A NOTE ON HYPONORMAL OPERATORS

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The last exercise in reference [4] is a question to which I did not know the answer: does there exist a hyponormal \((TT^* \leq T^*T)\) completely continuous operator which is not normal? Recently Tsuyoshi Andô has answered this question in the negative, by proving that every hyponormal completely continuous operator is necessarily normal ([1]). The key to Andô’s solution is a direct calculation with vectors, showing that a hyponormal operator \(T\) satisfies the relation \(\| T^* \| = \| T \| \) for every positive integer \(n\) (for “subnormal” operators, this was observed by P. R. Halmos on page 196 of [6]). It then follows, from Gelfand’s formula for spectral radius, that the spectrum of \(T\) contains a scalar \(\mu\) such that \(\| \mu \| = \| T \|\) (see [9], Theorem 1.6.3.).

The purpose of the present note is to obtain this result from another direction, via the technique of approximate proper vectors ([3]); in this approach, the nonemptiness of the spectrum of a hyponormal operator \(T\) is made to depend on the elementary case of a self-adjoint operator, and a simple calculation with proper vectors leads to a scalar \(\mu\) in the spectrum of \(T\) such that \(\| \mu \| = \| T \|\). This is the Theorem below, and its Corollaries 1 and 2 are due also to Andô. In the remaining corollaries, we note several applications to completely continuous operators.

We consider operators (= continuous linear mappings) defined in a Hilbert space. As in [3], the spectrum of an operator \(T\) is denoted \(s(T)\), and the approximate point spectrum is \(a(T)\). We note for future use that every boundary point of \(s(T)\) belongs to \(a(T)\); see, for example, ([4], hint to Exercise VIII. 3.4).

**Lemma 1.** Suppose \(T\) is a hyponormal operator, with \(\| T \| \leq 1\), and let \(\mathcal{M}\) be the set of all vectors which are fixed under the operator \(TT^*\). Then,

(i) \(\mathcal{M}\) is a closed linear subspace,
(ii) the vectors in \(\mathcal{M}\) are fixed under \(T^*T\),
(iii) \(\mathcal{M}\) is invariant under \(T\), and
(iv) the restriction of \(T\) to \(\mathcal{M}\) is an isometric operator in \(\mathcal{M}\).

**Proof.** Since \(\mathcal{M} = \{ x : TT^*x = x \} \) is the null space of \(I - TT^*\), it is a closed linear subspace. The relation \(TT^* \leq T^*T \leq I\) implies \(0 \leq I - T^*T \leq I - TT^*\), and from this it is clear that the null space of \(I - TT^*\) is contained in the null space of \(I - T^*T\). That is, \(TT^*x = x\)

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implies $T^* Tx = x$. This proves (ii). (Alternatively, given $TT^* x = x$, one can calculate directly that $|| T^* Tx - x ||^2 \leq 0$.) If $x \in \mathcal{H}$, that is if $TT^* x = x$, then the calculation $TT^*(Tx) = T(T^* Tx) = Tx$ shows that $Tx \in \mathcal{H}$; moreover, $|| Tx ||^2 = (T^* Tx | x) = || x ||^2$.

**Lemma 2.** Every isometric operator has an approximate proper value of absolute value 1.

**Proof.** Let $U$ be an isometric operator in a nonzero Hilbert space. Suppose first that the spectrum of $U$ contains 1; since $|| U || = 1$, it follows that 1 is a boundary point of $s(U)$ (see [4], part (ix) of Exercise VII.3.12), hence 1 is an approximate proper value for $U$.

If the spectrum of $U$ does not contain 1, that is if $I - U$ is invertible, we may form the Cayley transform $A$ of $U$; thus,

$$A = i(I + U)(I - U)^{-1} = i(I - U)^{-1}(I + U).$$

Using the hypothesis $U^* U = I$, let us show that $A$ is self-adjoint. Left-multiplying the relation $(I - U)A = i(I + U)$ by $U^*$, we have $(U^* - I)A = i(U^* + I)$, thus $(I - U)^* A = -i(I + U)^*$. Since $(I - U)^*$ is invertible, with inverse $[(I - U)^*]^{-1}$, we have

$$A = -i[(I - U)^*]^{-1}(I + U)^* = -i[(I + U)(I - U)^{-1}]^* = A^*.$$ 

It follows that the operators $A + iI$ and $A - iI$ are invertible, and solving the relation $(I - U)A = i(I + U)$ for $U$, we have

$$U = (A - iI)(A + iI)^{-1} = (A + iI)^{-1}(A - iI).$$

Incidentally, since $U$ is the product of invertible operators, we conclude that $U$ is unitary.

Since $A$ is self-adjoint, we know from an elementary argument that the approximate point spectrum of $A$ is non empty ([7], Theorem 34.2). Let $\alpha \in a(A)$, and let $x_n$ be a sequence of unit vectors such that $|| Ax_n - \alpha x_n || \to 0$. Define $\mu = (\alpha + i)^{-1}(\alpha - i)$; since $\alpha$ is real, $\mu$ has absolute value 1. It will suffice to show that $\mu$ is an approximate proper value for $U$; indeed, $||(U - \mu I)x_n|| \to 0$ results from the calculation

$$U - \mu I = (A + iI)^{-1}(A - iI) - (\alpha + i)^{-1}(\alpha - i)I$$

$$= (\alpha + i)^{-1}(A + iI)^{-1}[(\alpha + i)(A - iI) - (\alpha - i)(A + iI)]$$

$$= 2i(\alpha + i)^{-1}(A + iI)^{-1}(A - \alpha I),$$

the fact that $|| (A - \alpha I)x_n || \to 0$, and the continuity of the operator $2i(\alpha + i)^{-1}(A + iI)^{-1}$.

Incidentally, if $U$ is an isometric operator such that the spectrum of $U$ excludes some complex number $\mu$ of absolute value 1, then $\mu^{-1}U$
A NOTE ON HYPONORMAL OPERATORS

is an isometric operator whose spectrum excludes 1. The proof of Lemma 2 then shows that $\mu U$ is unitary, hence so is $U$. In other words: the spectrum of a nonnormal isometry must include the unit circle $|\mu| = 1$; indeed, Putnam has shown that the spectrum is the unit disc $|\mu| \leq 1$ ([8], Corollary 1). The latter result is also an immediate consequence of ([5], Lemma 2.1), and the fact that the spectrum of any unilateral shift operator is the unit disc.

**Theorem.** (Andô) Every hyponormal operator $T$ has an approximate proper value $\mu$ such that $|\mu| = ||T||$.

**Proof.** We may assume $||T|| = 1$ without loss of generality. Since $TT^* \geq 0$ and $||TT^*|| = 1$, we know that 1 is an approximate proper value for $TT^*$. Since the property of hyponormality is preserved under $*$-isomorphism, we may assume, after a change of Hilbert space, that 1 is a proper value for $TT^*$ ([3], Theorem 1). Form the nonzero closed linear subspace $\mathcal{M} = \{x : TT^*x = x\}$; according to Lemma 1, $\mathcal{M}$ is invariant under $T$, and the restriction of $T$ to $\mathcal{M}$ is an isometric operator $U$ in the Hilbert space $\mathcal{H}$. By Lemma 2, $U$ has an approximate proper value $\mu$ of absolute value 1. Let $x_n$ be any sequence of unit vectors in $\mathcal{M}$ such that $||Ux_n - \mu x_n|| \to 0$. Since $Ux_n = Tx_n$, obviously $\mu$ is an approximate proper value for $T$, and $|\mu| = 1 = ||T||$.

**Corollary 1.** A generalized nilpotent hyponormal operator is necessarily zero.

**Proof.** If $T$ is hyponormal, then $s(T)$ contains a scalar $\mu$ such that $|\mu| = ||T||$. For every positive integer $n$, it follows that $s(T^n)$ contains $\mu^n$ (see [7], Theorem 33.1); then $||T^n|| = |\mu^n| = ||T^n|| = ||T^n||$, and so $||T^n|| = ||T||^n$. If moreover $T$ is a generalized nilpotent, that is if $\lim ||T^n||^{1/n} = 0$, then $||T|| = 0$.

**Corollary 2.** If $T$ is a completely continuous hyponormal operator, then $T$ is normal.

**Proof.** The proof to be given is essentially the same as Andô's. The proper subspaces of $T$ are mutually orthogonal, and reduce $T$ ([4], Exercise VII. 2.5). Let $\mathcal{M}$ be the smallest closed linear subspace which contains every proper subspace of $T$, and let $\mathcal{N} = \mathcal{M}^\perp$; clearly $\mathcal{N}$ reduces $T$, and the restriction $T|\mathcal{N}$ is a completely continuous hyponormal operator in $\mathcal{N}$ ([4], Exercise VI. 9.18). If the spectrum of $T|\mathcal{N}$ were different from $\{0\}$, it would have a nonzero boundary point $\mu$, hence $\mu$ would be a proper value for $T|\mathcal{N}$ (see [4], Theorem VIII. 3.2); this is impossible since $\mathcal{N}^\perp = \mathcal{M}$ already contains every proper vector for $T$. 


We conclude from the Theorem that $T|\mathcal{N} = 0$, and this forces $\mathcal{N} = \{0\}$ (recall that $\mathcal{N}^\perp$ contains the null space of $T$). Thus, the proper subspaces of $T$ are a total family, hence $T$ is normal by ([4], Exercise VII. 2.5).

Suppose $T$ is a normal operator whose spectrum (a) has empty interior, and (b) does not separate the complex plane. Wermer has shown that the invariant subspaces of $T$ reduce $T$ ([10], Theorem 7). It is well known that the conditions (a) and (b) are fulfilled by the spectrum of any completely continuous operator. In particular: if $T$ is a completely continuous normal operator, then every invariant subspace of $T$ reduces $T$. A more elementary proof of this may be based on Corollary 2:

**Corollary 3.** If $T$ is a completely continuous normal operator, and $\mathcal{N}$ is a closed linear subspace invariant under $T$, then $\mathcal{N}$ reduces $T$.

**Proof.** Indeed, it suffices to assume that $T$ is hyponormal and $\mathcal{N}$ is an invariant subspace such that $T|\mathcal{N}$ is completely continuous. Since $T|\mathcal{N}$ is hyponormal ([4], Exercise VI. 9.10), it follows from Corollary 2 that $T|\mathcal{N}$ is normal, hence $\mathcal{N}$ reduces $T$ by ([4], Exercise VI. 9.9). Quoting ([4], Theorem VII. 3.1), we have:

**Corollary 4.** If $T$ is a hyponormal operator, then

$$|| T || = LUB \{ |(Tx| x) | : || x || \leq 1 \} .$$

Incidentally, if $T$ is hyponormal, it is clear from Corollary 4 that $|| T^* || = LUB \{ |(T^*x| x) | : || x || \leq 1 \} .$

**Corollary 5.** If the completely continuous operator $T$ is semi-normal in the sense of [8], then $T$ is normal.

**Proof.** The definition of semi-normality is that either $TT^* \leq T^*T$ or $TT^* \geq T^*T$, in other words, either $T$ or $T^*$ is hyponormal; since both are completely continuous (see [4], Exercise VIII. 1.6), our assertion follows from Corollary 2.

Let us say that an operator $T$ is nearly normal in case $T$ commutes with $T^*T$. The structure of nearly normal operators has been determined by Brown, and it is a consequence of his results that a completely continuous nearly normal operator is in fact normal (see the concluding remarks in [5]). This may also be proved as follows. An elementary calculation with square roots shows that a nearly normal operator is hyponormal (see [2], proof of Corollary 1 of Theorem 8); assuming also complete continuity and citing Corollary 2, we have:
**Corollary 6.** If $T$ is a completely continuous nearly normal operator, then $T$ is normal.

Finally,

**Corollary 7.** If $S = T + \lambda I$, where $T$ is a completely continuous operator, and if $S$ is hyponormal, then $S$ is normal.

*Proof.* Since $S$ is hyponormal, so is $T$ ([4], hint to Exercise VII. 1.6), hence $T$ is normal by Corollary 2; therefore $S$ is normal. So to speak, the $C^*$-algebra of all operators of the from $T + \lambda I$, with $T$ completely continuous, is of "finite class".

We close with an elementary remark about the adjoint of a hyponormal operator: if $T$ is hyponormal, then $s(T^*) = a(T^*)$. For, suppose $\lambda$ does not belong to $a(T^*)$, and let $\mu = \lambda^*$. Then, $(T - \mu I)^* = T^* - \lambda I$ is bounded below ([4], Exercise VII. 3.8), and since $T - \mu I$ is also hyponormal, the relation $(T - \mu I)(T - \mu I)^* \leq (T - \mu I)^*(T - \mu I)$ shows that $T - \mu I$ is also bounded below. Then $T - \mu I$ is invertible ([4], Exercise VI. 8.11), hence so is $T^* - \lambda I$, thus $\lambda$ does not belong to $s(T^*)$.

**References**


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Tsuyoshi Andô, *On fundamental properties of a Banach space with a cone* ........ 1163
Sterling K. Berberian, *A note on hyponormal operators* ............................ 1171
Errett Albert Bishop, *Analytic functions with values in a Frechet space* ........ 1177
(Sherman) Elwood Bohn, *Equicontinuity of solutions of a quasi-linear equation* .................................................. 1193
Andrew Michael Bruckner and E. Ostrow, *Some function classes related to the class of convex functions* .................. 1203
J. H. Curtiss, *Limits and bounds for divided differences on a Jordan curve in the complex domain* .......................... 1217
P. H. Doyle, III and John Gilbert Hocking, *Dimensional invertibility* ........... 1235
David G. Feingold and Richard Steven Varga, *Block diagonally dominant matrices and generalizations of the Gerschgorin circle theorem* .................. 1241
Leonard Dubois Fountain and Lloyd Kenneth Jackson, *A generalized solution of the boundary value problem for y'' = f(x, y, y')* ........................... 1251
Robert William Gilmer, Jr., *Rings in which semi-primary ideals are primary* .......... 1273
Ruth Goodman, *K-polar polynomials* .......................................................... 1277
Israel Halperin and Maria Wonenburger, *On the additivity of lattice completeness* ............................................................................ 1289
Isidore Heller and Alan Jerome Hoffman, *On unimodular matrices* .............. 1321
Robert G. Heyneman, *Duality in general ergodic theory* ............................... 1329
Charles Ray Hobby, *Abelian subgroups of p-groups* .................................... 1343
Kenneth Myron Hoffman and Hugo Rossi, *The minimum boundary for an analytic polyhedron* ............................................. 1347
Adam Koranyi, *The Bergman kernel function for tubes over convex cones* ........ 1355
Pesi Rustom Masani and Jack Max Robertson, *The time-domain analysis of a continuous parameter weakly stationary stochastic process* ........ 1361
William Schumacher Massey, *Non-existence of almost-complex structures on quaternionic projective spaces* .................. 1379
Deane Montgomery and Chung-Tao Yang, *A theorem on the action of SO(3)* .... 1385
Ronald John Nunke, *A note on Abelian group extensions* ............................ 1401
Carl Mark Pearcy, *A complete set of unitary invariants for operators generating finite W*-algebras of type I* ......................... 1405
Duane Sather, *Asymptotics. III. Stationary phase for two parameters with an application to Bessel functions* .................... 1423
J. Śladkowska, *Bounds of analytic functions of two complex variables in domains with the Bergman-Shilov boundary* ........ 1435
Joseph Gail Stampfli, *Hyponormal operators* ............................................... 1453
George Gustave Weill, *Some extremal properties of linear combinations of kernels on Riemann surfaces* ............................ 1459
Edward Takashi Kobayashi, *Errata: “A remark on the Nijenhuis tensor”* ....... 1467