( K \) is generated by the root of unity whose order is the least common multiple of \( n_1, n_2, \ldots, n_k \). Since this least common multiple is 2, 3, 4, or 6, by the fundamental theorem on units \( K \) contains no units of infinite order and hence each \( \varepsilon_i \) is a root of unity. By Lemma 2,

\[
T a' = n^{-1/2} \varepsilon' .
\]

Since the first row \( T \) consists of ones only, \( \varepsilon_1 \) is rational. In (4) we replace, if necessary, each \( a_i \) with \( -a_i \) and each \( \varepsilon_i \) with \( -\varepsilon_i \) to ensure that \( \varepsilon_1 = 1 \). Since \( T \) is unitary,

\[
a' = n^{-1/2} T^* \varepsilon' = n^{-1} \Omega^* \varepsilon' .
\]
Let \( \Omega = (\tau_{ij}), 1 \leq i, j \leq n \). Then, using (5), the triangle inequality, and the fact that each \( |\tau_{j,i}| \) and each \( |\varepsilon_j| \) is one, we find that

\[(6) \quad |a_i| \leq n^{-1} \sum_{j=1}^{n} |\tau_{j,i}\varepsilon_j| = 1, \quad 1 \leq i \leq n. \]

If we have \( a_q \neq 0 \) for some \( q \), then \( |a_q| \geq 1 \), so that in (6) for \( i = q \) we have equality. Since \( \tau_{1,q} = \varepsilon_1 = 1 \), the condition for equality in the triangle inequality forces \( \tau_{j,q}\varepsilon_j = 1 \) for each \( j \) so that \( \varepsilon_j = \tau_{j,q} \) for \( j = 1, 2, \ldots, n \). Then, for \( i \neq q \),

\[na_i = \sum_{j=1}^{n} \tau_{j,i}\tau_{j,q} = 0\]

since the columns of \( \Omega \) are pairwise orthogonal. Thus, in \( A \), there is but a single nonzero entry in each row.

**Theorem 1.** The normal basis for the integers of \( F \) is unique (up to permutation and change of sign) precisely when either (i) or (ii) below is satisfied:

(i) \( G \) is the direct product of cyclic groups of order 4 and/or order 2;

(ii) \( G \) is the direct product of cyclic groups of order 3 and/or order 2.

Another form of this theorem is given in [1, Theorem 6].

4. Normal discriminant matrices. Let \( \alpha^{a_1}, \ldots, \alpha^{a_n} \) be a normal integral basis of \( F \) and let \( A \) be any normal discriminant matrix. Permuting the row and columns of \( A \) in the same way (this preserves normality) we may assume \( A = (\beta_{ij}^{a_{ij}})1 \leq i, j \leq n \), where \( G_1, \ldots, G_n \) are given by (1). Now \( A = (a_{ij})D \) where \( D = (\alpha^{q(a_{ij})}1 \leq i, j \leq n \), and where \( (a_{ij}) \) is a unimodular matrix of rational integers. From \( AD^* = D^*A \) we get \( (a_{ij})DD^*(a_{ij})' = D^*(a_{ij})'(a_{ij})D \). As in [4], \( DD^* \) is rational so that \( D^*(a_{ij})'(a_{ij})D \) is left fixed by every element of \( G \). Let

\[P_s = I_{n_0n_1\cdots n_{k-1}} \times F_s \times I_{n_{s+1}n_{s+2}\cdots n_{k+1}}, \quad 1 \leq \varepsilon \leq k,\]

where, here and henceforth, \( n_0 = n_{k+1} = 1 \). The effect of replacing \( \alpha \) with \( \alpha^s \), in \( D \) may be determined by noting that

\[S_s(D_1 \times \cdots \times D_k) = D_1 \times \cdots \times (S_sD_s) \times \cdots \times D_k = D_1 \times \cdots \times (F_sD_s) \times \cdots \times D_k = I_{n_1} \times \cdots \times I_{n_{s-1}} \times F_s \times I_{n_{s+1}} \times \cdots \times I_{n_k}D_1 \times \cdots \times D_k = P_s(D_1 \times \cdots \times D_k).\]

Hence, replacing \( \alpha \) with \( \alpha^s \) in \( D \) changes \( D \) into \( P_sD \). Therefore \( D^*(a_{ij})'(a_{ij})D = (P_sD)^*(a_{ij})'(a_{ij})(P_sD) \) so that \( P_s(a_{ij})'(a_{ij})P_s = (a_{ij})'(a_{ij}), \)
for \( s = 1, 2, \ldots, k \). Following [4] we define a generalized permutation matrix to be a permutation matrix in which the nonzero entries are permitted to be \( \pm 1 \). Then Lemma 3 below shows that 
\[
(a_{ij}) = QC
\]
where \( Q \) is a generalized permutation matrix and \( C \) is a circulant of type \( (n_1, n_2, \ldots, n_k) \). Since \( (\beta_1, \ldots, \beta_n)' = (a_{i,j})(\alpha^{n_1}, \ldots, \alpha^{n_k})' \), this implies (by remarks made in § 2) that \( \beta_1, \ldots, \beta_n \) is a generalized permutation of a normal basis.

**THEOREM 2.** Let \( F \) be a field with a normal integral basis. Then only generalized permutations of a normal basis can give rise to normal discriminant matrices.

**THEOREM 3.** If \( A \) is a unimodular matrix of rational integers such that \( AA' \) is a circulant of type \( (n_1, n_2, \ldots, n_k) \), then \( A = CQ \) where \( C \) is a unimodular circulant of type \( (n_1, n_2, \ldots, n_k) \) and \( Q \) is a generalized permutation matrix.

**Proof.** Since each \( P_i \) is a circulant of type \( (n_1, n_2, \ldots, n_k) \), it follows from Lemma 1 that 
\[
P_i AA' P_i' = AA'
\]
for \( i = 1, 2, \ldots, k \), so that Theorem 3 follows from Lemma 3.

**LEMMA 3.** If \( A \) is a unimodular matrix of rational integers such that 
\[
P_i AA' P_i' = AA'
\]
for \( i = 1, 2, \ldots, k \), then \( A = CQ \) where \( C \) and \( Q \) are as in Theorem 3.

**Proof.** Let \( A_0 = A \) and \( Q_0 = I_n \). We shall prove by induction on \( i \) that, for \( 1 \leq i \leq k \), \( A = A_i Q_i \) where \( Q_i \) is a generalized permutation matrix and \( A_i \) may be so partitioned that \( A_i = (X_{s,t}) \), \( 1 \leq s, t \leq n_i \), \( n_i+1 \cdot n_{i+2} \cdot \ldots \cdot n_{k+1} \), where each \( X_{s,t} \) is a circulant of type \( (n_1, n_2, \ldots, n_i) \). The case \( i = k \) is the statement of the lemma. To avoid having to give a special discussion of the case \( i = 1 \) we make the following definitions and changes in notation. Recall that \( n_0 = n_{k+1} = 1 \).

A one row, one column matrix is said to be a circulant of type \( (n_0) \). A circulant of type \( (n_1, \ldots, n_i) \) will now be called a circulant of type \( (n_0, n_1, \ldots, n_i) \). We then know that \( A = A_0 Q_0 \) where \( A_0 \) is composed of one row, one column blocks which are circulants of type \( (n_0) \) and where \( Q_0 \) is a generalized permutation matrix. Our induction assumption is that for a fixed value of \( i \) with \( 1 \leq i \leq k \) we have \( A = A_i-1 Q_{i-1} \) where we may partition \( A_{i-1} = (A_{s,t}) \), \( 1 \leq s, t \leq n_i \), \( n_{i+1} \cdot \ldots \cdot n_{k+1} \), so that each \( A_{s,t} \) is a circulant of type \( (n_1, n_2, \ldots, n_{i-1}) \), and where \( Q_{i-1} \) is a generalized permutation matrix. We shall then deduce that \( A = A_i Q_i \). For notational simplicity we set \( f = n_0 n_1 \cdot \ldots n_{i-1}, g = n_i n_{i+1} \cdot \ldots n_k, h = n_{i+1} n_{i+2} \cdot \ldots n_{k+1}, m = n_1 n_2 \cdot \ldots n_i. \)
Now \( AA' = A_{i-1}A'_{i-1} \) so that from \( P_iAA'P_i' = AA' \) we deduce that \( M_iM_i' = I_n \), where \( M_i = A_{i-1}P_iA_{i-1} \). Since \( M_i \) is a matrix of rational integers it follows that \( M_i \) is a generalized permutation matrix. Since \( P_i \) and \( A_{i-1} \) may, after partitioning, be viewed as matrices with \( g \) rows and columns in elements which are circulants of type \( (n_0, n_1, \ldots, n_{i-1}) \), it follows from Lemma 1 that \( M_i \) is also a matrix with \( g \) rows and columns in elements which are circulants of type \( (n_0, n_i, \ldots, n_{i-1}) \). From this point of view \( M_i \) must be a “generalized permutation matrix” in that it has but a single nonzero entry in each of its \( g \) rows and columns. Each of these nonzero entries is of course both a circulant of type \( (n_0, n_1, \ldots, n_{i-1}) \) and a generalized permutation matrix.

We now digress for a moment to note that if \( M \) is a permutation matrix whose coefficients lie in a ring with identity then a permutation matrix \( R \) exists with coefficients in the same ring such that \( R'MR \) is a direct sum of one row identity matrices and/or matrices like \([0, 1, 0, \ldots, 0]\) for \( t > 1 \). This assertion is a consequence of the fact that a permutation may be decomposed into disjoint cycles.

Applying this fact to the “generalized permutation matrix” \( M_i \), we find that a permutation matrix \( R_i \) exists with \( g \) rows and columns in elements which are either 0 or 1 such that \( R_i^N_i = N_i \) is a direct sum of \( r \) matrices of the following type:

\[
E_j = \begin{bmatrix}
0 & E_{j,1} & 0 & 0 & \cdots & 0 \\
0 & 0 & E_{j,2} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & E_{j,e_j-1} \\
E_{j,e_j} & 0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\]

if \( e_j > 1 \), and \( E_j = (E_{j,1}) \) if \( e_j = 1 \). Here each 0 = 0 and each \( E_{j,e} \) is both a circulant of type \( (n_0, n_1, \ldots, n_{i-1}) \) (since \( R_i \) has circulants of this type as “elements”) and a generalized permutation matrix. Moreover, \( e_1 + e_2 + \cdots + e_r = g \). Since \( N_i \) is similar to \( P_i \) and \( P_i^{n_i} = I_n \), then \( N_i^{n_i} = I_n \). This implies that each \( e_j \leq n_i \). We shall prove that each \( e_j = n_i \). The proof is by contradiction. Suppose for at least one \( j \) that \( e_j < n_i \). We know that \( f(e_1 + e_2 + \cdots + e_r) = fg = n \). Hence \( fn_i.r > n \) and so \( r > h \). Now

\[
P_i = [0_f, I_f, 0_f, \ldots, 0_f]_{n_i} \times I_h
\]

and \( P_iA_{i-1} = A_{i-1}M_i \). Let \( H_i = \{A_{i,1}, A_{i,2}, \ldots, A_{i,e_i}\} \) for \( 1 \leq s \leq g \). Then from \( P_iA_{i-1} = A_{i-1}M_i \) it follows that:

\[
H_3 = H_3M_i, H_3 = H_3M_i, \ldots, H_{n_i} = H_{n_i}M_i
\]

\( H_{n_i} = H_{n_i}M_i, H_{n_i+1} = H_{n_i+1}M_i, H_{n_i+2} = H_{n_i+2}M_i, \ldots, H_{2n_i} = H_{2n_i}M_i, \ldots; H_{(h-1)n_i+1} = H_{(h-1)n_i+1}M_i, \ldots, H_{hn_i} = H_{hn_i}M_i \). Hence, if \( B_j = H_{(j-1)n_i+1} \) for \( 1 \leq j \leq h \), then \( H_{(j-1)n_i+q} = B_jM_i^{-1} \) for \( 2 \leq q \leq n_i \).
Consequently,

\[
A_{i-1}R_i = \begin{bmatrix}
B_1 \\
B_1M_i \\
B_1M_i^2 \\
\vdots \\
B_1M_i^{s-1} \\
B_h \\
B_hM_i \\
\vdots \\
B_hM_i^{s-1}
\end{bmatrix}
R_i = \begin{bmatrix}
B_1R_i \\
B_1M_iR_i \\
B_1M_i^2R_i \\
\vdots \\
B_1M_i^{s-1}R_i \\
B_hR_i \\
B_hM_iR_i \\
\vdots \\
B_hM_i^{s-1}R_i
\end{bmatrix}
= \begin{bmatrix}
B_1R_i \\
B_1R_iN_i \\
B_1R_iN_i^2 \\
\vdots \\
B_1R_iN_i^{s-1} \\
B_hR_i \\
B_hR_iN_i \\
\vdots \\
B_hR_iN_i^{s-1}
\end{bmatrix}.
\]

Here each \(B_jR_i\), \(1 \leq j \leq h\), may also be regarded as a row vector with \(g\) coordinates in elements which are circulants of type \((n_0, n_1, \cdots, n_{i-1})\). This is so because both \(B_j\) and \(R_i\) have circulants of this type as "elements".

Let \(X = (X_1, X_2, \cdots, X_g)\) be a row vector in which the \(X_i\) are square matrices with \(f\) rows and columns. Then

\[
XN_i = (X_1E_{1,1}, X_1E_{1,2}, X_1E_{1,3}, \cdots, X_1E_{1,n-1},
X_2E_{2,0}, X_2E_{2,1}, X_2E_{2,2}, \cdots, X_2E_{2,n-1},
\cdots, X_gE_{g,0}, \cdots, X_gE_{g,n-1})
\]

Since each \(E_{j,s}\) is a generalized permutation matrix, it follows that the first \(fe_1\) columns of \(XN_i\) are, apart from order and possible change of sign, just the first \(fe_1\) columns of \(X\); the next \(fe_2\) columns of \(XN_i\) are, up to order and sign, just the next \(fe_2\) columns of \(X\); and, in general, columns

\[
f(e_0 + e_1 + \cdots + e_{s-1}) + 1, f(e_0 + e + \cdots + e_{s-1}) + 2, \cdots,
\]

of \(XN_i\) are, apart from order and sign, just these same columns in \(X\). Here \(e_0 = 0\). This holds for \(s = 1, 2, \cdots, r\).

Hence, in \(B_jR_iN_i\), for \(1 \leq v \leq n_i - 1\) and fixed \(j\), columns \((7)\) (for a fixed value of \(s\)) are just a generalized permutation of columns \((7)\) in \(B_jR_i\). Moreover, the elements appearing in columns \((7)\) and row \(q\) of \(B_jR_i\) for \(2 \leq q \leq f\) are just a permutation of the elements in columns \((7)\) and the first row of \(B_jR_i\), since \(B_jR_i\) is composed of blocks which are circulants of type \((n_0, n_1, \cdots, n_{i-1})\). All this means that the elements in columns \((7)\) (for a fixed value of \(s\)) and row \(q\) (for \(2 \leq q \leq m\)) of the matrix
are generalized permutations of the elements in columns (7) and the first row of this matrix. Hence the integers in row $q$ (for $2 \leq q \leq m$) and columns (7) of the matrix (8) are congruent (modulo $2$) to a permutation of the integers in column (7) and the first row of (8).

In the matrix $A_{i-1}R_i$, add columns $f(e_0 + e_1 + \cdots + e_{s-1}) + 1$, $f(e_0 + e_1 + \cdots + e_s) - 1$ to column $f(e_0 + e_1 + \cdots + e_s)$ for $s = 1, 2, \ldots, r$. The integers appearing in rows $mp + 2$, $mp + 3, \ldots, m(p + 1)$ of column $f(e_0 + e_1 + \cdots + e_s)$ are now congruent (modulo $2$) to the integer in row $mp + 1$ and column $f(e_0 + e_1 + \cdots + e_s)$. This holds for $p = 0, 1, \ldots, h - 1$, and $s = 1, 2, \ldots, r$. Now add row $mp + 1$ to rows $mp + 2, mp + 3, \ldots, m(p + 1)$ for $p = 0, 1, \ldots, h - 1$. The integer in row $mp + q$ and column $f(e_1 + e_2 + \cdots + e_s)$ is now congruent to zero (modulo $2$), for $2 \leq q \leq m; 0 \leq p \leq h - 1; 1 \leq s \leq r$. Hence columns $f(e_1 + e_2 + \cdots + e_s)$ for $1 \leq s \leq r$ may be regarded as lying in the same vector space of dimension $h$ over the field of two elements. Since $r > h$, these vectors are dependent. Consequently the determinant of $A_{i-1}R_i$ is congruent to zero (modulo $2$). This is a contradiction as the determinant of $A_{i-1}R_i$ is $\pm 1$.

Hence each $e_j = n_i$. Let $Z_i$ be the block diagonal matrix diag $(I_f, E_{j,1}, E_{j,2}, E_{j,3}, \cdots, E_{j,n_{i-1}})$. Since $E_{j,1}E_{j,2} \cdots E_{j,n_{i-1}}$ is a diagonal block in $E_{j}^i$ and since $E_{j}^i = I_m$, it follows that $E_{j,1}E_{j,2} \cdots E_{j,n_{i-1}} = I_f$. From this fact and the fact that the $E_{j,1}$ are generalized permutation matrices we find that $Z_iE_iZ_i' = [0_f, I_f, 0_r, \cdots, 0_{l_{n_{i-1}}}]$. Hence, if $Z = \text{diag}(Z_1, Z_2, \cdots, Z_r)$, then $ZNZ' = P_i$. Moreover, $Z$ is a matrix with $g$ rows and columns in elements which are circulants of type $(n_0, n_1, \cdots, n_{i-1})$. We now have $M_i = U'_iP_iU_i$ where $U'_i = R_iZ'$ is a generalized permutation matrix and a matrix with $g$ rows and columns in elements which are circulants of type $(n_0, n_1, \cdots, n_{i-1})$. Then

$$A_{i-1} = \begin{bmatrix} B_iU'_iU_i \\ B_iU'_iP_iU_i \\ \cdots \\ B_iU'_iP_{n_{i-1}}U_i \\ \cdots \\ B_hU'_iU_i \\ \cdots \\ B_hU'_iP_{n_{i-1}}U_i \end{bmatrix} = \begin{bmatrix} B_iU'_i \\ B_iU'_iP_i \\ \cdots \\ B_iU'_iP_{n_{i-1}} \\ \cdots \\ B_hU'_i \\ \cdots \\ B_hU'_iP_{n_{i-1}} \end{bmatrix} U_i = A_iU_i,$$
say. Here each $B_j U'_i$ is a vector with $g$ coordinates in elements which are circulants of type $(n_0, n_1, \ldots, n_{i-1})$. From the form of $A_{ij}$ it follows that $A_i$ may be partitioned into blocks which are circulants of type $(n_0, n_1, \ldots, n_i)$.

The proof is now complete.

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