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# ANALYTIC FUNCTIONS WITH VALUES IN A FRECHET SPACE

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We wish to extend certain results in the theory of analytic functions of several complex variables to the case of analytic functions with values in a Frechet space F. To do this, we prove (Theorem 1 below) that such a function  $\varphi$  has an expansion of the form

$$(*) \qquad \qquad \varphi = \sum_{n=1}^{\infty} P_n \circ \varphi ,$$

where  $\{P_n\}$  is a sequence of continuous mutually annihilating projections on F whose ranges are all one-dimensional subspaces of F. This representation reduces the study of  $\varphi$ , for many purposes, to the study of the functions  $P_n \circ \varphi$ , which are essentially scalar-valued analytic functions. We actually prove the stronger (and more useful) result that if  $\{\varphi_k\}$  is a sequence of analytic functions with values in F then a single sequence  $\{P_n\}$  can be found to give an expansion (\*) for every  $\varphi_k$ . Expansions of vector-valued functions of a different type have been considered by Grothendick [6].

Theorem 1 is applied to generalize Theorem B of H. Cartan [3]. We consider a coherent analytic sheaf S on a Stein manifold M and introduce the notion of the *vectorization*  $S_F$  of S (relative to a given Frechet space F).

If 0 denotes the sheaf of locally-defined analytic functions and  $0_F$ denotes the sheaf of locally-defined analytic functions with values in F, then  $S_F$  is defined to be the tensor product  $S \otimes 0_F$  of the 0-modules S and  $0_F$ . For the important case of a coherent analytic subsheaf Sof the sheaf  $0^k$  of locally-defined k-tuples of analytic functions,  $S_F$  turns out to be canonically isomorphic to the sheaf  $S'_F$  determined by assigning to each open set U the module of all k-tuples  $(f_1, \dots, f_k)$  of analytic functions from U to F which have the property that for each u in  $F^*$ the k-tuple  $(u \circ f_1, \dots, u \circ f_k)$  is a cross-section of S over U. For instance, if S is the sheaf of all locally-defined analytic functions which vanish on a given analytic set A then it is evident that  $S'_F$  is the sheaf of all locally-defined analytic functions with values in F which vanish on A.

One of the main results, an extension of Theorem B of [3], will be that the cohomology groups  $H^{N}(M, S_{F})$  vanish in all dimensions  $N \geq 1$ , where  $S_{F}$  is the vectorization of a coherent analytic sheaf S on a Stein manifold M. Using this theorem and the isomorphism of  $S_{F}$  to the sheaf  $S'_{F}$  defined above one could show, for instance, that the usual

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sheaf—theoretic solutions to Cousin's problems carry over to the case of analytic functions with values in a Frechet space. Special cases were treated by totally different methods in [2], but the techniques of that paper seem to be inadequate to obtain general results.

The proofs are all Banach-space theoretic. That is, only Banach space theory is necessary to obtain the above extension of Theorem B and to prove the necessary facts about vectorizations. We begin with a theorem which is given without proof on p. 278 of Banach [1], who attributes it to H. Auerbach. A proof can be found in Taylor [7]. Since complex Banach spaces are considered here, we give the proof.

THEOREM (Auerbach). An n-dimensional Banach space B has a basis of unit vectors whose dual basis also consists of unit vectors.

*Proof.* Choose a basis  $(b^1, \dots, b^n)$  of B and for any x in B let  $(x_1, \dots, x_n)$  be the coordinates of x relative to the chosen basis. Let T be the set of all *n*-tuples  $(x^1, \dots, x^n)$  of unit vectors in B. For each  $(x^1, \dots, x^n)$  in T let  $\alpha(x^1, \dots, x^n)$  be the absolute value of the determinant det  $(x_i^i)$ . Thus  $\alpha$  is a continuous function on the compact space T. Now  $\alpha(x^1, \dots, x^n) \neq 0$  if and only if  $(x^1, \dots, x^n)$  is a basis. Thus  $\alpha$  attains its maximum for T at some point  $(y^1, \dots, y^n)$  in T which is a basis of unit vectors. Let  $(u^1, \dots, u^n)$  be the dual basis in  $B^*$ . Now  $||u^i|| \ge 1$  because  $\langle y^i, u^i \rangle = 1$ . Assume  $||u^i|| > 1$  for some *i*. Thus there exists t in B with ||t|| = 1 and  $\langle t, u^i \rangle = c > 1$ . Thus  $\langle t - cy^i, u^i \rangle = c$ 0, so that  $t - cy^i$  is a linear combination of the vectors of the basis  $(y^1, \dots, y^n)$  other than  $y^i$ . If we let  $(z^1, \dots, z^n)$  be the basis  $(y^1, \dots, y^n)$ with  $y^i$  replaced by t it follows that  $\alpha(z^1, \dots, z^n) = c\alpha(y^1, \dots, y^n)$ . Since the basis  $(z^1, \dots, z^n)$  consists of unit vectors this contradicts the choice of  $(y^1, \dots, y^n)$ . Thus  $||u^i|| = 1$  for all *i*, and the theorem is proved.

COROLLARY. If  $B_0$  is a finite-dimensional subspace of dimension n of a Banach space B there exist n mutually annihilating projections (idempotent continuous linear operators) on B, each of norm 1, whose ranges are one-dimensional subspaces of  $B_0$  and whose sum is a projection of B onto  $B_0$  of norm at most n.

*Proof.* Let  $(y^1, \dots, y^n)$  be a basis of unit vectors of  $B_0$  such that the dual basis  $(u^1, \dots, u^n)$  of  $B_0^*$  also consists of unit vectors. Let  $v^i$  be an extension of  $u^i$  to a linear functional on B of norm 1. The operators  $P_1, \dots, P_n$  on B defined by

$$P_i x = \langle x, v^i \rangle y^i$$

are the desired projections.

We recall that a Frechet space is a locally convex topological linear

space F which admits a countable family  $\{|| \ ||_k\}$  of continuous seminorms such that a basis for the neighborhoods of 0 in F is given by the sets

$$\{x \in F : ||x||_k < 1\}$$
.

If  $|| \quad ||$  is any continuous semi-norm on F it follows that for some k  $||x|| \leq ||x||_k$  for all x in F. If necessary it may be assumed that  $\{|| \quad ||_k\}$  is a monotonely nondecreasing sequence of semi-norms, in which case we shall call it a *defining sequence* of semi-norms for F.

LEMMA 1. Let F be a Frechet space with a defining sequence  $\{|| \ ||_k\}$  of semi-norms. Let  $\{a_n\}$  be a sequence of vectors in F,  $\{\delta_k\}$  a sequence of nonnegative real numbers, and  $\{k_j\}$  a strictly increasing sequence of positive integers. Then there exists a sequence  $\{P_n\}$  of mutually annihilating continuous projections on F, whose ranges are subspaces of F of dimensions at most 1, and a sequence  $\{\varepsilon_k\}$ , with  $0 < \varepsilon_k < \delta_k$  for all k, with the following properties. For each positive integer j the operator

$$Q_j = \sum\limits_{n=1}^{k_j} P_n$$

is a projection on the subspace  $B_j$  of F spanned by the vectors  $a_1, \dots, a_{k_j}$ . For each positive integer n the sum

$$||]a||_0 = \sum_{k=1}^{\infty} \varepsilon_k ||a||_k$$

is finite for  $a = a_n$ . For each positive integer j and all  $n \leq k_j$  we have  $||P_n||_0 \leq (1 + k_1^2) \cdots (1 + k_j^2)$ , where

$$|| P_n ||_0 = \sup \{ || P_n b ||_0 : b \in F, || b ||_0 = 1 \}$$
 .

*Proof.* We may assume the  $\delta_k$  to be so small that  $\sum_{k=1}^{\infty} \delta_k || a_n ||_k < \infty$  for all n. By induction we construct a sequence  $\{P_n\}$  of mutually annihilating continuous projections, a sequence  $\{\varepsilon_k\}$  of positive real numbers, and an increasing sequence  $\{N_j\}$  of positive integers such that

- (a)  $0 < \varepsilon_k < \delta_k$ ,
- (b) For each j the operator  $Q_j$  is a projection onto  $B_j$ ,
- (c)  $||P_n||^j < (1+k_1^2) \cdots (1+k_i^2)$  for  $1 \le n \le k_i$  and all  $i \le j$ .

We explain what is meant by (c). First of all,  $|| \quad ||^{j}$  is the continuous semi-norm on F defined by

$$||\,b\,||^{\scriptscriptstyle J} = \sum\limits_{k=1}^{N_j} arepsilon_k\,||\,b\,||_k$$
 .

Secondly,  $|| P_n ||^j$  is defined by

$$||P_n||^j = \sup \{||P_n b||^j : ||b||^j = 1\}$$
 .

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Assuming that  $P_1, \dots, P_{k_j}$  and  $N_1 \dots, N_j$ , and  $\varepsilon_1, \dots, \varepsilon_{N_j}$  have been found with the relevant properties, we show how to continue to the next stage j + 1. First choose  $N_{j+1} > N_j$  so large that  $|| \quad ||_{N_{j+1}}$  is a norm (and not merely a semi-norm) on  $B_{j+1}$ . Choose then  $\varepsilon_i, N_j < i \leq N_{j+1}$ , so small that  $0 < \varepsilon_i < \delta_i$  and  $|| P_n ||^{j+1} < (1 + k_1^2) \cdots (1 + k_i^2)$  for  $n \leq k_j$  and all  $i \leq j$ . To see that this can be done, notice that because  $|| \quad ||_{N_j}$  is a norm on  $B_j$  there exists r > 0 so that  $r \mid| a \mid|^j > || a \mid|_m$ for all a in  $B_j$  and all  $m \leq N_{j+1}$ . Thus

$$|| P_n ||^{j+1} \leq \sup \{ || P_n b ||^{j+1} : || b ||^j = 1 \} \leq (1 + \sum_{m=N_j+1}^{N_j+1} \varepsilon_m) || P_n ||^j.$$

Now use (c).

Now let  $Q'_j$  be the restriction of  $Q_j$  to  $B_{j+1}$  and let  $I_{j+1}$  be the identity operator on  $B_{j+1}$ . Thus  $I_{j+1} - Q'_j$  is a projection of  $B_{j+1}$  onto a subspace  $S_{j+1}$ . Clearly  $B_j$  and  $S_{j+1}$  are complementary subspaces of  $B_{j+1}$ , so that dim  $S_{j+1} \leq k_{j+1} - k_j$ . By the above corollary there exists a projection  $E_{j+1}$  with  $||E_{j+1}||^{j+1} \leq k_{j+1}$  of F onto  $B_{j+1}$ . Also by the above corollary there exist mutually annihilating projections  $R_n$ ,  $k_j < n \leq k_{j+1}$ , of  $S_{j+1}$  onto subspaces of dimensions at most 1 such that  $||R_n||^{j+1} \leq 1$  for all n and such that  $\Sigma R_n$  is the identity projection of  $S_{j+1}$  onto itself. For  $k_j < n \leq k_{j+1}$  we define

$$P_n = R_n (I_{j+1} - Q'_j) E_{j+1}$$
.

Thus the  $P_n$  are mutually annihilating projections for  $1 \leq n \leq k_{j+1}$ . Also  $Q_{j+1}$  is a projection onto  $B_{j+1}$ . Finally for  $k_j < n \leq k_{j+1}$  we have

$$egin{aligned} &|| \, P_n \, ||^{j+1} & \leq || \, R_n \, ||^{j+1} \, || \, I_{j+1} - Q_j' \, ||^{j+1} \, || \, E_{j+1} \, ||^{j+1} \ & \leq (1 + \sum\limits_{n=1}^{k_j} || \, P_n \, ||^{j+1}) k_{j+1} \ & < [1 + k_j (1 + k_1^2) \, \cdots \, (1 + k_j^2)] k_{j+1} \ & \leq (1 + k_1^2) \, \cdots \, (1 + k_{j+1}^2) \, . \end{aligned}$$

The same is true for  $n \leq k_j$ , by the above construction. Thus the construction has been continued another step. By induction it follows that sequences  $\{P_n\}$ ,  $\{N_j\}$ , and  $\{\varepsilon_k\}$  can be chosen satisfying properties (a), (b), and (c). It is immediate that the sequences  $\{P_n\}$  and  $\{\varepsilon_k\}$  satisfy the requirements of the lemma.

LEMMA 2. Let  $\{a_n\}$  be a sequence of elements of a Frechet space F,  $\{|| \quad ||_k\}$  a defining sequence of semi-norms on F, and  $\{\delta_k\}$  a sequence of positive real numbers. Then there exist a sequence  $\{\varepsilon_k\}$  of positive real numbers and a sequence  $\{P_n\}$  of mutually annihilating projections on F whose ranges are subspaces of F of dimensions at most 1 having the following properties.

(i)  $0 < \varepsilon_k < \delta_k$  for all k,

(ii) For  $a = a_n$  the norm  $||a||_0 = \sum_{k=1}^{\infty} \varepsilon_k ||a||_k$  is finite for all n,

(iii)  $R_m a_n = a_n$  for all positive integers m and n with  $m \ge 2n$ , where  $R_m = \sum_{j=1}^m P_j$ ,

(vi) For all t > 1 and  $\varepsilon > 0$  the sum  $\sum_{n=1}^{\infty} ||P_n||_0 t^{-n^{\varepsilon}}$  converges, where  $||P_n||_0$  is defined as above.

*Proof.* Define the sequence  $\{k_j\}$  by  $k_j = 2^j$ . Choose the sequences  $\{P_n\}$  and  $\{\varepsilon_k\}$  as in lemma 1. Clearly (i) and (ii) are satisfied. Now for each positive integer n there is a positive integer j with  $2^{j-1} \leq n < 2^j$ . It follows that  $a_n \in B_j$ . Thus  $R_{2^j}a_n = Q_ja_n = a_n$ , so that  $R_ma_n = a_n$  for all  $m \geq 2^j$  and therefore for all  $m \geq 2n$ . This proves (iii).

Now for each n choose j with  $2^{j-1} \leq n < 2^j$ . Thus

$$egin{array}{l} \| \, P_n \, \|_{_0} & \leq (1 \, + \, k_j^2)^{\jmath} = (1 \, + \, 2^{2j})^{\jmath} \ & \leq (5n^2)^{\jmath} \leq (5n^2)^{lpha} \, , \end{array}$$

where  $\alpha = 1 + \log_2 n$ . From this it follows from elementary calculus that (iv) holds, thereby proving the lemma.

LEMMA 3. Let

$$\sum_{n_1\geq 0,\cdots,n_{\alpha}\geq 0} a_i(n_1,\cdots,n_{\alpha}) z_1^{n_1}\cdots z_{\alpha}^{n_{\alpha}}$$

where  $\alpha = \alpha_i$  and  $1 \leq i < \infty$ , be a sequence of formal power series with coefficients in a Frechet space F. Let  $\{\delta_k\}$  be a sequence of positive real numbers. Then there exists a sequence  $\{\varepsilon_k\}$  with  $0 < \varepsilon_k < \delta_k$  for all k and a sequence  $\{P_n\}$  of mutually annihilating continuous projections of F onto subspaces of dimensions at most 1 such that

(a)  $R_m a_i(n_1, \dots, n_{\alpha}) = a_i(n_1, \dots, n_{\alpha})$  whenever  $m \ge 2^{i+2}n^{\alpha}$ , where  $\alpha = \alpha_i$ ,  $n = n_1 + \dots + n_{\alpha}$ , and  $R_m = \sum_{j=1}^m P_j$ ,

(b)  $P_m a_i(n_1, \dots, n_{\alpha}) = 0$  whenever  $m > 2^{i+2}n^{\alpha}$ ,

(c)  $\sum_{n=1}^{\infty} ||P_n||_0 t^{-n^{\varepsilon}} < \infty$  for all t > 1 and  $\varepsilon > 0$ , where  $|| ||_0$  is defined as above.

*Proof.* For each *i* order the coefficients  $a_i(n_1, \dots, n_{\alpha})$  into a sequence  $\{\alpha_k^k\}_{k=1}^{\infty}$  according to the size of *n*. We now define a sequence  $\{a_k\}$  of elements of *F* which is an ordering of the totality of the  $a_i(n_1, \dots, n_{\alpha})$ . For *k* given let  $2^i$  be the largest power of 2 dividing *k* and let  $j = 1/2(k2^{-i} + 1)$ . Let  $a_k = \alpha_i^j$ . Now choose the sequences  $\{\varepsilon_k\}$  and  $\{P_n\}$  as in Lemma 2. Clearly (c) holds. Since (b) is a consequence of (a) we need only check (a). To this end consider a fixed  $a_i(n_1, \dots, n_{\alpha})$ . Now there exists  $j \leq n^{\alpha}$  with  $a_i(n_1, \dots, n_{\alpha}) = \alpha_i^j$ . In turn  $\alpha_i^j = a_k$  for some  $k \leq 2^{i+1}n^{\alpha}$ . By (iii) of Lemma 2 it follows that  $R_m a_k = a_k$  for  $m \geq 2k$  and therefore for  $m \geq 2^{i+2}n^{\alpha}$ , as was to be proved.

We are now prepared to prove a series representation for analytic functions with values in a Frechet space which will be the principal tool in subsequent proofs.

THEOREM 1. Let F be a Frechet space and let  $\{M_i\}$  be a sequence of complex analytic manifolds. For each i let  $\varphi_i$  be an analytic function on  $M_i$  with values in F. Then there exists a sequence of vectors  $\{b_n\}$ in F and a sequence  $\{P_n\}$  of continuous mutually annihilating projections of F onto one-dimensional subspaces having the following properties. For each i the series  $\sum_{n=1}^{\infty} P_n \circ \varphi_i$  converges to  $\varphi_i$  on  $M_i$ . For each n we have  $P_n b_n = b_n$ , so that  $P_n \circ \varphi_i = \varphi_i^n b_n$ , for some analytic function  $\varphi_i^n$  on  $M_i$ . For each i the series  $\sum_{n=1}^{\infty} \varphi_i^n$  converges absolutely and uniformly on all compact subsets of  $M_i$ . For each continuous semi-norm || = 0 on F the sequence  $\{||b_n||\}$  is bounded.

*Proof.* For each *i* let dim  $M_i = \alpha = \alpha_i$ , so that  $M_i$  is coverable by a countable family of analytic homeomorphs  $\Gamma$  of the unit polycylinder

$$U^{\alpha} = \{z = (z_1, \cdots, z_{\alpha}) : |z_j| < 1, 1 \leq j \leq \alpha\}$$
.

Thus in the proof of the theorem we may replace the sequence  $\{M_i\}$  by the totality of all such  $\Gamma$ . There is therefore no loss of generality in assuming that each  $M_i$  is a polycylinder  $U^{\alpha}$  of dimension  $\alpha = \alpha_i$ . Let  $\{|| \quad ||_k\}$  be a defining sequence of semi-norms on F. Now for each i the analytic function  $\varphi_i$  has a power series expansion

$$\varphi_i = \sum_{n_1 \ge 0, \dots, n_{\alpha} \ge 0} a_i(n_1, \dots, n_{\alpha}) z_1^{n_1} \cdots z_{\alpha}^{n_{\alpha}}$$

on the polycylinder  $M_i = U^{\alpha}$ . This expansion converges absolutely and uniformly on each compact subset of  $M_i$  in each semi-norm  $|| \quad ||_k$ . By the diagonal process there therefore exist constants  $\delta_k > 0$  such that the power series for each  $\varphi_i$  converges absolutely and uniformly on each compact subset of  $M_i$  in the norm  $\sum_{k=1}^{\infty} \delta_k || \quad ||_k$ , so that in particular this norm is finite for each coefficient  $a_i(n_1, \dots, n_{\alpha})$ . Now choose the sequences  $\{\varepsilon_k\}$  and  $\{P_n\}$  as in Lemma 3 relative to the power series expansions of the  $\varphi_i$  and to the  $\delta_k$  just obtained. Thus the power series for  $\varphi_i$  converges absolutely and uniformly on compact subsets of  $M_i$  in the norm  $|| \quad ||_0$  defined above. If some of the projections  $P_n$ are zero, these may be omitted from the sequence. Thus for each nthere is a vector  $b_n$  in F with  $|| b_n ||_0 = 1$  spanning the range of  $P_n$ . To show that the sequences  $\{P_n\}$  and  $\{b_n\}$  have the desired properties, consider a fixed compact subset T of a fixed  $M_i$ . For each n write

$$\gamma_n = \sum_{n_1+\cdots+n_{\alpha}=n} \max \left\{ || a_i(n_1, \cdots, n_{\alpha}) z_1^{n_1} \cdots z_{\alpha}^{n_{\alpha}} ||_0 : z \in T \right\} .$$

By the usual convergence criteria we see that there exist r > 1 and c > 0 such that  $r^n \gamma_n < c$  for all n.

If j is any positive integer let k be the largest integer such that  $2^{i+2}k^{x} < j$ . Thus for each z in T we have

$$egin{aligned} &\|P_j arphi_i(z)\,\|_0\ &= \left\| \left|P_j \sum\limits_{n_1 + \cdots + n_lpha \ge k} a_i(n_1, \, \cdots, \, n_lpha) z_1^{n_1} \cdots z_lpha^{n_lpha} 
ight\|_0\ &\leq \|P_j\,\|_0 \sum\limits_{n \ge k} \gamma_n \le c \, \|P_j\,\|_0 \sum\limits_{n \ge k} r^{-n}\ &= c(1 - r^{-1})^{-1} \,\|P_j\,\|_0 \, r^{-k} \;. \end{aligned}$$

Thus

$$egin{split} &\mathcal{A} = \max \left\{ \sum\limits_{j=1}^\infty ||\, P_j arphi_i(z)\,||_{\scriptscriptstyle 0} \, \colon z \in T 
ight\} \ &\leq c (1 - r^{-1})^{-1} \sum\limits_{j=1}^\infty r^{-k}\,||\, P_j\,||_{\scriptscriptstyle 0} \;. \end{split}$$

Now by the definition of k we see that k is the integral part of  $(j2^{-i-2})^{1/\sigma}$ , so that  $k \ge j^{1/2\sigma}$  for all j sufficiently large. Thus  $\varDelta$  is finite if the sum  $\sum_{j=1}^{\infty} r^{-j^{\varepsilon}} ||P_j||_0$  converges, where  $\varepsilon = (2\alpha)^{-1}$ . By the choice of the sequence  $\{P_j\}$  this series converges so that  $\varDelta$  is finite. Now since  $||b_n||_0 = 1$ ,

$$\max \{ |\varphi_i^n(z)| : z \in T \} = \max \{ || P_n \varphi_i(z)||_0 : z \in T \}.$$

Therefore the series  $\sum_{n=1}^{\infty} \varphi_i^n(z)$  converges absolutely and uniformly on T. If  $|| \quad ||$  is a continuous semi-norm on F then  $|| \quad || \leq K || \quad ||_0$  for some K > 0, so that  $\{|| b_n ||\}$  is bounded by K. Finally, we must show that  $\sum_{n=1}^{\infty} P_n \circ \varphi_i$  actually converges to  $\varphi_i$  (and not to something else). To see this, note by (a) and (b) of Lemma 3 that  $R_m \circ \varphi_i$  and  $\varphi_i$  have power series expansions in the coordinates  $z_1, \dots, z_{\alpha}$  which agree up to terms of total order n, whenever  $m \geq 2^{i+2}n^{\alpha}$ . This completes the proof of Theorem 1.

Before giving the definition of the vectorization of an analytic sheaf, we indicate the terminology to be used, following Godement [5]. A presheaf S on a topological space X assigns to each open  $U \subset X$  a set S(U) and to each open set  $V \subset U \subset X$  a map  $r_{vv}: S(U) \to S(V)$ satisfying  $r_{wv} \circ r_{vv} = r_{wv}$  for  $W \subset V \subset U$ . In particular the same terminology will be used if S is a sheaf, that is, a presheaf satisfying axioms (F1) and (F2) on page 109 of [5]. To any presheaf S is canonically associated a sheaf S', and each element f in S(U) gives rise to a unique element in S'(U) which will also be denoted by f. If X is a complex analytic manifold a sheaf S on X is called analytic if it is a module over the sheaf 0 of locally defined analytic functions, that is, if for each U the set S(U) is an 0(U)-module, and if the usual commutation relations between module multiplication and the restriction maps  $S(U) \rightarrow S(V)$  and  $O(U) \rightarrow O(V)$  hold.

DEFINITION 1. Let S be an analytic sheaf on a complex analytic manifold M and let F be a Frechet space. Let 0 be the sheaf of locally-defined analytic functions on M and let  $0_F$  be the sheaf of locallydefined analytic functions on M with values in F, where by definition a continuous function f from an open set  $U \subset M$  to F is called analytic if  $u \circ f$  is analytic for all u in  $F^*$ . Clearly  $0_F$  is an 0-module, i.e., an analytic sheaf. The vectorization  $S_F$  of S (relative to F) is defined to be the sheaf  $S \otimes 0_F$ , the tensor product of the 0-modules S and  $0_F$ . This is defined in [5] as the sheaf determined by the presheaf data

$$U \rightarrow S(U) \otimes 0_F(U)$$
 ,

where S(U) and  $0_F(U)$  are considered as 0(U)-modules, together with the obvious restriction maps.

Note that if T is a continuous linear operator from a Frechet space F into a Frechet space G then the natural homomorphism  $T_0$  of  $0_F$  into  $0_G$  induced by T gives rise to a homomorphism  $T' = 1 \otimes T_0$  of  $S_F$  into  $S_G$ . In particular, if u is an element of  $F^*$  (and so a continuous linear operator from F into C) then u induces a homomorphism of  $S_F$  into  $S_G$ . But  $S_G$  is canonically isomorphic to S, in virtue of the canonical isomorphism between the 0(U)-modules  $S(U) \otimes 0(U)$  and S(U). (See [5] p. 8.) If we identify  $S_G$  with S it follows that each u in  $F^*$  induces a homomorphism u' of  $S_F$  onto S.

DEFINITION 2. If S is an analytic subsheaf of the Cartesian product  $0^n$  we define

$$S'_F(U) = \{f \in (0_F(U))^n : u \circ f \in S(U) \text{ for all } u \text{ in } F^*\}.$$

Clearly  $S'_F$  so defined is an analytic subsheaf of the Cartesian product  $(0_F)^n$ .

THEOREM 2. If S is a coherent analytic subsheaf of  $0^n$  then to each p in  $U \subset M$  and each f in  $S'_F(U)$  there exists a neighborhood V of p, functions  $H_1, \dots, H_k$  in S(V) and functions  $G_1, \dots, G_k$  in  $0_F(V)$  such that

$$r_{\scriptscriptstyle V \overline{\scriptscriptstyle U}} f = \sum\limits_{m=1}^k G_m H_m$$
 .

*Proof.* Since S is coherent, there exists a neighborhood  $V_0 \subset U$  of p and functions  $H_1, \dots, H_k$  in  $S(V_0)$  which generate S at each point of  $V_0$ . We may assume that  $\overline{V}_0$  is a compact subset of U. Let  $V_0 \supset V_1 \supset V_2 \supset \cdots$ 

be a basis for the neighborhoods of p. Let  $\mathcal{Q}$  be the subset of  $S(V_0)$  consisting of all elements in  $S(V_0)$  which as elements of  $(0(V_0))^*$  are bounded on  $V_0$ . Thus to each h in  $\mathcal{Q}$  there exists  $G = (G_1, \dots, G_k)$  in  $(0(V_i))^k$  for some i such that the restriction of h to  $V_i$  has the form

$$h=\sum\limits_{i=1}^k G_i H_i$$
 .

By choosing i large enough we may assume that

$$||G||_i = \sup \{|G_j(q)| : q \in V_i, 1 \le j \le k\}$$

is finite. Thus if for each pair (i, N) of positive integers we let  $\Omega_{iN}$  be the family of all h in  $\Omega$  such that G can be chosen in  $(0(V_i))^k$  with  $||G||_i \leq N$ , we see that  $\Omega = \bigcup \Omega_{iN}$  and that each  $\Omega_{iN}$  is a closed subset of  $\Omega$ , where  $\Omega$  has the norm defined by

$$||\,h\,||_{\scriptscriptstyle 0} = \sup\,\{|\,h_i(q)\,|: 1 \leqq i \leqq n, q \in V_{\scriptscriptstyle 0}\}$$

for each  $h = (h_1, \dots, h_n) \in \Omega \subset (0(V_0))^n$ . By the Baire category theorem there exists (i, N) such that  $\Omega_{iN}$  has a nonvoid interior. From this it follows as usual that there exists a constant K > 0 such that for each h in  $\Omega$  there exists G in  $(0(V_i))^k$  as above with  $||G||_i \leq K ||h||_0$ . Now consider f as in the statement of the theorem, so that  $f \in S'_F(U) \subset (0_F(U))^n$ . By Theorem 1 there exists a sequence of vectors  $\{b_j\}$  in F which is bounded in each continuous semi-norm on F and a sequence  $\{P_j\}$  of continuous projections on F having one-dimensional ranges such that  $\sum_{j=1}^{\infty} P_j \circ f$  converges uniformly to f on all compact subsets of U and such that for each j we have  $P_j \circ f = f_j b_j$  with  $f_j \in (0(U))^n$ , where  $\sum_{j=1}^{\infty} |f_j|$  converges uniformly on all compact subsets of U. Thus  $\sum_{j=1}^{\infty} |f_j||_0$  is finite, since  $\overline{V}_0 \subset U$ .

Now for each j there exists u in  $F^*$  with  $\langle b_j, u \rangle = 1$ . Thus

$$f_j = u \circ (f_j b_j) = u \circ (P_j \circ f) = (u \circ P_j) \circ f$$

is in S(U) because  $f \in S'_{F}(U)$  and  $u \circ P_{j} \in F^{*}$ . Thus  $f_{j} \in S(U)$  for all j. By the above for each j there exists  $G^{j} = (G_{1}^{j}, \dots, G_{k}^{j})$  in  $(0(V_{i}))^{k}$  such that on  $V_{i}$  we have

$${f}_j = {\sum\limits_{m = 1}^k {{G_m^j}{H_m}}}$$
 ,

with  $||G^j||_i \leq K ||f_j||_0$ . It follows that the series  $\sum_{j=1}^{\infty} G^j b_j$  converges uniformly and absolutely on  $V_i$  in each continuous semi-norm on F. Thus the sum of this series is an element  $G = (G_1, \dots, G_k)$  in  $(0_F(V_i))^k$ . Thus in the topology of uniform and absolute convergence on compact subsets of  $V_i$  in each continuous semi-norm on F we have

$$egin{aligned} f &= \lim_{t o \infty} \sum_{j=1}^{ au} f_j b_j \ &= \lim_{t o \infty} \sum_{j=1}^{t} \sum_{m=1}^{k} G_m^j H_m b_j \ &= \sum_{m=1}^{k} \left(\lim_{t o \infty} \sum_{j=1}^{t} G_m^j b_j 
ight) H_m \ &= \sum_{m=1}^{k} G_m H_m \ , \end{aligned}$$

as was to be proved.

The following consequence of Theorem 2 will be useful later.

LEMMA 4. If the element f of  $S_F(U)$  has the property that u'f is the zero element of S(U) for all u in  $F^*$  then f = 0.

*Proof.* By taking a covering of U by small open sets we reduce to the case in which f has a representation

$$f = \sum\limits_{i=1}^k h_i \bigotimes g_i$$
 ,

with  $h_i$  in S(U) and  $g_i$  in  $0_F(U)$ . Let R be the sheaf on U of relations of  $h_1, \dots, h_k$ . Thus for each u in  $F^*$  we see that

$$egin{aligned} 0 &= u'f = \sum\limits_{i=1}^k h_i \otimes \langle g_i, u 
angle \ &= \sum\limits_{i=1}^k \langle g_i, u 
angle h_i \;. \end{aligned}$$

Thus by Definition 2 we see that  $g = (g_1, \dots, g_k) \in R'_F(U)$ . By Theorem 2 it follows that each p in U has a neighborhood  $V \subset U$  such that there exist  $H_1, \dots, H_t$  in R(V) and  $G_1, \dots, G_t$  in  $0_F(V)$  with

$$r_{\scriptscriptstyle VV}g = \sum_{j=1}^t G_j H_j$$
 .

Thus for each i with  $1 \leq i \leq k$  we have

$$r_{{\scriptscriptstyle {
m V}}{\scriptscriptstyle {
m V}}}g_i = \sum\limits_{j=1}^t G_j H_j^i$$
 ,

where  $H_j = (H_j^1, \dots, H_j^k)$ . Therefore on V we have

$$egin{aligned} f &= \sum\limits_{i=1}^k h_i \otimes g_i = \sum\limits_{i=1}^k h_i \otimes \left(\sum\limits_{j=1}^t G_j H_j^i
ight) \ &= \sum\limits_{i=1}^k \left(\sum\limits_{j=1}^t h_i \otimes (G_i H_j^i)
ight) \ &= \sum\limits_{j=1}^t \left(\sum\limits_{i=1}^k H_j^i h_i
ight) \otimes G_j = 0 \end{aligned}$$

since  $H_j \in R(V)$  for all j. This proves Lemma 4.

We next give an important characterization of  $S_F$  in case S is a coherent analytic subsheaf of  $0^n$  for some positive integer n.

THEOREM 3. Let M be a Stein manifold and S a coherent analytic subsheaf of  $0^n$ . Let F be a Frechet space. For each open  $U \subset M$  there is a mapping  $\tau(U)$  from  $S(U) \otimes 0_F(U)$  into  $(0_F(U))^n$  which for each  $h = (h_1, \dots, h_n)$  in S(U) and g in  $0_F(U)$  maps  $h \otimes g$  onto  $gh = (gh_1, \dots, gh_n)$ in  $(0_F(U))^n$ . For each such g and h the image gh of  $h \otimes g$  actually lies in the subset  $S'_F(U)$  of  $(0_F(U))^n$ . The family of such mappings  $\tau(U)$  induces an isomorphism  $\tau$  of the sheaf  $S_F$  (which was defined above to be the sheaf determined by the presheaf data  $U \to S(U) \otimes 0_F(U)$ ) onto the sheaf  $S'_F$ . Thus  $S'_F$  and  $S_F$  are isomorphic.

*Proof.* Clearly the map of the Cartesian product  $S(U) \times 0_F(U)$ into  $(0_F(U))^n$  defined by  $(h, g) \to gh$  induces a group homomorphism of  $(S(U), 0_F(U))$ —the free abelian group generated by the elements of the Cartesian product  $S(U) \times 0_F(U)$ —into  $(0_F(U))^n$ . It is trivial to check that  $N(S(U), 0_F(U))$ : belongs to the kernel of this map, where  $N(S(U), 0_F(U))$  is defined as in [5] p. 8 to be the subgroup of  $(S(U), 0_F(U))$ generated by elements of the forms

- (i)  $(x_1 + x_2, y) (x_1, y) (x_2, y)$
- (ii)  $(x, y_1 + y_2) (x, y_1) (x, y_2)$
- (iii) (ax, y) (x, ay)

where  $x, x_1$ , and  $x_2$  are in  $S(U), y, y_1$ , and  $y_2$  are in  $0_F(U)$ , and  $a \in O(U)$ . Thus this map induces a homomorphism  $\tau(U)$  of the quotient  $(S(U), 0_F(U))/N(S(U), 0_F(U)) = S(U) \otimes 0_F(U)$  into  $(0_F(U))^n$ . It is trivial to check that  $\tau(U)$  is an O(U)-homomorphism. Now with g and h as above and u in  $F^*$  we have

$$u \circ \tau(U)(h \otimes g) = u \circ (gh) = (u \circ g)h \in S(U)$$
.

Thus  $\tau(U)(h \otimes g) \in S'_F(U)$ . It follows that the range of  $\tau(U)$  is a subset of  $S'_F(U)$ . It is now clear that the family of mappings  $\tau(U)$  induces an 0-homomorphism  $\tau$  of  $S_F$  into  $S'_F$ . To show that  $\tau$  is one-to-one we must prove

(a) If  $\tau(U)(\sum_{i=1}^{N} h_i \otimes g_i) = 0$  then each p in U has a neighborhood V such that  $r_{vU}(\sum_{i=1}^{N} h_i \otimes g_i) = 0$ .

To show that au is onto we must prove

(b) If  $f \in S_F'(U)$  then each p in U has a neighborhood V such that  $r_{vv}f = \tau(V)(\sum_{i=1}^{N} h_i \otimes g_i)$  for some elements  $h_i$  in S(V) and  $g_i$  in  $0_F(V)$ . We first prove (a). If we let R be the sheaf of relations on U of  $h_1, \dots, h_N$  by the coherence of R there exists a neighborhood  $V_0$  of p and elements  $r_1 = (r_1^1, \dots, r_1^N), \dots, r_n = (r_n^1, \dots, r_n^N)$  of  $R(V_0)$  which

generate R at each point of  $V_0$ . Now

$$\sum\limits_{i=1}^N g_i h_i = au(U) \Bigl( \sum\limits_{i=1}^N h_i \otimes g_i \Bigr) = 0$$
 .

Thus for each u in  $F^*$  we have

$$\sum_{i=1}^{N} (u \circ g_i) h_i = 0$$

so that  $(u \circ g_1, \dots, u \circ g_N) \in R(U)$  for all u in  $F^*$ . By definition this means that  $(g_1, \dots, g_N) \in R'_F(U)$ . Therefore by Theorem 2 we see that there exists a neighborhood V of p and  $G = (G_1, \dots, G_n)$  in  $(0_F(V))^n$  such that  $(g_1, \dots, g_N) = G_1r_1 + \dots + G_nr_n$ . Thus on V we have

$$\sum\limits_{i=1}^{N}h_i\otimes g_i = \sum\limits_{i=1}^{N}h_i\otimes \left(\sum\limits_{j=1}^{n}G_jr_j^i
ight) 
onumber \ = \sum\limits_{j=1}^{n}\left(\sum\limits_{i=1}^{N}(r_j^ih_i)
ight)\otimes G_j = 0$$

since  $r_j \in R(V)$  for each j. This proves (a).

To prove (b) notice by Theorem 2 that there exists a neighborhood V of p, elements  $h_1, \dots, h_N$  in S(V), and elements  $g_1, \dots, g_N$  in  $O_F(V)$  such that on V we have

$$f = \sum\limits_{i=1}^N g_i h_i = au(V) \left( \sum\limits_{i=1}^N h_i \otimes g_i 
ight)$$
 .

This completes the proof of Theorem 3.

We state for future reference a version of a theorem of Banach, first giving a definition.

DEFINITION 3. If  $\{g_n\}$  is a sequence of vectors in a Frechet space  $F_{\infty}$  the series  $\sum_{n=1}^{\infty} g_n$  is called *absolutely convergent* if the series  $\sum_{n=1}^{\infty} ||g_n||$  converges for each continuous semi-norm || || on F.

Notice that a continuous linear transformation from a Frechet space F to a Frechet space G takes absolutely convergent sequences into absolutely convergent sequences.

LEMMA 5. Let  $\sigma$  be a continuous linear map of a Frechet space F onto a Frechet space G. Let  $\{g_i\}$  be an absolutely convergent sequence from G. Then there exists an absolutely convergent sequence  $\{f_i\}$  in F such that  $\sigma(f_i) = g_i$  for all i.

*Proof.* Let  $\{|| \quad ||_k\}$  be a defining sequence of semi-norms on F. Since the map  $\sigma$  is continuous, we see ([1] p. 40) that for each k the set  $\sigma\{f: ||f||_k \leq 1\}$  contains a neighborhood  $\{g: ||g||'_k \leq 1\}$  of 0 in G, where  $|| \quad ||'_k$  is some continuous semi-norm on G. Thus for each g in

G and each k there exists f in F with  $\sigma(f) = g$  and  $||f||_k \leq ||g||'_k$ . Now for each k choose j = j(k) such that

$$\sum\limits_{n=j}^{\infty}||\,g_{\,n}\,||_{^{k}}<2^{-k}$$
 ,

so that

$$\sum_{k=1}^{\infty}\sum_{n=j(k)}^{\infty}||g_{n}||_{k}^{\prime}<\infty$$

We may assume that  $j(1) < j(2) < \cdots$ . For each n with  $j(k) \le n < j(k+1)$  choose  $f_n$  in F with  $\sigma(f_n) = g_n$  and  $||f_n||_k \le ||g_n||'_k$ . If for each n we let k(n) be the smallest value of k for which n < j(k+1), it follows that

$$\sum_{n=1}^{\infty} ||f_n||_{k(n)} < \infty$$
 .

Since for each t we have  $||f_n||_t \leq ||f_n||_k$  for all  $k \geq t$  it follows that

$$\sum_{n=1}^{\infty} ||f_n||_t$$

is finite for all t. This proves the lemma.

THEOREM 4. If S is a coherent analytic sheaf on a Stein manifold M and if F is a Frechet space then  $H^{N}(M, S_{F}) = 0$  for all  $N \geq 1$ .

*Proof.* Let f be an element of  $H^{N}(M, S_{F})$ . Consider a locally finite covering  $\{U_i\}$  of M by holomorphically convex open sets  $U_i$ , so fine that f is represented by an element of  $H^{N}(\{U_i\}, S_F)$ . For each finite sequence  $K = (i_1, \dots, i_k)$  of positive integers let  $U_K = U_{i_1} \cap \dots \cap U_{i_k}$ . The element f of  $H^{N}(M, S_{F})$  can be considered to belong to  $H^{N}(\{U_{i}\}, S_{F})$  and therefore can be represented by a cocycle  $f = \{f_i\}$  of  $Z^N(\{U_i\}, S_F)$ . Here I is any sequence of N+1 positive integers, and, for each I,  $f_I$  is an element of  $S_{\mathbb{F}}(U_{\mathbb{I}})$ . Also  $\delta f = 0$ , where  $\delta$  is the coboundary operator from  $C^{N}(\{U_i\}, S_F)$  into  $C^{N+1}(\{U_i\}, S_F)$  and  $Z^{N}(\{U_i\}, S_F)$  is the kernel of  $\delta$ . By choosing the covering  $\{U_i\}$  fine enough we may assume that for each K there exist elements  $h_{1K}, \dots, h_{\alpha K}$ , with  $\alpha$  depending on K, in  $S(U_{\kappa})$  which generate S at each point of  $U_{\kappa}$ . This implies ([3], expose XVIII, p. 9) that every h in  $S(U_{\kappa})$  has a representation of the form  $h = \sum_{i=1}^{\alpha} g_i h_{i\kappa}$ , with  $g_i \in O(U_{\kappa})$ . We may also choose the covering  $\{U_i\}$ so fine that, for each I,  $f_I$  can be represented in the form

$$f_{\scriptscriptstyle I} = \sum\limits_{i=1}^{lpha} h_{i \scriptscriptstyle I} \bigotimes g_{i \scriptscriptstyle I}$$

with  $h_{iI}$  as above and with  $g_{iI}$  in  $0_F(U_I)$ .

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By Theorem 1 there exists a sequence  $\{P_n\}$  of continuous mutually annihilating projections on F whose ranges are one dimensional and a sequence  $\{b_n\}$  of vectors in F bounded in each continuous semi-norm on F having the following properties. For each I and i the series  $\sum_{n=1}^{\infty} P_n \circ g_{iI}$ converges to  $g_{iI}$  on  $U_I$ . For each I and i we have  $P_n \circ g_{iI} = g_{iI}^n b_n$ , where  $g_{iI}^n \in O(U_I)$ . For each I and i the series  $\sum_{n=1}^{\infty} g_{iI}^n$  converges absolutely in the Frechet space  $O(U_I)$ . Now since for each n the projection  $P_n$ induces a homomorphism of the sheaf  $S_F$  onto itself, the element  $\{P_n f_I\}$ of  $C^N(\{U_i\}, S_F)$  is in  $Z^N(\{U_i\}, S_F)$ . Also

$$egin{aligned} P_n f_I &= \sum\limits_{i=1}^lpha h_{iI} \otimes P_n g_{iI} \ &= \sum\limits_{i=1}^lpha h_{iI} \otimes g_{iI}^n b_n = \left( \sum\limits_{i=1}^lpha g_{iI}^n h_{iI} 
ight) \otimes b_n \ . \end{aligned}$$

If for each *n* and *I* we let  $f_I^n$  be the element  $\sum_{i=1}^{\alpha} g_{iI}^n h_{iI}$  of  $S(U_I)$  it follows that for each *n* the element  $f^n = \{f_I^n\}^{i=1}$  of  $C^N(\{U_i\}, S)$  belongs to  $Z^N(\{U_i\}, S)$ . It is also clear that  $f^n b_n = P_n f$ .

Now there exists a natural Frechet space topology on each S(U), described in [4], expose XVII. This topology has the property that for each h in S(U) the map  $g \rightarrow gh$  of O(U) into S(U) is continuous. We therefore see that for each I the series

$$\sum\limits_{n=1}^{\infty} f_{I}^{n} = \sum\limits_{n=1}^{\infty} \left( \sum\limits_{i=1}^{lpha} g_{iI}^{n} h_{iI} 
ight)$$

converges absolutely in  $S(U_I)$  because for each I and i the series  $\sum_{n=1}^{\infty} g_{iI}^n$  converges absolutely in  $O(U_I)$ . Now the space  $C^N(\{U_i\}, S)$  is the Cartesian product of the Frechet spaces  $S(U_I)$ , and therefore possesses a Frechet space structure. Moreover  $Z^N(\{U_i\}, S)$  is closed in  $C^N(\{U_i\}, S)$  and is therefore also a Frechet space. Since for each I the series  $\sum_{n=1}^{\infty} f_I^n$  converges absolutely in  $S(U_I)$  it follows that  $\sum_{n=1}^{\infty} f^n$  converges absolutely in  $Z^N(\{U_i\}, S)$ . By Theorem B of [3] and Leray's theorem (see [5] p. 213) we see that the coboundary map  $\delta$  of the Frechet space  $C^{N-1}(\{U_i\}, S)$  into  $Z^N(\{U_i\}, S)$  is onto. From [4] we also see that  $\delta$  is continuous.

Let J stand for an arbitrary N-tuple of positive integers. Thus for each J, by the above, there is a continuous homomorphism.

$$\tau_J: (G_1, \cdots, G_{\alpha}) \to \sum_{i=1}^{\alpha} G_i h_{iJ}$$

of the Frechet space  $(0(U_J))^{\alpha}$  onto the Frechet space  $S(U_J)$ . These maps induce a continuous homomorphism  $\tau$  of the Frechet space  $\emptyset$  onto the Frechet space  $C^{N-1}(\{U_i\}, S)$ , where  $\emptyset$  is defined to be the product  $\prod_J (0(U_J))^{\alpha}$ , with  $\alpha$  depending as above on J, of the Frechet spaces  $(0(U_J))^{\alpha}$ . Thus

$$\sigma = \delta \circ \tau$$

is a continuous homomorphism of  $\Phi$  onto  $Z^{N}(\{U_{i}\}, S)$ . Since  $\sum_{n=1}^{\infty} f^{n}$  converges absolutely in  $Z^{N}(\{U_{i}\}, S)$  it follows from Lemma 5 that there exists an absolutely convergent sequence  $\{G^{n}\}$  in  $\Phi$  with  $\sigma(G^{n}) = f^{n}$  for all n. For each n write  $G^{n} = \{G_{i}^{n}\}$ , where

$$G_{\scriptscriptstyle J}^n=(G_{\scriptscriptstyle 1J}^n,\,\cdots,\,G_{\scriptscriptstyle lpha J}^n)\in (0(U_{\scriptscriptstyle J}))^{lpha}$$
 .

Thus for each J we see that the series  $\sum_{n=1}^{\infty} G_J^n$  converges absolutely and uniformly on every compact subset of  $U_J$ , so that the series  $\sum_{n=1}^{\infty} G_J^n b_n$  converges absolutely in  $(0_F(U_J))^{\alpha}$  to an element

$$G_J = (G_{1J}, \cdots, G_{\alpha J})$$

in  $(0_F(U_J))^{\alpha}$ . Thus for each *i* and *J* we have  $G_{iJ} = \sum_{n=1}^{\infty} G_{iJ}^n b_n$ . For each *J* let  $e_J$  be the element

$$e_{\scriptscriptstyle J} = \sum\limits_{i=1}^{a} h_{iJ} \bigotimes G_{iJ}$$

of  $S_F(U_J)$ . Thus  $e = \{e_J\} \in C^{N-1}(\{U_i\}, S_F)$ . We shall finish the proof by showing that  $\delta e = f$ . To this end it is sufficient by Lemma 4 to show  $u'(\delta e) = u'(f)$  for all u in  $F^*$ . We compute:

$$egin{aligned} u'(e_J) &= \sum\limits_{i=1}^lpha \langle G_{iJ}, u 
angle h_{iJ} \ &= \sum\limits_{i=1}^lpha \langle \sum\limits_{n=1}^lpha G_{iJ}^n b_n, u 
angle h_{iJ} \ &= \sum\limits_{n=1}^\infty \left( \sum\limits_{i=1}^lpha G_{iJ}^n h_{iJ} 
ight) \langle b_n, u 
angle \ &= \sum\limits_{n=1}^\infty ( au_J(G_J^n)) \langle b_n, u 
angle \end{aligned}$$

absolutely in  $S(U_J)$ . Thus

$$u'(e) = \sum_{n=1}^{\infty} \left( \tau(G^n) \right) \langle b_n, u 
angle$$

absolutely in  $C^{N-1}(\{U_i\}, S)$ . Thus

$$u'(\delta e) = \delta(u'(e)) = \sum_{n=1}^{\infty} (\delta \circ \tau)(G^n) \langle b_n, u \rangle$$
  
 $= \sum_{n=1}^{\infty} \sigma(G^n) \langle b_n, u \rangle = \sum_{n=1}^{\infty} f^n \langle b_n, u \rangle.$ 

Also for each I we have

$$u'(f_{\scriptscriptstyle I}) = \sum\limits_{i=1}^{lpha} \langle g_{i_{\scriptscriptstyle I}}, u 
angle h_{i_{\scriptscriptstyle I}}$$

$$=\sum_{i=1}^{a}\left\langle\sum_{n=1}^{\infty}g_{iI}^{n}b_{n},u\right\rangle h_{iI}$$
$$=\sum_{n=1}^{\infty}\left(\sum_{i=1}^{\alpha}g_{iI}^{n}h_{iI}\right)\left\langle b_{n},u\right\rangle =\sum_{n=1}^{\infty}f_{I}^{n}\left\langle b_{n},u\right\rangle.$$

Therefore  $u'(f) = \sum_{n=1}^{\infty} f^n \langle b_n, u \rangle$ . It follows that  $u'(f) = u'(\delta e)$  for all u in  $F^*$ , so that  $f = \delta e$ . This completes the proof of Theorem 4.

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