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JAMES HOWARD WELLS

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1. Introduction. Let V denote the family of all finite complex-valued and conuntably additive set functions on the Borel subsets of $R_+ = [0\ \infty)$ (hereafter called measures); $L^1(R_+)$ the set of all complex-valued functions on R_+ which are integrable in the sense of Lebesgue, identifying functions which are 0 almost everywhere; and A the elements in V which are absolutely continuous with respect to Lebesgue measure. For each $\mu \in V$ there exists an $f \in L^1(R_+)$ such that

$$\mu(E) = \int_E f(x)dx \tag{1.1}$$

for each Borel subset E of R_+. And, conversely, if $f \in L^1(R_+)$ the set function μ defined by (1.1) is a measure.

We introduce a norm into V by the formula

$$\|\mu\| = \sup \Sigma \mid \mu(E_i) \mid \qquad (\mu \in V)\ , \tag{1.2}$$

the supremum being taken over all finite partitions of R_+ into pairwise disjoint Borel sets E_i. It is well known ([6], p. 142 or [7]) that V becomes a commutative Banach algebra under the convolution operation

$$\nu(E) = \int_0^\infty \mu(E - x)d\lambda(x) \qquad (\mu, \lambda \in V)\ , \tag{1.3}$$

where E is any Borel subset of R_+; in symbols

$$\nu = \mu * \lambda\ . \tag{1.4}$$

The Laplace-Stieltjes transform of $\mu \in V$ will be denoted by $\hat{\mu}$:

$$\hat{\mu}(z) = \int_0^\infty e^{-zx}d\mu(x) \qquad (Re(z) \geqq 0)\ . \tag{1.5}$$

The relation (1.4) is equivalent to

$$\hat{\nu}(z) = \hat{\mu}(z)\hat{\lambda}(z) \qquad (Re(z) \geqq 0)\ . \tag{1.6}$$

The *identity* in V is the measure u such that $u(E) = 1$ if $0 \in E$ and 0 otherwise. A measure μ is *invertible* provided there exists a measure μ^{-1} such that $\mu * \mu^{-1} = u$; and the measure λ is a *divisor* of the measure μ, in symbols $\lambda \mid \mu$, provided there exists a measure ν such that $\mu = \lambda * \nu$. It follows from basic properties of the Laplace-Stieltjes

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transform that V is an integral domain and a semi-simple Banach algebra (see for example [6], p. 149).

The central problem under consideration here is that of determining the prime measures, that is, those noninvertible measures μ such that

(i) $\mu = \lambda * \nu$ always implies that one of the measures λ, ν is invertible.

It is clear that every prime measure μ satisfies the condition

(ii) $V * \mu \subset V * \lambda$ implies that either λ is invertible or $\mu \mid \lambda$.

And (i) follows from (ii) since V is an integral domain. Here $V * \mu$ denotes the ideal $\{\nu * \mu \mid \nu \in V\}$.

We give a partial solution by showing that all measures of the form

$$\mu_a = \frac{1}{1+a} u - \eta \qquad (Re(a) > 0)\,, \tag{1.7}$$

where $d\eta(x) = e^{-x}dx$, are primes. Stated in terms of the ideal structure of V, the result is that the maximal ideals $m_a = \{\mu \mid \hat{\mu}(a) = 0\}$, $Re(a) > 0$, are principal.

A related problem is the following: Given a fixed measure μ, for what measures λ is it true that $\lambda \mid \mu$? Climaxing a sequence of papers on this problem, notably [4] and [8], Fuchs [3] proved that $\lambda \mid \mu$ if and only if the Hausdorff method of summability $[H, \mu]$ includes the method $[H, \lambda]$. In this paper we make use of recent results on the representation of linear transformations by convolution to give a simple, and apparently unnoticed, alternative formulation in terms of the range of a convolution transform.[1]

THEOREM 1. *Every measure μ_a, $Re(a) > 0$, is a prime; and if there exists a prime μ essentially different from μ_a, $Re(a) > 0$ (two primes are essentially different if one cannot be obtained from the other by convolution with an invertible measure) then either $\hat{\mu}(z)$ has a root with real part 0 or the hull of the ideal $V * \mu$ consists only of maximal ideals in V which contain A.*

THEOREM 2. *Let T_μ, $\mu \in V$, be the linear operator from $L^1(R_+)$ into $L^1(R_+)$ defined by*

$$T_\mu f(t) = f * \mu(t) = \int_0^t f(t-x)\,d\mu(x) \tag{1.8}$$

for $f \in L^1(R_+)$. Let R_μ denote the range of T_μ. Then the measure λ is a divisor of the measure μ if and only if $R_\mu \subset R_\lambda$.

[1] The author is indebted to the referee for his helpful suggestions.

2. Proofs of the Theorems.

Proof of Theorem 1. The positive result of this theorem depends on the obvious fact (see condition (ii)) that if the maximal ideal m in V is principal and μ is a generator, that is $m = V * \mu$, then μ is a prime.

Fix $Re(a) \geqq 0$ and set $h(\mu) = \hat{\mu}(a)$. It follows from (1.5) and (1.6) that h defines a multiplicative linear functional on V. Hence $m_a = \{\mu \in V \mid \hat{\mu}(a) = 0\}$ is a maximal ideal in V. That $V * \mu_a \subset m_a$ follows from (1.4), (1.6) and the fact that $\hat{\mu}_a(z) = (1 + a)^{-1} - (1 + z)^{-1}$ vanishes at a.

The reverse inclusion requires that if $\mu \in m_a$, then $\mu = \nu * \mu_a$ for some $\nu \in V$. To this end we use a device suggested by [9] and define

$$\nu = (1 + a)\mu + (1 + a)^2\theta_a \tag{2.1}$$

where

$$d\theta_a = \int_0^x e^{a(x-t)} d\mu(t)dx = -\int_x^\infty e^{-a(t-x)} d\mu(t)dx = f(x)dx\ . \tag{2.2}$$

The equality of the two integrals is a consequence of $\hat{\mu}(a) = 0$. In case $\sigma = Re(a) > 0$, an application of the Fubini theorem using the second integral in (2.2) yields

$$\begin{aligned}\int_0^\infty |f(x)|\,dx &= \int_0^\infty \left|\int_x^\infty e^{-a(t-x)} d\mu(t)\right| dx \leqq \int_0^\infty\int_x^\infty e^{-\sigma(t-x)} d\,|\mu(t)|\,dx \\ &= \int_0^\infty\int_0^t e^{-\sigma(t-x)} dx\,d\,|\mu(t)| = \frac{1}{\sigma}\int_0^\infty [1 - e^{-\sigma t}]d\,|\mu(t)| \\ &\leqq \frac{1}{\sigma}\int_0^\infty |\,d\mu(t)\,| < \infty\ .\end{aligned}$$

This proves $f \in L^1(R_+)$ so that, in view of (1.1), $\theta_a \in A$ when $Re(a) > 0$.

It remains to verify that

$$\begin{aligned}\mu = \nu * \mu_a &= (1 + a)[\mu + (1 + a)\theta_a] * [(1 + a)^{-1}u - \eta] \\ &= (1 + a)[(1 + a)^{-1}\mu - \mu * \eta + \theta_a - (1 + a)\theta_a * \eta]\ .\end{aligned}$$

But integration by parts yields the relation

$$\int_0^t e^{-(t-x)}\int_x^\infty e^{-a(y-x)} d\mu(y)dx = (1 + a)^{-1}\left[\int_0^t e^{-(t-y)} d\mu(y) + \int_t^\infty e^{-a(y-t)} d\mu(y)\right]$$

which, together with the fact that $d(\phi * \gamma)(x) = (f * \gamma)(x)dx$ whenever $d\phi(x) = f(x)dx, f \in L^1(R_+)$ and $\gamma \in V$, shows that $(1 + a)\theta_a * \eta = -\mu * \eta + \theta_a$. This establishes the result.

If μ is a prime essentially different from μ_a, $Re(a) > 0$, and $\hat{\mu}(z)$ has no roots with real part 0, then $\hat{\mu}(z)$ has no roots. To see this note that $\hat{\mu}(a) = 0$ for $Re(a) > 0$ implies that $V * \mu \subset V * \mu_a = m_a$. Hence

$\mu = \nu * \mu_a$ for some $\nu \in V$ which, because of condition (ii), forces ν to be invertible; so μ is not essentially different from μ_a. Thus $V * \mu$ is not contained in m_a for any a, $Re(a) \geqq 0$. Phillips ([6], p. 148 or [7]) has shown that in the space Δ of maximal ideals in V, $\Delta_1 = \{m_a \mid Re(a) \geqq 0\}$ is precisely those maximal ideals which *omit* an element of A so that $\Delta_2 = \Delta - \Delta_1$ consists of all those maximal ideals which contain A. It is clear, then, that the hull of $V * \mu$, i.e., all maximal ideals which contain it, must be a subset of Δ_2.

Proof of Theorem 2. First suppose that $\lambda \mid \mu$. Then $\mu = \nu * \lambda$ for some $\nu \in V$ and, therefore,

$$L^1(R_+) * \mu = L^1(R_+) * \nu * \lambda \subset L^1(R_+) * \lambda ,$$

i.e., $R_\mu \subset R_\lambda$.

For the converse we note that the inclusion $R_\mu \subset R_\lambda$ implies that for each $f \in L^1(R_+)$ there exists a $g \in L^1(R_+)$ such that

$$f * \mu = g * \lambda . \tag{2.1}$$

But the fact that V is an integral domain insures the uniqueness of g. Hence the relation (2.1) defines a mapping T: $f \rightarrow g$ which is linear, commutes with convolution in the sense that $T(f * \gamma) = T(f) * \gamma$ for $f \in L^1(R_+)$, $\gamma \in V$, and, via an application of the closed graph theorem, bounded in the norm topology of $L^1(R_+)$. It follows using the type of argument given in [2], that every such mapping has the form $T(f) = f * \nu$ for some measure ν. Thus

$$f * \mu = (f * \nu) * \lambda = f * (\nu * \lambda) \tag{2.2}$$

for every $f \in L^1(R_+)$. A second application of the fact that V is an integral domain yields $\mu = \nu * \lambda$, that is $\lambda \mid \mu$, and the theorem is proved.

3. A remark and a question. Let $Re(a) > 0$, $Re(b) > 0$. It is easy to verify that $(z + 1)/(z + b)$ is the Laplace-Stieltjes transform of an invertible measure. Consequently the measure defined by

$$\hat{\mu}(z) = \frac{z - a}{z - b} = \hat{\mu}_a(z) \frac{(1 + a)(z + 1)}{z + b} \; (Re(z) \geqq 0) \tag{3.1}$$

is a prime not essentially different from μ_a. The primes given by relation (3.1) coincide with those given in [4]. Existence of other primes remains an open question.

Repeated application of Theorem 1 yields the relation

$$V * \mu_{a_1} * \mu_{a_2} * \cdots * \mu_{a_n} = \bigcap_{i=1}^{n} m_{a_i} , \qquad n = 2, 3, \cdots \tag{3.2}$$

where $Re(a_i) > 0$, $i = 1, 2, 3, \cdots$. On the other hand, it is known [1] that the closed ideal $m = \bigcap_{i=1}^{\infty} m_{a_i}$ is not trivial in case $\sum_{i=1}^{\infty} 1/|a_i| < \infty$. A natural question to ask is the following: Does there exist a measure μ such that $V * \mu = m$?

REFERENCES

1. T. Carleman, *Über die approximation analytischer funktionen durch lineare aggregate von vorgegeben potenzen,* Arkiv för mat., astronomi och fysik, 17, **9** (1923), 1-30.
2. R. E. Edwards, *Representation theorems for certain functional operators,* Pacific J. Math., **7** (1957), 1333-1339.
3. W. H. J. Fuchs, *A theorem on Hausdorff's methods of summation,* Quarterly J. of Math., **16** (1945), 64-77.
4. H. L. Garabedian, Einar Hille and H. S. Wall, *Formulations of the Hausdorff inclusion problem,* Duke Math. J., **8** (1941), 193-213.
5. Einar Hille and J. D. Tamarkin, *Questions of relative inclusion in the domain of Hausdorff means,* Proc. Nat. Acad. Sci., **19** (1933), 573-577.
6. Einar Hille and R. S. Phillips, *Functional analysis and semi-groups,* Amer. Math. Soc. Coll. Publ., XXXI, 1957.
7. R. S. Phillips, *Spectral theory for semi-groups of operators,* Trans. Amer. Math. Soc., **71** (1951), 393-415.
8. W. W. Rogosinski, *On Hausdorff's method of summability,* Proc. Cambridge Phil. Soc., **38** (1942), 166-192.
9. L. L. Silverman and J. D. Tamarkin, *On the generalization of Abel's theorem for certain definitions of summability,* Math. Z., **29** (1928), 161-170.

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