ANALYTIC FUNCTIONS WITH VALUES IN A FRECHET SPACE

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We wish to extend certain results in the theory of analytic functions of several complex variables to the case of analytic functions with values in a Frechet space $F$. To do this, we prove (Theorem 1 below) that such a function $\varphi$ has an expansion of the form

$$\varphi = \sum_{n=1}^{\infty} P_n \circ \varphi,$$

where $\{P_n\}$ is a sequence of continuous mutually annihilating projections on $F$ whose ranges are all one-dimensional subspaces of $F$. This representation reduces the study of $\varphi$, for many purposes, to the study of the functions $P_n \circ \varphi$, which are essentially scalar-valued analytic functions.

We actually prove the stronger (and more useful) result that if $\{\varphi_k\}$ is a sequence of analytic functions with values in $F$ then a single sequence $\{P_n\}$ can be found to give an expansion (*) for every $\varphi_k$. Expansions of vector-valued functions of a different type have been considered by Grothendick [6].

Theorem 1 is applied to generalize Theorem B of H. Cartan [3]. We consider a coherent analytic sheaf $S$ on a Stein manifold $M$ and introduce the notion of the vectorization $S_{\mathcal{F}}$ of $S$ (relative to a given Frechet space $F$).

If $0$ denotes the sheaf of locally-defined analytic functions and $0_{\mathcal{F}}$ denotes the sheaf of locally-defined analytic functions with values in $F$, then $S_{\mathcal{F}}$ is defined to be the tensor product $S \otimes 0_{\mathcal{F}}$ of the $0$-modules $S$ and $0_{\mathcal{F}}$. For the important case of a coherent analytic subsheaf $S$ of the sheaf $0_{\mathcal{F}}$ of locally-defined $k$-tuples of analytic functions, $S_{\mathcal{F}}$ turns out to be canonically isomorphic to the sheaf $S'_{\mathcal{F}}$ determined by assigning to each open set $U$ the module of all $k$-tuples $(f_1, \ldots, f_k)$ of analytic functions from $U$ to $F$ which have the property that for each $u$ in $F^*$ the $k$-tuple $(u \circ f_1, \ldots, u \circ f_k)$ is a cross-section of $S$ over $U$. For instance, if $S$ is the sheaf of all locally-defined analytic functions which vanish on a given analytic set $A$ then it is evident that $S'_{\mathcal{F}}$ is the sheaf of all locally-defined analytic functions with values in $F$ which vanish on $A$.

One of the main results, an extension of Theorem B of [3], will be that the cohomology groups $H^N(M, S_{\mathcal{F}})$ vanish in all dimensions $N \geq 1$, where $S_{\mathcal{F}}$ is the vectorization of a coherent analytic sheaf $S$ on a Stein manifold $M$. Using this theorem and the isomorphism of $S_{\mathcal{F}}$ to the sheaf $S'_{\mathcal{F}}$ defined above one could show, for instance, that the usual

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sheaf—theoretic solutions to Cousin's problems carry over to the case of analytic functions with values in a Frechet space. Special cases were treated by totally different methods in [2], but the techniques of that paper seem to be inadequate to obtain general results.

The proofs are all Banach-space theoretic. That is, only Banach space theory is necessary to obtain the above extension of Theorem B and to prove the necessary facts about vectorizations. We begin with a theorem which is given without proof on p. 278 of Banach [1], who attributes it to H. Auerbach. A proof can be found in Taylor [7]. Since complex Banach spaces are considered here, we give the proof.

**THEOREM (Auerbach).** An n-dimensional Banach space $B$ has a basis of unit vectors whose dual basis also consists of unit vectors.

**Proof.** Choose a basis $(b_1, \ldots, b_n)$ of $B$ and for any $x$ in $B$ let $(x_1, \ldots, x_n)$ be the coordinates of $x$ relative to the chosen basis. Let $T$ be the set of all $n$-tuples $(x_1, \ldots, x_n)$ of unit vectors in $B$. For each $(x_1, \ldots, x_n)$ in $T$ let $\alpha(x_1, \ldots, x_n)$ be the absolute value of the determinant $\det(x_1)$. Thus $\alpha$ is a continuous function on the compact space $T$. Now $\alpha(x_1, \ldots, x_n) \neq 0$ if and only if $(x_1, \ldots, x_n)$ is a basis. Thus $\alpha$ attains its maximum for $T$ at some point $(y_1, \ldots, y_n)$ in $T$ which is a basis of unit vectors. Let $(u_1, \ldots, u_n)$ be the dual basis in $B^*$. Now $\|u_i\| \geq 1$ because $\langle y_i, u_i \rangle = 1$. Assume $\|u_i\| > 1$ for some $i$. Thus there exists $t$ in $B$ with $\|t\| = 1$ and $\langle t, u_i \rangle = c > 1$. Thus $\langle t - cy_i, u_i \rangle = 0$, so that $t - cy_i$ is a linear combination of the vectors of the basis $(y_1, \ldots, y_n)$ other than $y_i$. If we let $(z_1, \ldots, z_n)$ be the basis $(y_1, \ldots, y_n)$ with $y_i$ replaced by $t$ it follows that $\alpha(z_1, \ldots, z_n) = c\alpha(y_1, \ldots, y_n)$. Since the basis $(z_1, \ldots, z_n)$ consists of unit vectors this contradicts the choice of $(y_1, \ldots, y_n)$. Thus $\|u_i\| = 1$ for all $i$, and the theorem is proved.

**COROLLARY.** If $B_0$ is a finite-dimensional subspace of dimension $n$ of a Banach space $B$ there exist $n$ mutually annihilating projections (idempotent continuous linear operators) on $B$, each of norm 1, whose ranges are one-dimensional subspaces of $B$, and whose sum is a projection of $B$ onto $B_0$ of norm at most $n$.

**Proof.** Let $(y_1, \ldots, y_n)$ be a basis of unit vectors of $B_0$ such that the dual basis $(u_1, \ldots, u_n)$ of $B_0^*$ also consists of unit vectors. Let $v_i$ be an extension of $u_i$ to a linear functional on $B$ of norm 1. The operators $P_1, \ldots, P_n$ on $B$ defined by

\[ P_i x = \langle x, v_i \rangle y_i \]

are the desired projections.

We recall that a Frechet space is a locally convex topological linear
space $F$ which admits a countable family $\{|| \cdot ||_k\}$ of continuous semi-norms such that a basis for the neighborhoods of 0 in $F$ is given by the sets

$$\{x \in F : || x ||_k < 1\}.$$ 

If $|| \cdot ||$ is any continuous semi-norm on $F$ it follows that for some $k$ $|| x || \leq || x ||_k$ for all $x$ in $F$. If necessary it may be assumed that $\{|| \cdot ||_k\}$ is a monotonely nondecreasing sequence of semi-norms, in which case we shall call it a defining sequence of semi-norms for $F$.

**Lemma 1.** Let $F$ be a Frechet space with a defining sequence $\{|| \cdot ||_k\}$ of semi-norms. Let $\{a_n\}$ be a sequence of vectors in $F$, $\{\delta_k\}$ a sequence of nonnegative real numbers, and $\{k_j\}$ a strictly increasing sequence of positive integers. Then there exists a sequence $\{P_n\}$ of mutually annihilating continuous projections on $F$, whose ranges are subspaces of $F$ of dimensions at most 1, and a sequence $\{\varepsilon_k\}$, with $0 < \varepsilon_k < \delta_k$ for all $k$, with the following properties. For each positive integer $j$ the operator

$$Q_j = \sum_{n=1}^{k_j} P_n$$

is a projection on the subspace $B_j$ of $F$ spanned by the vectors $a_1, \ldots, a_{k_j}$. For each positive integer $n$ the sum

$$||a||_0 = \sum_{k=1}^{\infty} \varepsilon_k ||a||_k$$

is finite for $a = a_n$. For each positive integer $j$ and all $n \leq k_j$ we have $||P_n||_0 \leq (1 + k_i^2) \cdots (1 + k_j^2)$, where

$$||P_n||_0 = \text{sup} \{||P_n b||_0 : b \in F, \ ||b||_0 = 1\}.$$ 

**Proof.** We may assume the $\delta_k$ to be so small that $\sum_{n=1}^{\infty} \delta_k ||a_n||_k < \infty$ for all $n$. By induction we construct a sequence $\{P_n\}$ of mutually annihilating continuous projections, a sequence $\{\varepsilon_k\}$ of positive real numbers, and an increasing sequence $\{N_j\}$ of positive integers such that

(a) $0 < \varepsilon_k < \delta_k$,

(b) For each $j$ the operator $Q_j$ is a projection onto $B_j$,

(c) $||P_n||_0 < (1 + k_i^2) \cdots (1 + k_j^2)$ for $1 \leq n \leq k_i$ and all $i \leq j$.

We explain what is meant by (c). First of all, $|| \cdot ||'$ is the continuous semi-norm on $F$ defined by

$$||b||' = \sum_{k=1}^{N_j} \varepsilon_k ||b||_k.$$ 

Secondly, $||P_n||'$ is defined by

$$||P_n||' = \text{sup} \{||P_n b||_0 : ||b||'_0 = 1\}.$$
Assuming that $P_1, \ldots, P_k$, and $N_1, \ldots, N_j$, and $\varepsilon_1, \ldots, \varepsilon_{N_j}$ have been found with the relevant properties, we show how to continue to the next stage $j + 1$. First choose $N_{j+1} > N_j$ so large that $\| P_n \|_{N_{j+1}}$ is a norm (not merely a semi-norm) on $B_{j+1}$. Choose then $\varepsilon_i, N_j < i \leq N_{j+1}$, so small that $0 < \varepsilon_i < \delta_i$ and $\| P_n \|_{j+1} < (1 + k_i) \cdot \cdots \cdot (1 + k_{j+1})$ for $n \leq k_j$ and all $i \leq j$. To see that this can be done, notice that because $\| P \|$ is a norm on $B_j$ there exists $r > 0$ so that $r \| a \| > \| a \|_m$ for all $a$ in $B_j$ and all $m \leq N_{j+1}$. Thus

$$\| P_n \|_{j+1} \leq \sup \{ \| P_n b \|^{j+1} : \| b \| = 1 \} \leq (1 + \sum_{m=N_{j+1}}^{N_{j+1}} \varepsilon_m) \| P_n \|^{j+1}.$$ 

Now use (c).

Now let $Q_j$ be the restriction of $Q_j$ to $B_{j+1}$ and let $I_{j+1}$ be the identity operator on $B_{j+1}$. Thus $I_{j+1} - Q_j$ is a projection of $B_{j+1}$ onto a subspace $S_{j+1}$. Clearly $B_j$ and $S_{j+1}$ are complementary subspaces of $B_{j+1}$, so that $\dim S_{j+1} = k_{j+1} - k_j$. By the above corollary there exists a projection $E_{j+1}$ with $\| E_{j+1} \|^{j+1} \leq k_{j+1}$ of $F$ onto $B_{j+1}$. Also by the above corollary there exist mutually annihilating projections $R_n, k_j < n \leq k_{j+1}$, of $S_{j+1}$ onto subspaces of dimensions at most 1 such that $\| R_n \|^{j+1} \leq 1$ for all $n$ and such that $\Sigma R_n$ is the identity projection of $S_{j+1}$ onto itself. For $k_j < n \leq k_{j+1}$ we define

$$P_n = R_n(I_{j+1} - Q_j)E_{j+1}.$$ 

Thus the $P_n$ are mutually annihilating projections for $1 \leq n \leq k_{j+1}$.

Also $Q_{j+1}$ is a projection onto $B_{j+1}$. Finally for $k_j < n \leq k_{j+1}$ we have

$$\| P_n \|^{j+1} \leq \| R_n \|^{j+1} \| I_{j+1} - Q_j \|^{j+1} \| E_{j+1} \|^{j+1}$$

$$\leq (1 + \sum_{m=1}^{k_j} \| P_n \|^{j+1})k_{j+1}$$

$$< [1 + k_j(1 + k_j^2) \cdots (1 + k_j^2)]k_{j+1}$$

$$\leq (1 + k_i^2) \cdots (1 + k_{j+1}).$$

The same is true for $n \leq k_j$, by the above construction. Thus the construction has been continued another step. By induction it follows that sequences $\{P_n\}$, $\{N_j\}$, and $\{\varepsilon_j\}$ can be chosen satisfying properties (a), (b), and (c). It is immediate that the sequences $\{P_n\}$ and $\{\varepsilon_j\}$ satisfy the requirements of the lemma.

**Lemma 2.** Let $\{a_n\}$ be a sequence of elements of a Frechet space $F$, $\{\| \|_k\}$ a defining sequence of semi-norms on $F$, and $\{\delta_k\}$ a sequence of positive real numbers. Then there exist a sequence $\{\varepsilon_k\}$ of positive real numbers and a sequence $\{P_n\}$ of mutually annihilating projections on $F$ whose ranges are subspaces of $F$ of dimensions at most 1 having the following properties.
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(i) $0 < \varepsilon_k < \delta_k$ for all $k$,
(ii) For $a = a_n$ the norm $\|a\|_0 = \sum_{k=1}^{\infty} \varepsilon_k \|a\|_k$ is finite for all $n$,
(iii) $R_m a_n = a_n$ for all positive integers $m$ and $n$ with $m \geq 2n$,

where $R_m = \sum_{j=1}^{m} P_j$,
(iv) For all $t > 1$ and $\varepsilon > 0$ the sum $\sum_{n=1}^{\infty} \|P_n\|_0 t^{-n^s}$ converges, where $\|P_n\|_0$ is defined as above.

Proof. Define the sequence $\{k_j\}$ by $k_j = 2^j$. Choose the sequences $\{P_n\}$ and $\{\varepsilon_k\}$ as in lemma 1. Clearly (i) and (ii) are satisfied. Now for each positive integer $n$ there is a positive integer $j$ with $2^{j+1} \leq n < 2^j$. It follows that $a_n \in B_j$. Thus $R_j a_n = Q_j a_n = a_n$, so that $R_m a_n = a_n$ for all $m \geq 2^j$ and therefore for all $m \geq 2n$. This proves (iii).

Now for each $n$ choose $j$ with $2^{j-1} \leq n < 2^j$. Thus

$$\|P_n\|_0 \leq (1 + k_j)^j = (1 + 2^j)^j \leq (5n^j)^j \leq (5n^j)^s,$$

where $\alpha = 1 + \log_2 n$. From this it follows from elementary calculus that (iv) holds, thereby proving the lemma.

LEMMA 3. Let

$$\sum_{n_1 \geq 0, \ldots, n_{\alpha} \geq 0} a_i(n_1, \ldots, n_{\alpha}) z_1^{n_1} \cdots z_{\alpha}^{n_\alpha},$$

where $\alpha = \alpha_i$ and $1 \leq i < \infty$, be a sequence of formal power series with coefficients in a Frechet space $F$. Let $\{\varepsilon_k\}$ be a sequence of positive real numbers. Then there exists a sequence $\{\varepsilon_k\}$ with $0 < \varepsilon_k < \delta_k$ for all $k$ and a sequence $\{P_n\}$ of mutually annihilating continuous projections of $F$ onto subspaces of dimensions at most 1 such that

(a) $R_m a_i(n_1, \ldots, n_{\alpha}) = a_i(n_1, \ldots, n_{\alpha})$ whenever $m \geq 2^{i+5} n^s$, where $\alpha = \alpha_i$, $n = n_1 + \cdots + n_{\alpha}$, and $R_m = \sum_{j=1}^{m} P_j$,
(b) $P_m a_i(n_1, \ldots, n_{\alpha}) = 0$ whenever $m > 2^{i+5} n^s$,
(c) $\sum_{j=1}^{\infty} \|P_n\|_0 t^{-n^s} < \infty$ for all $t > 1$ and $\varepsilon > 0$, where $\| \|_0$ is defined as above.

Proof. For each $i$ order the coefficients $a_i(n_1, \ldots, n_{\alpha})$ into a sequence $\{a_i^j\}_{j=1}^{\infty}$ according to the size of $n$. We now define a sequence $\{a_k\}$ of elements of $F$ which is an ordering of the totality of the $a_i(n_1, \ldots, n_{\alpha})$. For $k$ given let $2^j$ be the largest power of 2 dividing $k$ and let $j = 1/2(k2^{-i} + 1)$. Let $a_k = a_i^j$. Now choose the sequences $\{\varepsilon_k\}$ and $\{P_n\}$ as in Lemma 2. Clearly (c) holds. Since (b) is a consequence of (a) we need only check (a). To this end consider a fixed $a_i(n_1, \ldots, n_{\alpha})$. Now there exists $j \leq n^s$ with $a_i(n_1, \ldots, n_{\alpha}) = a_i^j$. In turn $\alpha^j = a_k$ for some $k \leq 2^{i+5} n^s$. By (iii) of Lemma 2 it follows that $R_m a_k = a_k$ for $m \geq 2k$ and therefore for $m \geq 2^{i+5} n^s$, as was to be proved.
We are now prepared to prove a series representation for analytic functions with values in a Frechet space which will be the principal tool in subsequent proofs.

**Theorem 1.** Let $F$ be a Frechet space and let $\{M_i\}$ be a sequence of complex analytic manifolds. For each $i$ let $\varphi_i$ be an analytic function on $M_i$ with values in $F$. Then there exists a sequence of vectors $\{b_n\}$ in $F$ and a sequence $\{P_n\}$ of continuous mutually annihilating projections of $F$ onto one-dimensional subspaces having the following properties. For each $i$ the series $\sum_{n=1}^{\infty} P_n \circ \varphi_i$ converges to $\varphi_i$ on $M_i$. For each $n$ we have $P_n b_n = b_n$, so that $P_n \circ \varphi_i = \varphi_i^* b_n$, for some analytic function $\varphi_i^*$ on $M_i$. For each $i$ the series $\sum_{n=1}^{\infty} \varphi_i^* b_n$ converges absolutely and uniformly on all compact subsets of $M_i$. For each continuous semi-norm $\| \|$ on $F$ the sequence $\{\| b_n \|\}$ is bounded.

**Proof.** For each $i$ let $\dim M_i = \alpha = \alpha_i$, so that $M_i$ is coverable by a countable family of analytic homeomorphs $\Gamma$ of the unit polycylinder

$$U^\alpha = \{z = (z_1, \ldots, z_\alpha) : |z_j| < 1, 1 \leq j \leq \alpha\}.$$  

Thus in the proof of the theorem we may replace the sequence $\{M_i\}$ by the totality of all such $\Gamma$. There is therefore no loss of generality in assuming that each $M_i$ is a polycylinder $U^\alpha$ of dimension $\alpha = \alpha_i$. Let $\{\| \|_{\delta_k}\}$ be a defining sequence of semi-norms on $F$. Now for each $i$ the analytic function $\varphi_i$ has a power series expansion

$$\varphi_i = \sum_{n_1 \geq 0, \ldots, n_\alpha \geq 0} a_i(n_1, \ldots, n_\alpha) z_1^{n_1} \cdots z_\alpha^{n_\alpha}$$

on the polycylinder $M_i = U^\alpha$. This expansion converges absolutely and uniformly on each compact subset of $M_i$ in each semi-norm $\| \|_{\delta_k}$. By the diagonal process there therefore exist constants $\delta_k > 0$ such that the power series for each $\varphi_i$ converges absolutely and uniformly on each compact subset of $M_i$ in the norm $\sum_{k=1}^{\infty} \delta_k \| \|_{\delta_k}$, so that in particular this norm is finite for each coefficient $a_i(n_1, \ldots, n_\alpha)$. Now choose the sequences $\{\varepsilon_k\}$ and $\{P_n\}$ as in Lemma 3 relative to the power series expansions of the $\varphi_i$ and to the $\delta_k$ just obtained. Thus the power series for $\varphi_i$ converges absolutely and uniformly on compact subsets of $M_i$ in the norm $\| \|_0$ defined above. If some of the projections $P_n$ are zero, these may be omitted from the sequence. Thus for each $n$ there is a vector $b_n$ in $F$ with $\|b_n\|_0 = 1$ spanning the range of $P_n$. To show that the sequences $\{P_n\}$ and $\{b_n\}$ have the desired properties, consider a fixed compact subset $T$ of a fixed $M_i$. For each $n$ write

$$\gamma_n = \sum_{n_1 \geq 0, \ldots, n_\alpha \geq 0} \max \{\| a_i(n_1, \ldots, n_\alpha) z_1^{n_1} \cdots z_\alpha^{n_\alpha} \|_0 : z \in T\}.$$
By the usual convergence criteria we see that there exist $r > 1$ and $c > 0$ such that $r^n \gamma_n < c$ for all $n$.

If $j$ is any positive integer let $k$ be the largest integer such that $2^{i+j} k^a < j$. Thus for each $z$ in $T$ we have

$$
\| P_j \varphi_i(z) \|_0 = \left\| P_j \sum_{n_1, \ldots, n_a \geq k} a_i(n_1, \ldots, n_a) z_1^{n_1} \cdots z_a^{n_a} \right\|_0 \\
\leq \| P_j \|_0 \sum_{a \geq k} r^n \gamma_a \leq c \| P_j \|_0 \sum_{a \geq k} r^{-n} \\
= c(1 - r^{-1})^{-1} \| P_j \|_0 r^{-k} .
$$

Thus

$$
\Delta = \max \left\{ \sum_{j=1}^\infty \| P_j \varphi_i(z) \|_0 : z \in T \right\} \\
\leq c(1 - r^{-1})^{-1} \sum_{j=1}^\infty r^{-k} \| P_j \|_0 .
$$

Now by the definition of $k$ we see that $k$ is the integral part of $(j 2^{-i-j})^{1/a}$, so that $k \geq j^{2^{i-j}}$ for all $j$ sufficiently large. Thus $\Delta$ is finite if the sum $\sum_{j=1}^\infty r^{-j} \| P_j \|_0$ converges, where $\varepsilon = (2a)^{-1}$. By the choice of the sequence $\{P_j\}$ this series converges so that $\Delta$ is finite. Now since $\| b_n \|_0 = 1$, 

$$
\max \{ \| \varphi_i(z) \| : z \in T \} = \max \{ \| P_n \varphi_i(z) \|_0 : z \in T \} .
$$

Therefore the series $\sum_{n=1}^\infty \varphi_i^n(z)$ converges absolutely and uniformly on $T$. If $\| \| \|$ is a continuous semi-norm on $F$ then $\| \| \| \| \leq K \| \|$ for some $K > 0$, so that $\{ \| b_n \| \}$ is bounded by $K$. Finally, we must show that $\sum_{n=1}^\infty P_n \varphi_i$ actually converges to $\varphi_i$ (and not to something else). To see this, note by (a) and (b) of Lemma 3 that $R_n \circ \varphi_i$ and $\varphi_i$ have power series expansions in the coordinates $z_1, \ldots, z_a$ which agree up to terms of total order $n$, whenever $m \geq 2^{i+j} k^a$. This completes the proof of Theorem 1.

Before giving the definition of the vectorization of an analytic sheaf, we indicate the terminology to be used, following Godement [5]. A presheaf $S$ on a topological space $X$ assigns to each open $U \subset X$ a set $S(U)$ and to each open set $V \subset U \subset X$ a map $r_{VU} : S(U) \to S(V)$ satisfying $r_{WV} \circ r_{UV} = r_{WU}$ for $W \subset V \subset U$. In particular the same terminology will be used if $S$ is a sheaf, that is, a presheaf satisfying axioms (F1) and (F2) on page 109 of [5]. To any presheaf $S$ is canonically associated a sheaf $S'$, and each element $f$ in $S(U)$ gives rise to a unique element in $S'(U)$ which will also be denoted by $f$. If $X$ is a complex analytic manifold a sheaf $S$ on $X$ is called analytic if it is a module over the sheaf $0$ of locally defined analytic functions, that is, if for each $U$ the set $S(U)$ is an $0(U)$-module, and if the usual com-
mutation relations between module multiplication and the restriction maps $S(U) \to S(V)$ and $0(U) \to 0(V)$ hold.

**Definition 1.** Let $S$ be an analytic sheaf on a complex analytic manifold $M$ and let $F$ be a Frechet space. Let $0$ be the sheaf of locally-defined analytic functions on $M$ and let $0_r$ be the sheaf of locally-defined analytic functions on $M$ with values in $F$, where by definition a continuous function $f$ from an open set $U \subset M$ to $F$ is called analytic if $u \circ f$ is analytic for all $u$ in $F^*$. Clearly $0_r$ is an $0$-module, i.e., an analytic sheaf. The vectorization $S \otimes 0_r$ of $S$ is defined to be the sheaf $S \otimes 0_r$, the tensor product of the $0$-modules $S$ and $0_r$. This is defined in [5] as the sheaf determined by the presheaf data

$$U \to S(U) \otimes 0_r(U),$$

where $S(U)$ and $0_r(U)$ are considered as $0(U)$-modules, together with the obvious restriction maps.

Note that if $T$ is a continuous linear operator from a Frechet space $F$ into a Frechet space $G$ then the natural homomorphism $T_0$ of $0_r$ into $0_g$ induces a homomorphism $T' = 1 \otimes T_0$ of $S_r$ into $S_g$. In particular, if $u$ is an element of $F^*$ (and so a continuous linear operator from $F$ into $C$) then $u$ induces a homomorphism of $S_r$ into $S_g$. But $S_g$ is canonically isomorphic to $S$, in virtue of the canonical isomorphism between the $0(U)$-modules $S(U) \otimes 0(U)$ and $S(U)$. (See [5] p. 8.) If we identify $S_g$ with $S$ it follows that each $u$ in $F^*$ induces a homomorphism $u'$ of $S_r$ onto $S$.

**Definition 2.** If $S$ is an analytic subsheaf of the Cartesian product $0^*$ we define

$$S_r'(U) = \{ f \in (0_r(U))^* : u \circ f \in S(U) \text{ for all } u \text{ in } F^* \}.$$ 

Clearly $S_r'$ so defined is an analytic subsheaf of the Cartesian product $(0_r)^*$.

**Theorem 2.** If $S$ is a coherent analytic subsheaf of $0^*$ then to each $p$ in $U \subset M$ and each $f$ in $S_r'(U)$ there exists a neighborhood $V$ of $p$, functions $H_1, \ldots, H_k$ in $S(V)$ and functions $G_1, \ldots, G_k$ in $0_r(V)$ such that

$$r_{V \circ f} = \sum_{m=1}^k G_m H_m.$$ 

**Proof.** Since $S$ is coherent, there exists a neighborhood $V_0 \subset U$ of $p$ and functions $H_1, \ldots, H_k$ in $S(V_0)$ which generate $S$ at each point of $V_0$. We may assume that $V_0$ is a compact subset of $U$. Let $V_0 \supset V_1 \supset V_2 \supset \cdots$
be a basis for the neighborhoods of \( p \). Let \( \Omega \) be the subset of \( S(V_o) \) consisting of all elements in \( S(V_o) \) which as elements of \((0(V_o))^k\) are bounded on \( V_o \). Thus to each \( h \) in \( \Omega \) there exists \( G = (G_1, \cdots, G_k) \) in \((0(V_o))^k\) for some \( i \) such that the restriction of \( h \) to \( V_i \) has the form

\[
h = \sum_{i=1}^k G_i H_i.
\]

By choosing \( i \) large enough we may assume that

\[
\|G\|_i = \sup \{|G_j(q)| : q \in V_i, 1 \leq j \leq k\}
\]

is finite. Thus if for each pair \((i, N)\) of positive integers we let \( \Omega_{i,N} \) be the family of all \( h \) in \( \Omega \) such that \( G \) can be chosen in \((0(V_i))^k\) with \( \|G\|_i \leq N \), we see that \( \Omega = \bigcup \Omega_{i,N} \) and that each \( \Omega_{i,N} \) is a closed subset of \( \Omega \), where \( \Omega \) has the norm defined by

\[
\|h\|_0 = \sup \{|h_i(q)| : 1 \leq i \leq n, q \in V_o\}
\]

for each \( h = (h_1, \cdots, h_n) \in \Omega \subset (0(V_o))^n \). By the Baire category theorem there exists \((i, N)\) such that \( \Omega_{i,N} \) has a nonvoid interior. From this it follows as usual that there exists a constant \( K > 0 \) such that for each \( h \) in \( \Omega \) there exists \( G \) in \((0(V_i))^k\) as above with \( \|G\|_i \leq K \|h\|_0 \). Now consider \( f \) as in the statement of the theorem, so that \( f \in S(U) \subset (0_r(U))^* \). By Theorem 1 there exists a sequence of vectors \( \{b_j\} \) in \( F \) which is bounded in each continuous semi-norm on \( F \) and a sequence \( \{P_j\} \) of continuous projections on \( F \) having one-dimensional ranges such that \( \sum_{j=1}^\infty P_j \circ f \) converges uniformly to \( f \) on all compact subsets of \( U \) and such that for each \( j \) we have \( P_j \circ f = f_j b_j \) with \( f_j \in (0(U))^n \), where \( \sum_{j=1}^\infty \|f_j\| \) converge uniformly on all compact subsets of \( U \). Thus \( \sum_{j=1}^\infty \|f_j\|_0 \) is finite, since \( V_o \subset U \).

Now for each \( j \) there exists \( u \) in \( F^* \) with \( \langle b_j, u \rangle = 1 \). Thus

\[
f_j = u \circ (f \circ b_j) = u \circ (P_j \circ f) = (u \circ P_j) \circ f.
\]

is in \( S(U) \) because \( f \in S_r(U) \) and \( u \circ P_j \in F^* \). Thus \( f_j \in S(U) \) for all \( j \). By the above for each \( j \) there exists \( G^j = (G^j_1, \cdots, G^j_k) \) in \((0(V_i))^k\) such that on \( V_i \) we have

\[
f_j = \sum_{m=1}^k G^j_m H_m,
\]

with \( \|G^j\|_i \leq K \|f_j\|_0 \). It follows that the series \( \sum_{j=1}^\infty G^j b_j \) converges uniformly and absolutely on \( V_i \) in each continuous semi-norm on \( F \). Thus the sum of this series is an element \( G = (G_1, \cdots, G_k) \) in \((0_r(V_i))^k \). Thus in the topology of uniform and absolute convergence on compact subsets of \( V_i \) in each continuous semi-norm on \( F \) we have
as was to be proved.

The following consequence of Theorem 2 will be useful later.

**Lemma 4.** If the element \( f \) of \( S_*(U) \) has the property that \( u'f \) is the zero element of \( S(U) \) for all \( u \) in \( F^* \) then \( f = 0 \).

**Proof.** By taking a covering of \( U \) by small open sets we reduce to the case in which \( f \) has a representation

\[
f = \sum_{i=1}^{k} h_i \otimes g_i ,
\]

with \( h_i \) in \( S(U) \) and \( g_i \) in \( 0_r(U) \). Let \( R \) be the sheaf on \( U \) of relations of \( h_1, \ldots, h_k \). Thus for each \( u \) in \( F^* \) we see that

\[
0 = u'f = \sum_{i=1}^{k} h_i \otimes \langle g_i, u \rangle = \sum_{i=1}^{k} \langle g_i, u \rangle h_i .
\]

Thus by Definition 2 we see that \( g = (g_1, \ldots, g_k) \in R'_r(U) \). By Theorem 2 it follows that each \( p \) in \( U \) has a neighborhood \( V \subset U \) such that there exist \( H_1, \ldots, H_i \) in \( R(V) \) and \( G_1, \ldots, G_i \) in \( 0_r(V) \) with

\[
r_{vv}g = \sum_{j=1}^{i} G_j H_j .
\]

Thus for each \( i \) with \( 1 \leq i \leq k \) we have

\[
r_{vv}g_i = \sum_{j=1}^{i} G_j H_j ,
\]

where \( H_j = (H_j^1, \ldots, H_j^i) \). Therefore on \( V \) we have

\[
f = \sum_{i=1}^{k} h_i \otimes g_i = \sum_{i=1}^{k} h_i \otimes \left( \sum_{j=1}^{i} G_j H_j \right) = \sum_{i=1}^{k} \left( \sum_{j=1}^{i} h_i \otimes (G_j H_j) \right) = \sum_{j=1}^{i} \left( \sum_{i=j}^{k} H_j h_i \right) \otimes G_j = 0
\]
since $H_j \in R(V)$ for all $j$. This proves Lemma 4.

We next give an important characterization of $S_F$ in case $S$ is a coherent analytic subsheaf of $0^n$ for some positive integer $n$.

**Theorem 3.** Let $M$ be a Stein manifold and $S$ a coherent analytic subsheaf of $0^n$ for some positive integer $n$. Let $F$ be a Frechet space. For each open $U \subset M$ there is a mapping $\tau(U)$ from $S(U) \otimes 0_F(U)$ into $(0_F(U))^n$ which for each $h = (h_1, \ldots, h_n)$ in $S(U)$ and $g$ in $0_F(U)$ maps $h \otimes g$ onto $gh = (gh_1, \ldots, gh_n)$ in $(0_F(U))^n$. For each such $g$ and $h$ the image $gh$ of $h \otimes g$ actually lies in the subset $S'_F(U)$ of $(0_F(U))^n$. The family of such mappings $\tau(U)$ induces an isomorphism $\tau$ of the sheaf $S'_F$ (which was defined above to be the sheaf determined by the presheaf data $U \rightarrow S(U) \otimes 0_F(U)$) onto the sheaf $S_F$. Thus $S'_F$ and $S_F$ are isomorphic.

**Proof.** Clearly the map of the Cartesian product $S(U) \times 0_F(U)$ into $(0_F(U))^n$ defined by $(h, g) \mapsto gh$ induces a group homomorphism of $(S(U), 0_F(U))$—the free abelian group generated by the elements of the Cartesian product $S(U) \times 0_F(U)$—into $(0_F(U))^n$. It is trivial to check that $N(S(U), 0_F(U))$ belongs to the kernel of this map, where $N(S(U), 0_F(U))$ is defined as in [5] p. 8 to be the subgroup of $(S(U), 0_F(U))$ generated by elements of the forms

(i) $(x_1 + x_2, y) - (x_1, y) - (x_2, y)$
(ii) $(x, y_1 + y_2) - (x, y_1) - (x, y_2)$
(iii) $(ax, y) - (x, ay)$

where $x, x_1,$ and $x_2$ are in $S(U)$, $y, y_1,$ and $y_2$ are in $0_F(U)$, and $a \in 0(U)$. Thus this map induces a homomorphism $\tau(U)$ of the quotient $(S(U), 0_F(U))/N(S(U), 0_F(U)) = S(U) \otimes 0_F(U)$ into $(0_F(U))^n$. It is trivial to check that $\tau(U)$ is an $0(U)$-homomorphism. Now with $g$ and $h$ as above and $u$ in $F^*$ we have

$$u \circ \tau(U)(h \otimes g) = u \circ (gh) = (u \circ g)h \in S(U).$$

Thus $\tau(U)(h \otimes g) \in S'_F(U)$. It follows that the range of $\tau(U)$ is a subset of $S'_F(U)$. It is now clear that the family of mappings $\tau(U)$ induces an $0$-homomorphism $\tau$ of $S_F$ into $S'_F$. To show that $\tau$ is one-to-one we must prove

(a) If $\tau(U)(\sum_{i=1}^n h_i \otimes g_i) = 0$ then each $p\in U$ has a neighborhood $V$ such that $r_{U,V}(\sum_{i=1}^n h_i \otimes g_i) = 0$.

To show that $\tau$ is onto we must prove

(b) If $f \in S'_F(U)$ then each $p\in U$ has a neighborhood $V$ such that $r_{U,V}f = \tau(V)(\sum_{i=1}^n h_i \otimes g_i)$ for some elements $h_i$ in $S(V)$ and $g_i$ in $0_F(V)$. We first prove (a). If we let $R$ be the sheaf of relations on $U$ of $h_1, \ldots, h_n$ by the coherence of $R$ there exists a neighborhood $V$ of $p$ and elements $r_i = (r_i^1, \ldots, r_i^n), \ldots, r_n = (r_n^1, \ldots, r_n^n)$ of $R(V)$ which
generate $R$ at each point of $V_0$. Now

$$\sum_{i=1}^{N} g_i h_i = \tau(U) \left( \sum_{i=1}^{N} h_i \otimes g_i \right) = 0.$$  

Thus for each $u$ in $F^*$ we have

$$\sum_{i=1}^{N} (u \circ g_i) h_i = 0$$

so that $(u \circ g_1, \cdots, u \circ g_N) \in R(U)$ for all $u$ in $F^*$. By definition this means that $(g_1, \cdots, g_N) \in R_*(U)$. Therefore by Theorem 2 we see that there exists a neighborhood $V$ of $p$ and $G = (G_1, \cdots, G_n)$ in $(0_r(V))^*$ such that $(g_1, \cdots, g_N) = G_1 r_1 + \cdots + G_n r_n$. Thus on $V$ we have

$$\sum_{i=1}^{N} h_i \otimes g_i = \sum_{i=1}^{N} h_i \otimes \left( \sum_{j=1}^{n} G_j r_j^i \right)$$

$$= \sum_{j=1}^{n} \left( \sum_{i=1}^{N} (r_j^i h_i) \right) \otimes G_j = 0$$

since $r_j \in R(V)$ for each $j$. This proves (a).

To prove (b) notice by Theorem 2 that there exists a neighborhood $V$ of $p$, elements $h_1, \cdots, h_N$ in $S(V)$, and elements $g_1, \cdots, g_N$ in $0_r(V)$ such that on $V$ we have

$$f = \sum_{i=1}^{N} g_i h_i = \tau(V) \left( \sum_{i=1}^{N} h_i \otimes g_i \right).$$

This completes the proof of Theorem 3.

We state for future reference a version of a theorem of Banach, first giving a definition.

**Definition 3.** If $\{g_n\}$ is a sequence of vectors in a Frechet space $F$, the series $\sum_{n=1}^{\infty} g_n$ is called **absolutely convergent** if the series $\sum_{n=1}^{\infty} ||g_n||$ converges for each continuous semi-norm $||||$ on $F$.

Notice that a continuous linear transformation from a Frechet space $F$ to a Frechet space $G$ takes absolutely convergent sequences into absolutely convergent sequences.

**Lemma 5.** Let $\sigma$ be a continuous linear map of a Frechet space $F$ onto a Frechet space $G$. Let $\{g_i\}$ be an absolutely convergent sequence from $G$. Then there exists an absolutely convergent sequence $\{f_i\}$ in $F$ such that $\sigma(f_i) = g_i$ for all $i$.

**Proof.** Let $\{||\|_k\}$ be a defining sequence of semi-norms on $F$. Since the map $\sigma$ is continuous, we see ([1] p. 40) that for each $k$ the set $\sigma(\{f : ||f||_k \leq 1\})$ contains a neighborhood $\{g : ||g||_k \leq 1\}$ of $0$ in $G$, where $||\|_k$ is some continuous semi-norm on $G$. Thus for each $g$ in
G and each \( k \) there exists \( f \) in \( F \) with \( \sigma(f) = g \) and \( \|f\|_k \leq \|g\|_k \). Now for each \( k \) choose \( j = j(k) \) such that
\[
\sum_{n=j}^{\infty} \|g_n\|_k^k < 2^{-k},
\]
so that
\[
\sum_{k=1}^{\infty} \sum_{n=j(k)}^{\infty} \|g_n\|_k^k < \infty.
\]
We may assume that \( j(1) < j(2) < \cdots \). For each \( n \) with \( j(k) \leq n < j(k+1) \) choose \( f_n \) in \( F \) with \( \sigma(f_n) = g_n \) and \( \|f_n\|_k \leq \|g_n\|_k \). If for each \( n \) we let \( k(n) \) be the smallest value of \( k \) for which \( n < j(k+1) \), it follows that
\[
\sum_{n=1}^{\infty} \|f_n\|_{k(n)} < \infty.
\]
Since for each \( t \) we have \( \|f_n\|_t \leq \|f_n\|_k \) for all \( k \geq t \) it follows that
\[
\sum_{n=1}^{\infty} \|f_n\|_t
\]
is finite for all \( t \). This proves the lemma.

**Theorem 4.** If \( S \) is a coherent analytic sheaf on a Stein manifold \( M \) and if \( F \) is a Frechet space then \( H^N(M, S_F) = 0 \) for all \( N \geq 1 \).

**Proof.** Let \( f \) be an element of \( H^N(M, S_F) \). Consider a locally finite covering \( \{U_i\} \) of \( M \) by holomorphically convex open sets \( U_i \), so fine that \( f \) is represented by an element of \( H^N(U_i, S_F) \). For each finite sequence \( K = (i_1, \ldots, i_k) \) of positive integers let \( U_K = U_{i_1} \cap \cdots \cap U_{i_k} \). The element \( f \) of \( H^N(M, S_F) \) can be considered to belong to \( H^N(U_i, S_F) \) and therefore can be represented by a cocycle \( f = \{f_I\} \) of \( Z^N(U_i, S_F) \). Here \( I \) is any sequence of \( N + 1 \) positive integers, and, for each \( I \), \( f_I \) is an element of \( S_F(U_i) \). Also \( \delta f = 0 \), where \( \delta \) is the coboundary operator from \( C^N(U_i, S_F) \) into \( C^{N+1}(U_i, S_F) \) and \( Z^N(U_i, S_F) \) is the kernel of \( \delta \). By choosing the covering \( \{U_i\} \) fine enough we may assume that for each \( K \) there exist elements \( h_{1K}, \ldots, h_{aK} \), with \( a \) depending on \( K \), in \( S(U_K) \) which generate \( S \) at each point of \( U_K \). This implies ([3], expose XVIII, p. 9) that every \( h \) in \( S(U_K) \) has a representation of the form \( h = \sum_{i=1}^{a} g_i h_{iK} \), with \( g_i \in 0(U_K) \). We may also choose the covering \( \{U_i\} \) so fine that, for each \( I \), \( f_I \) can be represented in the form
\[
f_I = \sum_{i=1}^{a} h_{iI} \otimes g_{iI}
\]
with \( h_{iI} \) as above and with \( g_{iI} \) in \( 0_F(U_I) \).
By Theorem 1 there exists a sequence \( \{P_n\} \) of continuous mutually annihilating projections on \( F \) whose ranges are one dimensional and a sequence \( \{b_n\} \) of vectors in \( F \) bounded in each continuous semi-norm on \( F \) having the following properties. For each \( I \) and \( i \) the series \( \sum_{n=1}^{\infty} P_n \circ g_{it} \) converges to \( g_{it} \) on \( U_i \). For each \( I \) and \( i \) we have \( P_n \circ g_{it} = g_{it}^n b_n \), where \( g_{it}^n \in 0(U_i) \). For each \( I \) and \( i \) the series \( \sum_{n=1}^{\infty} g_{it}^n \) converges absolutely in the Frechet space \( 0(U_i) \). Now since for each \( n \) the projection \( P_n \) induces a homomorphism of the sheaf \( S_F \) onto itself, the element \( \{P_n f_I\} \) of \( C^\infty(U_i, S_F) \) is in \( Z^\infty(U_i, S_F) \). Also

\[
P_n f_I = \sum_{i=1}^{a} h_{ii} \otimes P_n g_{ii} = \sum_{i=1}^{a} h_{ii} \otimes g_{ii}^n b_n = \left( \sum_{i=1}^{a} g_{ii}^n h_{ii} \right) \otimes b_n.
\]

If for each \( n \) and \( I \) we let \( f^n_I \) be the element \( \sum_{i=1}^{a} g_{ii}^n h_{ii} \) of \( S(U_i) \) it follows that for each \( n \) the element \( f^n_I = \{f^n_I\}_{i=1}^{a} \) of \( C^\infty(U_i, S) \) belongs to \( Z^\infty(U_i, S) \). It is also clear that \( f^n_I b_n = P_n f_I \).

Now there exists a natural Frechet space topology on each \( S(U) \), described in [4], expose XVII. This topology has the property that for each \( h \) in \( S(U) \) the map \( g \rightarrow gh \) of \( 0(U) \) into \( S(U) \) is continuous. We therefore see that for each \( I \) the series

\[
\sum_{n=1}^{a} f^n_I = \sum_{n=1}^{a} \left( \sum_{i=1}^{a} g_{ii}^n h_{ii} \right)
\]

converges absolutely in \( S(U_i) \) because for each \( I \) and \( i \) the series \( \sum_{n=1}^{a} g_{ii}^n \) converges absolutely in \( 0(U_i) \). Now the space \( C^\infty(U_i, S) \) is the Cartesian product of the Frechet spaces \( S(U_i) \), and therefore possesses a Frechet space structure. Moreover \( Z^\infty(U_i, S) \) is closed in \( C^\infty(U_i, S) \) and is therefore also a Frechet space. Since for each \( I \) the series \( \sum_{n=1}^{a} f^n_I \) converges absolutely in \( S(U_i) \) it follows that \( \sum_{n=1}^{a} f^n_I \) converges absolutely in \( Z^\infty(U_i, S) \). By Theorem B of [3] and Leray's theorem (see [5] p. 213) we see that the coboundary map \( \delta \) of the Frechet space \( C^\infty(U_i, S) \) into \( Z^\infty(U_i, S) \) is onto. From [4] we also see that \( \delta \) is continuous.

Let \( J \) stand for an arbitrary \( N \)-tuple of positive integers. Thus for each \( J \), by the above, there is a continuous homomorphism

\[
\tau_J : (G_1, \cdots, G_a) \rightarrow \sum_{i=1}^{a} G_j h_{ij}
\]

of the Frechet space \( (0(U_i))^{a} \) onto the Frechet space \( S(U_i) \). These maps induce a continuous homomorphism \( \tau \) of the Frechet space \( \Phi \) onto the Frechet space \( C^\infty(U_i, S) \), where \( \Phi \) is defined to be the product \( \prod_I (0(U_i))^{a} \), with \( a \) depending as above on \( J \), of the Frechet spaces \( (0(U_i))^{a} \). Thus
\[ \sigma = \delta \circ \tau \]

is a continuous homomorphism of \( \Phi \) onto \( \mathbb{Z}^N(U_i, S) \). Since \( \sum_{n=1}^{\infty} f_n \) converges absolutely in \( \mathbb{Z}^n(U_i, S) \) it follows from Lemma 5 that there exists an absolutely convergent sequence \( \{G^n\} \) in \( \Phi \) with \( \sigma(G^n) = f^n \) for all \( n \). For each \( n \) write \( G^n = \{G^n_j\} \), where

\[ G^n_j = (G^n_{1j}, \ldots, G^n_{aj}) \in \mathcal{O}(U_j)^a . \]

Thus for each \( J \) we see that the series \( \sum_{n=1}^{\infty} G^n_j \) converges absolutely and uniformly on every compact subset of \( U_j \), so that the series \( \sum_{n=1}^{\infty} G^n_j b_n \) converges absolutely in \( \mathcal{O}(U_j)^a \) to an element

\[ G^J = (G^J_{1j}, \ldots, G^J_{aj}) \]

in \( \mathcal{O}(U_j)^a \). Thus for each \( i \) and \( J \) we have \( G^J = \sum_{n=1}^{\infty} G^J_{ij} b_n \).

For each \( J \) let \( e_J \) be the element

\[ e_J = \sum_{i=1}^{a} h_{ij} \otimes G_{ij} \]

of \( \mathcal{O}(U_j) \). Thus \( e = \{e_J\} \in C^{\infty-1}(U_i, S) \). We shall finish the proof by showing that \( \delta(e) = f \). To this end it is sufficient by Lemma 4 to show \( u'(\delta e) = u'(f) \) for all \( u \) in \( F^* \). We compute:

\[ u'(e_J) = \sum_{i=1}^{a} \langle G^J_{ij}, u \rangle h_{ij} \]

\[ = \sum_{i=1}^{a} \left( \sum_{n=1}^{\infty} G^J_{ij} b_n, u \right) h_{ij} \]

\[ = \sum_{n=1}^{\infty} \left( \sum_{i=1}^{a} G^J_{ij} h_{ij} \right) \langle b_n, u \rangle \]

\[ = \sum_{n=1}^{\infty} (\tau_J(G^n)) \langle b_n, u \rangle \]

absolutely in \( \mathcal{O}(U_j) \). Thus

\[ u'(e) = \sum_{n=1}^{\infty} (\tau(G^n)) \langle b_n, u \rangle \]

absolutely in \( C^{\infty-1}(U_i, S) \). Thus

\[ u'(\delta e) = \delta(u'(e)) = \sum_{n=1}^{\infty} (\delta \circ \tau)(G^n) \langle b_n, u \rangle \]

\[ = \sum_{n=1}^{\infty} \sigma(G^n) \langle b_n, u \rangle = \sum_{n=1}^{\infty} f^n \langle b_n, u \rangle . \]

Also for each \( I \) we have

\[ u'(f_I) = \sum_{i=1}^{a} \langle g_{ii}, u \rangle h_{ii} \]
\[ = \sum_{i=1}^{n} \left\langle \sum_{k=1}^{m} g_{ik} b_{n}, u \right\rangle h_{i1} \]
\[ = \sum_{n=1}^{\infty} \left( \sum_{i=1}^{n} g_{ik} h_{i1} \right) \langle b_{n}, u \rangle = \sum_{n=1}^{\infty} f^{*}_{n} \langle b_{n}, u \rangle . \]

Therefore \( u'(f) = \sum_{n=1}^{\infty} f^{*}_{n} \langle b_{n}, u \rangle \). It follows that \( u'(f) = u'(\delta e) \) for all \( u \) in \( F^{*} \), so that \( f = \delta e \). This completes the proof of Theorem 4.

REFERENCES

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