

Pacific Journal of Mathematics

ABSOLUTE CONTINUITY OF INFINITELY DIVISIBLE DISTRIBUTIONS

HOWARD GREGORY TUCKER

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1. Introduction and summary. A probability distribution function F is said to be infinitely divisible if and only if for every integer n there is a distribution function F_n whose n -fold convolution is F . If F is infinitely divisible, its characteristic function f is necessarily of the form

$$(1) \quad f(u) = \exp \left\{ iu\gamma + \int_{-\infty}^{\infty} \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) \frac{1+x^2}{x^2} dG(x) \right\},$$

where $u \in (-\infty, \infty)$, γ is some constant, and G is a bounded, non-decreasing function. J. R. Blum and M. Rosenblatt [1] have found necessary and sufficient conditions that F be continuous and necessary and sufficient conditions that F be discrete. The purpose of this note is to add to the results of Blum and Rosenblatt by giving sufficient conditions under which an infinitely divisible probability distribution F is absolutely continuous. These conditions are that G be discontinuous at 0 or that $\int_{-\infty}^{\infty} (1/x^2) dG_{ac}(x) = \infty$, where G_{ac} is the absolutely continuous component of G . In §2 some lemmas will be proved, and in §3 the proof of the sufficiency of these conditions will be given. All notation used here is standard and may be found, for example, in Loève [2].

2. Some lemmas. In this section three lemmas are proved which will be used in the following section.

LEMMA 1. *If F and H are probability distribution functions, and if F is absolutely continuous, then the convolution of F and H , $F * H$, is absolutely continuous.*

This lemma is well known, and the proof is omitted.

LEMMA 2. *If $\{F_n\}$ is a sequence of absolutely continuous distribution functions, and if $p_n \geq 1$ and $\sum_{n=1}^{\infty} p_n = 1$, then $\sum_{n=1}^{\infty} p_n F_n$ is an absolutely continuous distribution function.*

Proof. By using the Lebesgue monotone convergence theorem it is easy to verify that $\sum_{n=1}^{\infty} p_n f_n$ is the density of $\sum_{n=1}^{\infty} p_n F_n$, where f_n is the density of F_n .

LEMMA 3. *Let $\{Y, X_1, X_2, \dots\}$ be independent random variables. Assume that the X_i 's have the same absolutely continuous distribution F , and assume that the distribution of Y is Poisson with expectation λ . Then $Z = X_1 + \dots + X_Y$ has a distribution function which has a saltus $e^{-\lambda}$ at 0 and is absolutely continuous elsewhere, and has as characteristic function*

$$f_z(u) = \exp \lambda \int_{-\infty}^{\infty} (e^{iux} - 1) dF(x) .$$

Proof. Let $E(x)$ be the distribution function degenerate at 0, and let $F^{*n}(x)$ denote the convolution of F with itself n times. Then it is easy to see that the distribution function of $Z, F_z(z)$, may be written as $F_z(z) = e^{-\lambda}E(z) + \sum_{n=1}^{\infty} e^{-\lambda}(\lambda^n/n!)F^{*n}(z)$. By lemma 1, each F^{*n} is absolutely continuous and has a density f^{*n} . We need only show that $F_z(z) - e^{-\lambda}E(z)$ is absolutely continuous. If we write

$$F_z(z) - e^{-\lambda}E(z) = \sum_{n=1}^{\infty} e^{-\lambda}(\lambda^n/n!) \int_{-\infty}^z f^{*n}(t) dt$$

and apply the Lebesgue monotone convergence theorem we obtain this conclusion.

3. The theorem. If G is a bounded nondecreasing function used in (1), then we may write $G(x) = G_s(x) + G_{ac}(x)$, where G_s is a singular nondecreasing function and $G_{ac}(x)$ is an absolutely continuous nondecreasing function.

THEOREM. *Let F be an infinitely divisible distribution function with characteristic function (1). Then F is absolutely continuous if at least one of the following two conditions is satisfied:*

- (i) G is not continuous at 0, or
- (ii) $\int_{-\infty}^{\infty} (1/x^2) dG_{ac}(x) = \infty$.

Proof. If condition (i) is satisfied, then by Lemma 1 it easily follows that F is absolutely continuous, since in that case F is a convolution of a normal distribution with another infinitely divisible distribution. We now prove that condition (ii) is sufficient. By Lemma 1 it is sufficient to prove that the distribution function F_0 whose characteristic function is

$$(2) \quad \exp \int_{-\infty}^{\infty} \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) \frac{1+x^2}{x^2} dG_{ac}(x)$$

is absolutely continuous. Let $\varepsilon_n > \varepsilon_{n+1} > 0$ for each n be such that $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ and such that

$$\lambda_n = \int_{S_n} ((1 + x^2)/x^2) dG_{ac}(x) > 0 ,$$

where

$$S_n = (-\varepsilon_{n-1}, -\varepsilon_n] \cup [\varepsilon_n, \varepsilon_{n-1}) , \quad n = 1, 2, \dots ,$$

and where $\varepsilon_0 = \infty$. Let U_n be a random variable with characteristic function

$$(3) \quad f_{U_n}(u) = \exp \int_{S_n} \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) \frac{1+x^2}{x^2} dG_{ac}(x) ,$$

and let

$$H_n(x) = (1/\lambda_n) \int_{(-\infty, x] \cap S_n} ((1 + x^2)/x^2) dG_{ac}(x) .$$

One easily sees that $\lambda_n < \infty$ and that $H_n(x)$ is an absolutely continuous distribution function of a bounded random variable. For each positive integer n we may write, by Lemma 3, that

$$U_n = X_{n,1} + X_{n,2} + \dots + X_{n,Z_n} - \int_{S_n} (1/x) dG_{ac}(x)$$

where Z_n is a random variable with Poisson distribution with expectation λ_n , where $\{X_{n,1}, X_{n,2}, \dots\}$ have the common absolutely continuous distribution function $H_n(x)$, and where $\{Z_n, X_{n,1}, X_{n,2}, \dots\}$ are independent. If we assume that

$$\{\{Z_n, X_{n,1}, X_{n,2}, \dots\}, n = 1, 2, \dots\}$$

are all independent, then the distribution function of

$$U_0 = \sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty} \left(\sum_{j=1}^{Z_n} X_{n,j} - \int_{S_n} (1/x) dG_{ac}(x) \right)$$

is equal to F_0 . Now let us define a sequence of events $\{C_n\}$ by

$$C_1 = [Z_1 \neq 0] , \quad C_2 = [Z_1 = 0][Z_2 \neq 0] ,$$

and, in general,

$$C_n = [Z_n \neq 0] \bigcap_{i=1}^{n-1} [Z_i = 0] .$$

These events are easily seen to be disjoint. If we define

$$(4) \quad C_0 = \left(\bigcup_{n=1}^{\infty} C_n \right)^c = \bigcap_{n=1}^{\infty} [Z_n = 0] ,$$

then $\Omega = \bigcup_{n=1}^{\infty} C_n$, where Ω is the sure event. The distribution function of U_0 is

$$F_{U_0}(u) = \sum_{n=1}^{\infty} P([U_0 \leq u] | C_n)P(C_n) + P([U_0 \leq u]C_0) .$$

By (4) and by hypothesis, we obtain

$$P([U_0 \leq u]C_0) \leq P(C_0) = \lim_{n \rightarrow \infty} \exp \left\{ - \int_{-\infty}^{-\varepsilon_n} + \int_{\varepsilon_n}^{\infty} (1/x^2) dG_{a_c}(x) \right\} = 0 .$$

Also, $P([U_0 \leq u] | C_n)$ is the distribution function of $X_{n,1} + W_n$, where $X_{n,1}$ and W_n are independent random variables. Since the distribution function of $X_{n,1}$ is absolutely continuous, it follows by Lemma 1 that $P([U_0 \leq u] | C_n)$ is absolutely continuous for each n . Lemma 2 then implies that $F_{U_0}(u)$ is absolutely continuous, which concludes the proof of the theorem.

The condition given in this theorem is not necessary, as is shown in the following example. Let $\gamma = 0$ in (1), and let α, β be real numbers which satisfy $\beta > 1, 1 > \alpha > \beta/2$. For $j = 1, 2, \dots$, let us denote

$$x_j = j^{-\alpha} \quad \text{and} \quad \rho_j = j^{-\beta} .$$

Let G be a pure jump function with jumps at x_j and $-x_j$ of size ρ_j for every j . (The total variation of G is $2 \sum \rho_j < \infty$.) In this case we obtain

$$f(u) = \exp 2 \sum_{n=1}^{\infty} \left(\cos \frac{u}{n^\alpha} - 1 \right) \frac{n^{2\alpha} - 1}{n^\beta} .$$

We shall show that there is a constant K such that for all $|u| \geq \pi$, the inequality

$$(5) \quad 0 < f(u) < \exp(-K|u|^{2-\beta/\alpha})$$

is true. This is equivalent to showing that

$$(6) \quad \sum_{n=1}^{\infty} \frac{n^{2\alpha} + 1}{n^\beta} \sin^2 \frac{|u|}{2n^\alpha} > K|u|^{2-\beta/\alpha} .$$

Let us consider, for each fixed $|u| \geq \pi$ the integer N defined by

$$N = \left[\frac{1}{2} \left(\frac{2|u|}{\pi} \right)^{1/\alpha} + 1 \right],$$

where the square brackets have their usual meaning. It is easy to verify that $0 < |u|/2N^\alpha < \pi/2$, and thus we may write

$$\begin{aligned} \frac{N^{2\alpha} + 1}{N^\beta} \sin^2 \frac{|u|}{2N^\alpha} &> N^{2\alpha-\beta} \sin^2 \frac{|u|}{2 \left[\left(\frac{2|u|}{\pi} \right)^{1/\alpha} \right]^\alpha} \\ &> K|u|^{2-\beta/\alpha} , \end{aligned}$$

where K does not depend on u . This inequality implies that inequality (6) is true, thus implying (5). Inequality (5) implies that $f(u) \in L_1(-\infty, +\infty)$, which in turn implies that $f(u)$ is the characteristic function of an absolutely continuous distribution. (See Theorem 3.2.2 on page 40 in [3].)

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