

# Pacific Journal of Mathematics

**A NOTE ON HYPONORMAL OPERATORS**

STERLING K. BERBERIAN

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The last exercise in reference [4] is a question to which I did not know the answer: does there exist a hyponormal ( $TT^* \leq T^*T$ ) completely continuous operator which is not normal? Recently Tsuyoshi Andô has answered this question in the negative, by proving that every hyponormal completely continuous operator is necessarily normal ([1]). The key to Andô's solution is a direct calculation with vectors, showing that a hyponormal operator  $T$  satisfies the relation  $\|T^n\| = \|T\|^n$  for every positive integer  $n$  (for "subnormal" operators, this was observed by P. R. Halmos on page 196 of [6]). It then follows, from Gelfand's formula for spectral radius, that the spectrum of  $T$  contains a scalar  $\mu$  such that  $|\mu| = \|T\|$  (see [9], Theorem 1.6.3.).

The purpose of the present note is to obtain this result from another direction, via the technique of approximate proper vectors ([3]); in this approach, the nonemptiness of the spectrum of a hyponormal operator  $T$  is made to depend on the elementary case of a self-adjoint operator, and a simple calculation with proper vectors leads to a scalar  $\mu$  in the spectrum of  $T$  such that  $|\mu| = \|T\|$ . This is the Theorem below, and its Corollaries 1 and 2 are due also to Andô. In the remaining corollaries, we note several applications to completely continuous operators.

We consider operators (=continuous linear mappings) defined in a Hilbert space. As in [3], the spectrum of an operator  $T$  is denoted  $s(T)$ , and the approximate point spectrum is  $a(T)$ . We note for future use that every boundary point of  $s(T)$  belongs to  $a(T)$ ; see, for example, ([4], hint to Exercise VIII. 3.4).

**LEMMA 1.** *Suppose  $T$  is a hyponormal operator, with  $\|T\| \leq 1$ , and let  $\mathcal{M}$  be the set of all vectors which are fixed under the operator  $TT^*$ . Then,*

- (i)  $\mathcal{M}$  is a closed linear subspace,
- (ii) the vectors in  $\mathcal{M}$  are fixed under  $T^*T$ ,
- (iii)  $\mathcal{M}$  is invariant under  $T$ , and
- (iv) the restriction of  $T$  to  $\mathcal{M}$  is an isometric operator in  $\mathcal{M}$ .

*Proof.* Since  $\mathcal{M} = \{x : TT^*x = x\}$  is the null space of  $I - TT^*$ , it is a closed linear subspace. The relation  $TT^* \leq T^*T \leq I$  implies  $0 \leq I - T^*T \leq I - TT^*$ , and from this it is clear that the null space of  $I - TT^*$  is contained in the null space of  $I - T^*T$ . That is,  $TT^*x = x$

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Received February 27, 1962.

implies  $T^*Tx = x$ . This proves (ii). (Alternatively, given  $TT^*x = x$ , one can calculate directly that  $\|T^*Tx - x\|^2 \leq 0$ .) If  $x \in \mathcal{M}$ , that is if  $TT^*x = x$ , then the calculation  $TT^*(Tx) = T(T^*Tx) = Tx$  shows that  $Tx \in \mathcal{M}$ ; moreover,  $\|Tx\|^2 = (T^*Tx|x) = \|x\|^2$ .

**LEMMA 2.** *Every isometric operator has an approximate proper value of absolute value 1.*

*Proof.* Let  $U$  be an isometric operator in a nonzero Hilbert space. Suppose first that the spectrum of  $U$  contains 1; since  $\|U\| = 1$ , it follows that 1 is a boundary point of  $s(U)$  (see [4], part (ix) of Exercise VII. 3. 12), hence 1 is an approximate proper value for  $U$ .

If the spectrum of  $U$  does not contain 1, that is if  $I - U$  is invertible, we may form the Cayley transform  $A$  of  $U$ ; thus,

$$A = i(I + U)(I - U)^{-1} = i(I - U)^{-1}(I + U).$$

Using the hypothesis  $U^*U = I$ , let us show that  $A$  is self-adjoint. Left-multiplying the relation  $(I - U)A = i(I + U)$  by  $U^*$ , we have  $(U^* - I)A = i(U^* + I)$ , thus  $(I - U)^*A = -i(I + U)^*$ . Since  $(I - U)^*$  is invertible, with inverse  $[(I - U)^{-1}]^*$ , we have

$$A = -i[(I - U)^{-1}]^*(I + U)^* = -i[(I + U)(I - U)^{-1}]^* = A^*.$$

It follows that the operators  $A + iI$  and  $A - iI$  are invertible, and solving the relation  $(I - U)A = i(I + U)$  for  $U$ , we have

$$U = (A - iI)(A + iI)^{-1} = (A + iI)^{-1}(A - iI).$$

Incidentally, since  $U$  is the product of invertible operators, we conclude that  $U$  is unitary.

Since  $A$  is self-adjoint, we know from an elementary argument that the approximate point spectrum of  $A$  is non empty ([7], Theorem 34.2). Let  $\alpha \in a(A)$ , and let  $x_n$  be a sequence of unit vectors such that  $\|Ax_n - \alpha x_n\| \rightarrow 0$ . Define  $\mu = (\alpha + i)^{-1}(\alpha - i)$ ; since  $\alpha$  is real,  $\mu$  has absolute value 1. It will suffice to show that  $\mu$  is an approximate proper value for  $U$ ; indeed,  $\|(U - \mu I)x_n\| \rightarrow 0$  results from the calculation

$$\begin{aligned} U - \mu I &= (A + iI)^{-1}(A - iI) - (\alpha + i)^{-1}(\alpha - i)I \\ &= (\alpha + i)^{-1}(A + iI)^{-1}[(\alpha + i)(A - iI) - (\alpha - i)(A + iI)] \\ &= 2i(\alpha + i)^{-1}(A + iI)^{-1}(A - \alpha I), \end{aligned}$$

the fact that  $\|(A - \alpha I)x_n\| \rightarrow 0$ , and the continuity of the operator  $2i(\alpha + i)^{-1}(A + iI)^{-1}$ .

Incidentally, if  $U$  is an isometric operator such that the spectrum of  $U$  excludes some complex number  $\mu$  of absolute value 1, then  $\mu^{-1}U$

is an isometric operator whose spectrum excludes 1. The proof of Lemma 2 then shows that  $\mu^{-1}U$  is unitary, hence so is  $U$ . In other words: the spectrum of a nonnormal isometry must include the unit circle  $|\mu| = 1$ ; indeed, Putnam has shown that the spectrum is the unit disc  $|\mu| \leq 1$  ([8], Corollary 1). The latter result is also an immediate consequence of ([5], Lemma 2.1), and the fact that the spectrum of any unilateral shift operator is the unit disc.

**THEOREM.** (Andô) *Every hyponormal operator  $T$  has an approximate proper value  $\mu$  such that  $|\mu| = \|T\|$ .*

*Proof.* We may assume  $\|T\| = 1$  without loss of generality. Since  $TT^* \geq 0$  and  $\|TT^*\| = 1$ , we know that 1 is an approximate proper value for  $TT^*$ . Since the property of hyponormality is preserved under \*-isomorphism, we may assume, after a change of Hilbert space, that 1 is a proper value for  $TT^*$  ([3], Theorem 1). Form the nonzero closed linear subspace  $\mathcal{M} = \{x : TT^*x = x\}$ ; according to Lemma 1,  $\mathcal{M}$  is invariant under  $T$ , and the restriction of  $T$  to  $\mathcal{M}$  is an isometric operator  $U$  in the Hilbert space  $\mathcal{M}$ . By Lemma 2,  $U$  has an approximate proper value  $\mu$  of absolute value 1. Let  $x_n$  be any sequence of unit vectors in  $\mathcal{M}$  such that  $\|Ux_n - \mu x_n\| \rightarrow 0$ . Since  $Ux_n = Tx_n$ , obviously  $\mu$  is an approximate proper value for  $T$ , and  $|\mu| = 1 = \|T\|$ .

**COROLLARY 1.** *A generalized nilpotent hyponormal operator is necessarily zero.*

*Proof.* If  $T$  is hyponormal, then  $s(T)$  contains a scalar  $\mu$  such that  $|\mu| = \|T\|$ . For every positive integer  $n$ , it follows that  $s(T^n)$  contains  $\mu^n$  (see [7], Theorem 33.1); then  $\|T\|^n = |\mu|^n = |\mu^n| \leq \|T^n\| \leq \|T\|^n$ , and so  $\|T^n\| = \|T\|^n$ . If moreover  $T$  is a generalized nilpotent, that is if  $\lim \|T^n\|^{1/n} = 0$ , then  $\|T\| = 0$ .

**COROLLARY 2.** *If  $T$  is a completely continuous hyponormal operator, then  $T$  is normal.*

*Proof.* The proof to be given is essentially the same as Andô's. The proper subspaces of  $T$  are mutually orthogonal, and reduce  $T$  ([4], Exercise VII. 2.5). Let  $\mathcal{M}$  be the smallest closed linear subspace which contains every proper subspace of  $T$ , and let  $\mathcal{N} = \mathcal{M}^\perp$ ; clearly  $\mathcal{N}$  reduces  $T$ , and the restriction  $T|_{\mathcal{N}}$  is a completely continuous hyponormal operator in  $\mathcal{N}$  ([4], Exercise VI. 9.18). If the spectrum of  $T|_{\mathcal{N}}$  were different from  $\{0\}$ , it would have a nonzero boundary point  $\mu$ , hence  $\mu$  would be a proper value for  $T|_{\mathcal{N}}$  (see [4], Theorem VIII. 3.2); this is impossible since  $\mathcal{N}^\perp = \mathcal{M}$  already contains every proper vector for  $T$ .

We conclude from the Theorem that  $T|_{\mathcal{N}} = 0$ , and this forces  $\mathcal{N} = \{0\}$  (recall that  $\mathcal{N}^\perp$  contains the null space of  $T$ ). Thus, the proper subspaces of  $T$  are a total family, hence  $T$  is normal by ([4], Exercise VII. 2.5).

Suppose  $T$  is a normal operator whose spectrum (a) has empty interior, and (b) does not separate the complex plane. Wermer has shown that the invariant subspaces of  $T$  reduce  $T$  ([10], Theorem 7). It is well known that the conditions (a) and (b) are fulfilled by the spectrum of any completely continuous operator. In particular: if  $T$  is a completely continuous normal operator, then every invariant subspace of  $T$  reduces  $T$ . A more elementary proof of this may be based on Corollary 2:

**COROLLARY 3.** *If  $T$  is a completely continuous normal operator, and  $\mathcal{N}$  is a closed linear subspace invariant under  $T$ , then  $\mathcal{N}$  reduces  $T$ .*

*Proof.* Indeed, it suffices to assume that  $T$  is hyponormal and  $\mathcal{N}$  is an invariant subspace such that  $T|_{\mathcal{N}}$  is completely continuous. Since  $T|_{\mathcal{N}}$  is hyponormal ([4], Exercise VI. 9.10), it follows from Corollary 2 that  $T|_{\mathcal{N}}$  is normal, hence  $\mathcal{N}$  reduces  $T$  by ([4], Exercise VI. 9.9).

Quoting ([4], Theorem VII. 3.1), we have:

**COROLLARY 4.** *If  $T$  is a hyponormal operator, then*

$$\|T\| = LUB\{\|(Tx|x)| : \|x\| \leq 1\}.$$

Incidentally, if  $T$  is hyponormal, it is clear from Corollary 4 that  $\|T^*\| = LUB\{\|(T^*x|x)| : \|x\| \leq 1\}$ .

**COROLLARY 5.** *If the completely continuous operator  $T$  is semi-normal in the sense of [8], then  $T$  is normal.*

*Proof.* The definition of semi-normality is that either  $TT^* \leq T^*T$  or  $TT^* \geq T^*T$ , in other words, either  $T$  or  $T^*$  is hyponormal; since both are completely continuous (see [4], Exercise VIII. 1.6), our assertion follows from Corollary 2.

Let us say that an operator  $T$  is *nearly normal* in case  $T$  commutes with  $T^*T$ . The structure of nearly normal operators has been determined by Brown, and it is a consequence of his results that a completely continuous nearly normal operator is in fact normal (see the concluding remarks in [5]). This may also be proved as follows. An elementary calculation with square roots shows that a nearly normal operator is hyponormal (see [2], proof of Corollary 1 of Theorem 8); assuming also complete continuity and citing Corollary 2, we have:

**COROLLARY 6.** *If  $T$  is a completely continuous nearly normal operator, then  $T$  is normal.*

Finally,

**COROLLARY 7.** *If  $S = T + \lambda I$ , where  $T$  is a completely continuous operator, and if  $S$  is hyponormal, then  $S$  is normal.*

*Proof.* Since  $S$  is hyponormal, so is  $T$  ([4], hint to Exercise VII. 1.6), hence  $T$  is normal by Corollary 2; therefore  $S$  is normal. So to speak, the  $C^*$ -algebra of all operators of the form  $T + \lambda I$ , with  $T$  completely continuous, is of "finite class".

We close with an elementary remark about the adjoint of a hyponormal operator: if  $T$  is hyponormal, then  $s(T^*) = a(T^*)$ . For, suppose  $\lambda$  does not belong to  $a(T^*)$ , and let  $\mu = \lambda^*$ . Then,  $(T - \mu I)^* = T^* - \lambda I$  is bounded below ([4], Exercise VII. 3.8), and since  $T - \mu I$  is also hyponormal, the relation  $(T - \mu I)(T - \mu I)^* \leq (T - \mu I)^*(T - \mu I)$  shows that  $T - \mu I$  is also bounded below. Then  $T - \mu I$  is invertible ([4], Exercise VI. 8.11), hence so is  $T^* - \lambda I$ , thus  $\lambda$  does not belong to  $s(T^*)$ .

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The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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