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We wish to extend certain results in the theory of analytic functions of several complex variables to the case of analytic functions with values in a Frechet space $F$. To do this, we prove (Theorem 1 below) that such a function $\varphi$ has an expansion of the form

$$(*) \qquad \varphi = \sum_{n=1}^{\infty} P_n \circ \varphi ,$$

where $\{P_n\}$ is a sequence of continuous mutually annihilating projections on $F$ whose ranges are all one-dimensional subspaces of $F$. This representation reduces the study of $\varphi$, for many purposes, to the study of the functions $P_n \circ \varphi$, which are essentially scalar-valued analytic functions. We actually prove the stronger (and more useful) result that if $\{\varphi_k\}$ is a sequence of analytic functions with values in $F$ then a single sequence $\{P_n\}$ can be found to give an expansion $(*)$ for every $\varphi_k$. Expansions of vector-valued functions of a different type have been considered by Grothendick [6].

Theorem 1 is applied to generalize Theorem B of H. Cartan [3]. We consider a coherent analytic sheaf $S$ on a Stein manifold $M$ and introduce the notion of the *vectorization* $S_F$ of $S$ (relative to a given Frechet space $F$).

If $0$ denotes the sheaf of locally-defined analytic functions and $0_F$ denotes the sheaf of locally-defined analytic functions with values in $F$, then $S_F$ is defined to be the tensor product $S \otimes 0_F$ of the 0-modules $S$ and $0_F$. For the important case of a coherent analytic subsheaf $S$ of the sheaf $0^k$ of locally-defined $k$-tuples of analytic functions, $S_F$ turns out to be canonically isomorphic to the sheaf $S'_F$ determined by assigning to each open set $U$ the module of all $k$-tuples $(f_1, \cdots, f_k)$ of analytic functions from $U$ to $F$ which have the property that for each $u$ in $F^*$ the $k$-tuple $(u \circ f_1, \cdots, u \circ f_k)$ is a cross-section of $S$ over $U$. For instance, if $S$ is the sheaf of all locally-defined analytic functions which vanish on a given analytic set $A$ then it is evident that $S'_F$ is the sheaf of all locally-defined analytic functions with values in $F$ which vanish on $A$.

One of the main results, an extension of Theorem B of [3], will be that the cohomology groups $H^N(M, S_F)$ vanish in all dimensions $N \geqq 1$, where $S_F$ is the vectorization of a coherent analytic sheaf $S$ on a Stein manifold $M$. Using this theorem and the isomorphism of $S_F$ to the sheaf $S'_F$ defined above one could show, for instance, that the usual

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sheaf—theoretic solutions to Cousin's problems carry over to the case of analytic functions with values in a Frechet space. Special cases were treated by totally different methods in [2], but the techniques of that paper seem to be inadequate to obtain general results.

The proofs are all Banach-space theoretic. That is, only Banach space theory is necessary to obtain the above extension of Theorem B and to prove the necessary facts about vectorizations. We begin with a theorem which is given without proof on p. 278 of Banach [1], who attributes it to H. Auerbach. A proof can be found in Taylor [7]. Since complex Banach spaces are considered here, we give the proof.

THEOREM (Auerbach). *An $n$-dimensional Banach space $B$ has a basis of unit vectors whose dual basis also consists of unit vectors.*

*Proof.* Choose a basis $(b^1, \cdots, b^n)$ of $B$ and for any $x$ in $B$ let $(x_1, \cdots, x_n)$ be the coordinates of $x$ relative to the chosen basis. Let $T$ be the set of all $n$-tuples $(x^1, \cdots, x^n)$ of unit vectors in $B$. For each $(x^1, \cdots, x^n)$ in $T$ let $\alpha(x^1, \cdots, x^n)$ be the absolute value of the determinant $\det(x^i_j)$. Thus $\alpha$ is a continuous function on the compact space $T$. Now $\alpha(x^1, \cdots, x^n) \neq 0$ if and only if $(x^1, \cdots, x^n)$ is a basis. Thus $\alpha$ attains its maximum for $T$ at some point $(y^1, \cdots, y^n)$ in $T$ which is a basis of unit vectors. Let $(u^1, \cdots, u^n)$ be the dual basis in $B^*$. Now $\|u^i\| \geqq 1$ because $\langle y^i, u^i\rangle = 1$. Assume $\|u^i\| > 1$ for some $i$. Thus there exists $t$ in $B$ with $\|t\| = 1$ and $\langle t, u^i\rangle = c > 1$. Thus $\langle t - cy^i, u^i\rangle = 0$, so that $t - cy^i$ is a linear combination of the vectors of the basis $(y^1, \cdots, y^n)$ other than $y^i$. If we let $(z^1, \cdots, z^n)$ be the basis $(y^1, \cdots, y^n)$ with $y^i$ replaced by $t$ it follows that $\alpha(z^1, \cdots, z^n) = c\alpha(y^1, \cdots, y^n)$. Since the basis $(z^1, \cdots, z^n)$ consists of unit vectors this contradicts the choice of $(y^1, \cdots, y^n)$. Thus $\|u^i\| = 1$ for all $i$, and the theorem is proved.

COROLLARY. *If $B_0$ is a finite-dimensional subspace of dimension $n$ of a Banach space $B$ there exist $n$ mutually annihilating projections (idempotent continuous linear operators) on $B$, each of norm* 1, *whose ranges are one-dimensional subspaces of $B_0$ and whose sum is a projection of $B$ onto $B_0$ of norm at most $n$.*

*Proof.* Let $(y^1, \cdots, y^n)$ be a basis of unit vectors of $B_0$ such that the dual basis $(u^1, \cdots, u^n)$ of $B_0^*$ also consists of unit vectors. Let $v^i$ be an extension of $u^i$ to a linear functional on $B$ of norm 1. The operators $P_1, \cdots, P_n$ on $B$ defined by

$$P_i x = \langle x, v^i\rangle y^i$$

are the desired projections.

We recall that a Frechet space is a locally convex topological linear space $F$ which admits a countable family $\{\| \ \|_k\}$ of continuous semi-norms such that a basis for the neighborhoods of 0 in $F$ is given by the sets

$$\{x \in F : \| x \|_k < 1\} .$$

If $\| \ \|$ is any continuous semi-norm on $F$ it follows that for some $k$ $\| x \| \leqq \| x \|_k$ for all $x$ in $F$. If necessary it may be assumed that $\{\| \ \|_k\}$ is a monotonely nondecreasing sequence of semi-norms, in which case we shall call it a *defining sequence* of semi-norms for $F$.

LEMMA 1. *Let $F$ be a Frechet space with a defining sequence $\{\| \ \|_k\}$ of semi-norms. Let $\{a_n\}$ be a sequence of vectors in $F$, $\{\delta_k\}$ a sequence of nonnegative real numbers, and $\{k_j\}$ a strictly increasing sequence of positive integers. Then there exists a sequence $\{P_n\}$ of mutually annihilating continuous projections on $F$, whose ranges are subspaces of $F$ of dimensions at most 1, and a sequence $\{\varepsilon_k\}$, with $0 < \varepsilon_k < \delta_k$ for all $k$, with the following properties. For each positive integer $j$ the operator*

$$Q_j = \sum_{n=1}^{k_j} P_n$$

*is a projection on the subspace $B_j$ of $F$ spanned by the vectors $a_1, \cdots, a_{k_j}$. For each positive integer $n$ the sum*

$$\| a \|_0 = \sum_{k=1}^{\infty} \varepsilon_k \| a \|_k$$

*is finite for $a = a_n$. For each positive integer $j$ and all $n \leqq k_j$ we have $\| P_n \|_0 \leqq (1 + k_1^2) \cdots (1 + k_j^2)$, where*

$$\| P_n \|_0 = \sup \{\| P_n b \|_0 : b \in F, \quad \| b \|_0 = 1\} .$$

*Proof.* We may assume the $\delta_k$ to be so small that $\sum_{k=1}^{\infty} \delta_k \| a_n \|_k < \infty$ for all $n$. By induction we construct a sequence $\{P_n\}$ of mutually annihilating continuous projections, a sequence $\{\varepsilon_k\}$ of positive real numbers, and an increasing sequence $\{N_j\}$ of positive integers such that

(a) $0 < \varepsilon_k < \delta_k$,

(b) For each $j$ the operator $Q_j$ is a projection onto $B_j$,

(c) $\| P_n \|^j < (1 + k_1^2) \cdots (1 + k_i^2)$ for $1 \leqq n \leqq k_i$ and all $i \leqq j$.

We explain what is meant by (c). First of all, $\| \ \|^j$ is the continuous semi-norm on $F$ defined by

$$\| b \|^j = \sum_{k=1}^{N_j} \varepsilon_k \| b \|_k .$$

Secondly, $\| P_n \|^j$ is defined by

$$\| P_n \|^j = \sup \{\| P_n b \|^j : \| b \|^j = 1\} .$$

Assuming that $P_1, \cdots, P_{k_j}$ and $N_1 \cdots, N_j$, and $\varepsilon_1, \cdots, \varepsilon_{N_j}$ have been found with the relevant properties, we show how to continue to the next stage $j+1$. First choose $N_{j+1} > N_j$ so large that $\|\ \|_{N_{j+1}}$ is a norm (and not merely a semi-norm) on $B_{j+1}$. Choose then $\varepsilon_i$, $N_j < i \leqq N_{j+1}$, so small that $0 < \varepsilon_i < \delta_i$ and $\| P_n \|^{j+1} < (1 + k_1^2) \cdots (1 + k_i^2)$ for $n \leqq k_j$ and all $i \leqq j$. To see that this can be done, notice that because $\|\ \|_{N_j}$ is a norm on $B_j$ there exists $r > 0$ so that $r \| a \|^j > \| a \|_m$ for all $a$ in $B_j$ and all $m \leqq N_{j+1}$. Thus

$$\| P_n \|^{j+1} \leqq \sup \{\| P_n b \|^{j+1} : \| b \|^j = 1\} \leqq (1 + \sum_{m=N_j+1}^{N_{j+1}} \varepsilon_m) \| P_n \|^j .$$

Now use (c).

Now let $Q_j'$ be the restriction of $Q_j$ to $B_{j+1}$ and let $I_{j+1}$ be the identity operator on $B_{j+1}$. Thus $I_{j+1} - Q_j'$ is a projection of $B_{j+1}$ onto a subspace $S_{j+1}$. Clearly $B_j$ and $S_{j+1}$ are complementary subspaces of $B_{j+1}$, so that $\dim S_{j+1} \leqq k_{j+1} - k_j$. By the above corollary there exists a projection $E_{j+1}$ with $\| E_{j+1} \|^{j+1} \leqq k_{j+1}$ of $F$ onto $B_{j+1}$. Also by the above corollary there exist mutually annihilating projections $R_n$, $k_j < n \leqq k_{j+1}$, of $S_{j+1}$ onto subspaces of dimensions at most 1 such that $\| R_n \|^{j+1} \leqq 1$ for all $n$ and such that $\Sigma R_n$ is the identity projection of $S_{j+1}$ onto itself. For $k_j < n \leqq k_{j+1}$ we define

$$P_n = R_n(I_{j+1} - Q_j')E_{j+1} .$$

Thus the $P_n$ are mutually annihilating projections for $1 \leqq n \leqq k_{j+1}$. Also $Q_{j+1}$ is a projection onto $B_{j+1}$. Finally for $k_j < n \leqq k_{j+1}$ we have

$$\begin{aligned} \| P_n \|^{j+1} &\leqq \| R_n \|^{j+1} \| I_{j+1} - Q_j' \|^{j+1} \| E_{j+1} \|^{j+1} \\ &\leqq (1 + \sum_{n=1}^{k_j} \| P_n \|^{j+1}) k_{j+1} \\ &< [1 + k_j(1 + k_1^2) \cdots (1 + k_j^2)] k_{j+1} \\ &\leqq (1 + k_1^2) \cdots (1 + k_{j+1}^2) . \end{aligned}$$

The same is true for $n \leqq k_j$, by the above construction. Thus the construction has been continued another step. By induction it follows that sequences $\{P_n\}$, $\{N_j\}$, and $\{\varepsilon_k\}$ can be chosen satisfying properties (a), (b), and (c). It is immediate that the sequences $\{P_n\}$ and $\{\varepsilon_k\}$ satisfy the requirements of the lemma.

LEMMA 2. *Let $\{a_n\}$ be a sequence of elements of a Frechet space $F$, $\{\|\ \|_k\}$ a defining sequence of semi-norms on $F$, and $\{\delta_k\}$ a sequence of positive real numbers. Then there exist a sequence $\{\varepsilon_k\}$ of positive real numbers and a sequence $\{P_n\}$ of mutually annihilating projections on $F$ whose ranges are subspaces of $F$ of dimensions at most* 1 *having the following properties.*

(i) $0 < \varepsilon_k < \delta_k$ *for all* $k$,

(ii) *For* $a = a_n$ *the norm* $\|a\|_0 = \sum_{k=1}^{\infty} \varepsilon_k \|a\|_k$ *is finite for all* $n$,

(iii) $R_m a_n = a_n$ *for all positive integers* $m$ *and* $n$ *with* $m \geqq 2n$, *where* $R_m = \sum_{j=1}^{m} P_j$,

(vi) *For all* $t > 1$ *and* $\varepsilon > 0$ *the sum* $\sum_{n=1}^{\infty} \|P_n\|_0 t^{-n^{\varepsilon}}$ *converges, where* $\|P_n\|_0$ *is defined as above.*

*Proof.* Define the sequence $\{k_j\}$ by $k_j = 2^j$. Choose the sequences $\{P_n\}$ and $\{\varepsilon_k\}$ as in lemma 1. Clearly (i) and (ii) are satisfied. Now for each positive integer $n$ there is a positive integer $j$ with $2^{j-1} \leqq n < 2^j$. It follows that $a_n \in B_j$. Thus $R_{2^j} a_n = Q_j a_n = a_n$, so that $R_m a_n = a_n$ for all $m \geqq 2^j$ and therefore for all $m \geqq 2n$. This proves (iii).

Now for each $n$ choose $j$ with $2^{j-1} \leqq n < 2^j$. Thus

$$\begin{aligned} \|P_n\|_0 &\leqq (1 + k_j^2)^j = (1 + 2^{2j})^j \\ &\leqq (5n^2)^j \leqq (5n^2)^{\alpha} , \end{aligned}$$

where $\alpha = 1 + \log_2 n$. From this it follows from elementary calculus that (iv) holds, thereby proving the lemma.

LEMMA 3. *Let*

$$\sum_{n_1 \geqq 0, \cdots, n_\alpha \geqq 0} a_i(n_1, \cdots, n_\alpha) z_1^{n_1} \cdots z_\alpha^{n_\alpha}$$

*where* $\alpha = \alpha_i$ *and* $1 \leqq i < \infty$, *be a sequence of formal power series with coefficients in a Frechet space* $F$. *Let* $\{\delta_k\}$ *be a sequence of positive real numbers. Then there exists a sequence* $\{\varepsilon_k\}$ *with* $0 < \varepsilon_k < \delta_k$ *for all* $k$ *and a sequence* $\{P_n\}$ *of mutually annihilating continuous projections of* $F$ *onto subspaces of dimensions at most* 1 *such that*

(a) $R_m a_i(n_1, \cdots, n_\alpha) = a_i(n_1, \cdots, n_\alpha)$ *whenever* $m \geqq 2^{i+2} n^{\alpha}$, *where* $\alpha = \alpha_i$, $n = n_1 + \cdots + n_\alpha$, *and* $R_m = \sum_{j=1}^{m} P_j$,

(b) $P_m a_i(n_1, \cdots, n_\alpha) = 0$ *whenever* $m > 2^{i+2} n^{\alpha}$,

(c) $\sum_{n=1}^{\infty} \|P_n\|_0 t^{-n^{\varepsilon}} < \infty$ *for all* $t > 1$ *and* $\varepsilon > 0$, *where* $\| \ \|_0$ *is defined as above.*

*Proof.* For each $i$ order the coefficients $a_i(n_1, \cdots, n_\alpha)$ into a sequence $\{\alpha_i^k\}_{k=1}^{\infty}$ according to the size of $n$. We now define a sequence $\{a_k\}$ of elements of $F$ which is an ordering of the totality of the $a_i(n_1, \cdots, n_\alpha)$. For $k$ given let $2^i$ be the largest power of 2 dividing $k$ and let $j = 1/2(k2^{-i} + 1)$. Let $a_k = \alpha_i^j$. Now choose the sequences $\{\varepsilon_k\}$ and $\{P_n\}$ as in Lemma 2. Clearly (c) holds. Since (b) is a consequence of (a) we need only check (a). To this end consider a fixed $a_i(n_1, \cdots, n_\alpha)$. Now there exists $j \leqq n^{\alpha}$ with $a_i(n_1, \cdots, n_\alpha) = \alpha_i^j$. In turn $\alpha_i^j = a_k$ for some $k \leqq 2^{i+1} n^{\alpha}$. By (iii) of Lemma 2 it follows that $R_m a_k = a_k$ for $m \geqq 2k$ and therefore for $m \geqq 2^{i+2} n^{\alpha}$, as was to be proved.

We are now prepared to prove a series representation for analytic functions with values in a Frechet space which will be the principal tool in subsequent proofs.

THEOREM 1. *Let $F$ be a Frechet space and let $\{M_i\}$ be a sequence of complex analytic manifolds. For each $i$ let $\varphi_i$ be an analytic function on $M_i$ with values in $F$. Then there exists a sequence of vectors $\{b_n\}$ in $F$ and a sequence $\{P_n\}$ of continuous mutually annihilating projections of $F$ onto one-dimensional subspaces having the following properties. For each $i$ the series $\sum_{n=1}^{\infty} P_n \circ \varphi_i$ converges to $\varphi_i$ on $M_i$. For each $n$ we have $P_n b_n = b_n$, so that $P_n \circ \varphi_i = \varphi_i^n b_n$, for some analytic function $\varphi_i^n$ on $M_i$. For each $i$ the series $\sum_{n=1}^{\infty} \varphi_i^n$ converges absolutely and uniformly on all compact subsets of $M_i$. For each continuous semi-norm $\|\ \|$ on $F$ the sequence $\{\|b_n\|\}$ is bounded.*

*Proof.* For each $i$ let $\dim M_i = \alpha = \alpha_i$, so that $M_i$ is coverable by a countable family of analytic homeomorphs $\Gamma$ of the unit polycylinder

$$U^\alpha = \{z = (z_1, \cdots, z_\alpha) : |z_j| < 1, 1 \leqq j \leqq \alpha\} .$$

Thus in the proof of the theorem we may replace the sequence $\{M_i\}$ by the totality of all such $\Gamma$. There is therefore no loss of generality in assuming that each $M_i$ is a polycylinder $U^\alpha$ of dimension $\alpha = \alpha_i$. Let $\{\|\ \|_k\}$ be a defining sequence of semi-norms on $F$. Now for each $i$ the analytic function $\varphi_i$ has a power series expansion

$$\varphi_i = \sum_{n_1 \geqq 0, \cdots, n_\alpha \geqq 0} a_i(n_1, \cdots, n_\alpha) z_1^{n_1} \cdots z_\alpha^{n_\alpha}$$

on the polycylinder $M_i = U^\alpha$. This expansion converges absolutely and uniformly on each compact subset of $M_i$ in each semi-norm $\|\ \|_k$. By the diagonal process there therefore exist constants $\delta_k > 0$ such that the power series for each $\varphi_i$ converges absolutely and uniformly on each compact subset of $M_i$ in the norm $\sum_{k=1}^{\infty} \delta_k \|\ \|_k$, so that in particular this norm is finite for each coefficient $a_i(n_1, \cdots, n_\alpha)$. Now choose the sequences $\{\varepsilon_k\}$ and $\{P_n\}$ as in Lemma 3 relative to the power series expansions of the $\varphi_i$ and to the $\delta_k$ just obtained. Thus the power series for $\varphi_i$ converges absolutely and uniformly on compact subsets of $M_i$ in the norm $\|\ \|_0$ defined above. If some of the projections $P_n$ are zero, these may be omitted from the sequence. Thus for each $n$ there is a vector $b_n$ in $F$ with $\|b_n\|_0 = 1$ spanning the range of $P_n$. To show that the sequences $\{P_n\}$ and $\{b_n\}$ have the desired properties, consider a fixed compact subset $T$ of a fixed $M_i$. For each $n$ write

$$\gamma_n = \sum_{n_1 + \cdots + n_\alpha = n} \max \{\|a_i(n_1, \cdots, n_\alpha) z_1^{n_1} \cdots z_\alpha^{n_\alpha}\|_0 : z \in T\} .$$

By the usual convergence criteria we see that there exist $r > 1$ and $c > 0$ such that $r^n \gamma_n < c$ for all $n$.

If $j$ is any positive integer let $k$ be the largest integer such that $2^{i+2}k^\alpha < j$. Thus for each $z$ in $T$ we have

$$\begin{aligned} &\| P_j \varphi_i(z) \|_0 \\ &= \left\| P_j \sum_{n_1 + \cdots + n_\alpha \geqq k} a_i(n_1, \cdots, n_\alpha) z_1^{n_1} \cdots z_\alpha^{n_\alpha} \right\|_0 \\ &\leqq \| P_j \|_0 \sum_{n \geqq k} \gamma_n \leqq c \| P_j \|_0 \sum_{n \geqq k} r^{-n} \\ &= c(1 - r^{-1})^{-1} \| P_j \|_0 r^{-k} . \end{aligned}$$

Thus

$$\begin{aligned} \Delta &= \max \left\{ \sum_{j=1}^{\infty} \| P_j \varphi_i(z) \|_0 : z \in T \right\} \\ &\leqq c(1 - r^{-1})^{-1} \sum_{j=1}^{\infty} r^{-k} \| P_j \|_0 . \end{aligned}$$

Now by the definition of $k$ we see that $k$ is the integral part of $(j 2^{-i-2})^{1/\alpha}$, so that $k \geqq j^{1/2\alpha}$ for all $j$ sufficiently large. Thus $\Delta$ is finite if the sum $\sum_{j=1}^{\infty} r^{-j^\varepsilon} \| P_j \|_0$ converges, where $\varepsilon = (2\alpha)^{-1}$. By the choice of the sequence $\{P_j\}$ this series converges so that $\Delta$ is finite. Now since $\| b_n \|_0 = 1$,

$$\max \{ | \varphi_i^n(z) | : z \in T \} = \max \{ \| P_n \varphi_i(z) \|_0 : z \in T \} .$$

Therefore the series $\sum_{n=1}^{\infty} \varphi_i^n(z)$ converges absolutely and uniformly on $T$. If $\| \quad \|$ is a continuous semi-norm on $F$ then $\| \quad \| \leqq K \| \quad \|_0$ for some $K > 0$, so that $\{ \| b_n \| \}$ is bounded by $K$. Finally, we must show that $\sum_{n=1}^{\infty} P_n \circ \varphi_i$ actually converges to $\varphi_i$ (and not to something else). To see this, note by (a) and (b) of Lemma 3 that $R_m \circ \varphi_i$ and $\varphi_i$ have power series expansions in the coordinates $z_1, \cdots, z_\alpha$ which agree up to terms of total order $n$, whenever $m \geqq 2^{i+2} n^\alpha$. This completes the proof of Theorem 1.

Before giving the definition of the vectorization of an analytic sheaf, we indicate the terminology to be used, following Godement [5]. A presheaf $S$ on a topological space $X$ assigns to each open $U \subset X$ a set $S(U)$ and to each open set $V \subset U \subset X$ a map $r_{VU} : S(U) \to S(V)$ satisfying $r_{WV} \circ r_{VU} = r_{WU}$ for $W \subset V \subset U$. In particular the same terminology will be used if $S$ is a sheaf, that is, a presheaf satisfying axioms (F1) and (F2) on page 109 of [5]. To any presheaf $S$ is canonically associated a sheaf $S'$, and each element $f$ in $S(U)$ gives rise to a unique element in $S'(U)$ which will also be denoted by $f$. If $X$ is a complex analytic manifold a sheaf $S$ on $X$ is called analytic if it is a module over the sheaf $0$ of locally defined analytic functions, that is, if for each $U$ the set $S(U)$ is an $0(U)$-module, and if the usual commutation relations between module multiplication and the restriction maps $S(U) \to S(V)$ and $0(U) \to 0(V)$ hold.

DEFINITION 1. Let $S$ be an analytic sheaf on a complex analytic manifold $M$ and let $F$ be a Frechet space. Let $0$ be the sheaf of locally-defined analytic functions on $M$ and let $0_F$ be the sheaf of locally-defined analytic functions on $M$ with values in $F$, where by definition a continuous function $f$ from an open set $U \subset M$ to $F$ is called analytic if $u \circ f$ is analytic for all $u$ in $F^*$. Clearly $0_F$ is an $0$-module, i.e., an analytic sheaf. The vectorization $S_F$ of $S$ (relative to $F$) is defined to be the sheaf $S \otimes 0_F$, the tensor product of the $0$-modules $S$ and $0_F$. This is defined in [5] as the sheaf determined by the presheaf data

$$U \to S(U) \otimes 0_F(U) ,$$

where $S(U)$ and $0_F(U)$ are considered as $0(U)$-modules, together with the obvious restriction maps.

Note that if $T$ is a continuous linear operator from a Frechet space $F$ into a Frechet space $G$ then the natural homomorphism $T_0$ of $0_F$ into $0_G$ induced by $T$ gives rise to a homomorphism $T' = 1 \otimes T_0$ of $S_F$ into $S_G$. In particular, if $u$ is an element of $F^*$ (and so a continuous linear operator from $F$ into $C$) then $u$ induces a homomorphism of $S_F$ into $S_C$. But $S_C$ is canonically isomorphic to $S$, in virtue of the canonical isomorphism between the $0(U)$-modules $S(U) \otimes 0(U)$ and $S(U)$. (See [5] p. 8.) If we identify $S_C$ with $S$ it follows that each $u$ in $F^*$ induces a homomorphism $u'$ of $S_F$ onto $S$.

DEFINITION 2. If $S$ is an analytic subsheaf of the Cartesian product $0^n$ we define

$$S'_F(U) = \{f \in (0_F(U))^n : u \circ f \in S(U) \text{ for all } u \text{ in } F^*\} .$$

Clearly $S'_F$ so defined is an analytic subsheaf of the Cartesian product $(0_F)^n$.

THEOREM 2. *If $S$ is a coherent analytic subsheaf of $0^n$ then to each $p$ in $U \subset M$ and each $f$ in $S'_F(U)$ there exists a neighborhood $V$ of $p$, functions $H_1, \cdots, H_k$ in $S(V)$ and functions $G_1, \cdots, G_k$ in $0_F(V)$ such that*

$$r_{VU} f = \sum_{m=1}^{k} G_m H_m .$$

*Proof.* Since $S$ is coherent, there exists a neighborhood $V_0 \subset U$ of $p$ and functions $H_1, \cdots, H_k$ in $S(V_0)$ which generate $S$ at each point of $V_0$. We may assume that $\bar{V}_0$ is a compact subset of $U$. Let $V_0 \supset V_1 \supset V_2 \supset \cdots$

be a basis for the neighborhoods of $p$. Let $\Omega$ be the subset of $S(V_0)$ consisting of all elements in $S(V_0)$ which as elements of $(0(V_0))^n$ are bounded on $V_0$. Thus to each $h$ in $\Omega$ there exists $G = (G_1, \cdots, G_k)$ in $(0(V_i))^k$ for some $i$ such that the restriction of $h$ to $V_i$ has the form

$$h = \sum_{i=1}^{k} G_i H_i \ .$$

By choosing $i$ large enough we may assume that

$$\| G \|_i = \sup \{ | G_j(q) | : q \in V_i, 1 \leqq j \leqq k \}$$

is finite. Thus if for each pair $(i, N)$ of positive integers we let $\Omega_{iN}$ be the family of all $h$ in $\Omega$ such that $G$ can be chosen in $(0(V_i))^k$ with $\| G \|_i \leqq N$, we see that $\Omega = \bigcup \Omega_{iN}$ and that each $\Omega_{iN}$ is a closed subset of $\Omega$, where $\Omega$ has the norm defined by

$$\| h \|_0 = \sup \{ | h_i(q) | : 1 \leqq i \leqq n, q \in V_0 \}$$

for each $h = (h_1, \cdots, h_n) \in \Omega \subset (0(V_0))^n$. By the Baire category theorem there exists $(i, N)$ such that $\Omega_{iN}$ has a nonvoid interior. From this it follows as usual that there exists a constant $K > 0$ such that for each $h$ in $\Omega$ there exists $G$ in $(0(V_i))^k$ as above with $\| G \|_i \leqq K \| h \|_0$. Now consider $f$ as in the statement of the theorem, so that $f \in S'_F(U) \subset (0_F(U))^n$. By Theorem 1 there exists a sequence of vectors $\{b_j\}$ in $F$ which is bounded in each continuous semi-norm on $F$ and a sequence $\{P_j\}$ of continuous projections on $F$ having one-dimensional ranges such that $\sum_{j=1}^{\infty} P_j \circ f$ converges uniformly to $f$ on all compact subsets of $U$ and such that for each $j$ we have $P_j \circ f = f_j b_j$ with $f_j \in (0(U))^n$, where $\sum_{j=1}^{\infty} | f_j |$ converges uniformly on all compact subsets of $U$. Thus $\sum_{j=1}^{\infty} \| f_j \|_0$ is finite, since $\bar{V}_0 \subset U$.

Now for each $j$ there exists $u$ in $F^*$ with $\langle b_j, u \rangle = 1$. Thus

$$f_j = u \circ (f_j b_j) = u \circ (P_j \circ f) = (u \circ P_j) \circ f$$

is in $S(U)$ because $f \in S'_F(U)$ and $u \circ P_j \in F^*$. Thus $f_j \in S(U)$ for all $j$. By the above for each $j$ there exists $G^j = (G_1^j, \cdots, G_k^j)$ in $(0(V_i))^k$ such that on $V_i$ we have

$$f_j = \sum_{m=1}^{k} G_m^j H_m \ ,$$

with $\| G^j \|_i \leqq K \| f_j \|_0$. It follows that the series $\sum_{j=1}^{\infty} G^j b_j$ converges uniformly and absolutely on $V_i$ in each continuous semi-norm on $F$. Thus the sum of this series is an element $G = (G_1, \cdots, G_k)$ in $(0_F(V_i))^k$. Thus in the topology of uniform and absolute convergence on compact subsets of $V_i$ in each continuous semi-norm on $F$ we have

$$\begin{aligned} f &= \lim_{t\to\infty} \sum_{j=1}^{t} f_j b_j \\ &= \lim_{t\to\infty} \sum_{j=1}^{t} \sum_{m=1}^{k} G_m^j H_m b_j \\ &= \sum_{m=1}^{k} \left( \lim_{t\to\infty} \sum_{j=1}^{t} G_m^j b_j \right) H_m \\ &= \sum_{m=1}^{k} G_m H_m \ , \end{aligned}$$

as was to be proved.

The following consequence of Theorem 2 will be useful later.

LEMMA 4. *If the element $f$ of $S_F(U)$ has the property that $u'f$ is the zero element of $S(U)$ for all $u$ in $F^*$ then $f = 0$.*

*Proof.* By taking a covering of $U$ by small open sets we reduce to the case in which $f$ has a representation

$$f = \sum_{i=1}^{k} h_i \otimes g_i \ ,$$

with $h_i$ in $S(U)$ and $g_i$ in $0_F(U)$. Let $R$ be the sheaf on $U$ of relations of $h_1, \cdots, h_k$. Thus for each $u$ in $F^*$ we see that

$$\begin{aligned} 0 = u'f &= \sum_{i=1}^{k} h_i \otimes \langle g_i, u \rangle \\ &= \sum_{i=1}^{k} \langle g_i, u \rangle h_i \ . \end{aligned}$$

Thus by Definition 2 we see that $g = (g_1, \cdots, g_k) \in R'_F(U)$. By Theorem 2 it follows that each $p$ in $U$ has a neighborhood $V \subset U$ such that there exist $H_1, \cdots, H_t$ in $R(V)$ and $G_1, \cdots, G_t$ in $0_F(V)$ with

$$r_{VU} g = \sum_{j=1}^{t} G_j H_j \ .$$

Thus for each $i$ with $1 \leqq i \leqq k$ we have

$$r_{VU} g_i = \sum_{j=1}^{t} G_j H_j^i \ ,$$

where $H_j = (H_j^1, \cdots, H_j^k)$. Therefore on $V$ we have

$$\begin{aligned} f = \sum_{i=1}^{k} h_i \otimes g_i &= \sum_{i=1}^{k} h_i \otimes \left( \sum_{j=1}^{t} G_j H_j^i \right) \\ &= \sum_{i=1}^{k} \left( \sum_{j=1}^{t} h_i \otimes (G_i H_j^i) \right) \\ &= \sum_{j=1}^{t} \left( \sum_{i=1}^{k} H_j^i h_i \right) \otimes G_j = 0 \end{aligned}$$

since $H_j \in R(V)$ for all $j$. This proves Lemma 4.

We next give an important characterization of $S_F$ in case $S$ is a coherent analytic subsheaf of $0^n$ for some positive integer $n$.

THEOREM 3. *Let $M$ be a Stein manifold and $S$ a coherent analytic subsheaf of $0^n$. Let $F$ be a Frechet space. For each open $U \subset M$ there is a mapping $\tau(U)$ from $S(U) \otimes 0_F(U)$ into $(0_F(U))^n$ which for each $h = (h_1, \cdots, h_n)$ in $S(U)$ and $g$ in $0_F(U)$ maps $h \otimes g$ onto $gh = (gh_1, \cdots, gh_n)$ in $(0_F(U))^n$. For each such $g$ and $h$ the image $gh$ of $h \otimes g$ actually lies in the subset $S'_F(U)$ of $(0_F(U))^n$. The family of such mappings $\tau(U)$ induces an isomorphism $\tau$ of the sheaf $S_F$ (which was defined above to be the sheaf determined by the presheaf data $U \to S(U) \otimes 0_F(U)$) onto the sheaf $S'_F$. Thus $S'_F$ and $S_F$ are isomorphic.*

*Proof*. Clearly the map of the Cartesian product $S(U) \times 0_F(U)$ into $(0_F(U))^n$ defined by $(h, g) \to gh$ induces a group homomorphism of $(S(U), 0_F(U))$—the free abelian group generated by the elements of the Cartesian product $S(U) \times 0_F(U)$—into $(0_F(U))^n$. It is trivial to check that $N(S(U), 0_F(U))$ belongs to the kernel of this map, where $N(S(U), 0_F(U))$ is defined as in [5] p. 8 to be the subgroup of $(S(U), 0_F(U))$ generated by elements of the forms

(i) $(x_1 + x_2, y) - (x_1, y) - (x_2, y)$

(ii) $(x, y_1 + y_2) - (x, y_1) - (x, y_2)$

(iii) $(ax, y) - (x, ay)$

where $x$, $x_1$, and $x_2$ are in $S(U)$, $y$, $y_1$, and $y_2$ are in $0_F(U)$, and $a \in 0(U)$. Thus this map induces a homomorphism $\tau(U)$ of the quotient $(S(U), 0_F(U))/N(S(U), 0_F(U)) = S(U) \otimes 0_F(U)$ into $(0_F(U))^n$. It is trivial to check that $\tau(U)$ is an $0(U)$-homomorphism. Now with $g$ and $h$ as above and $u$ in $F^*$ we have

$$u \circ \tau(U)(h \otimes g) = u \circ (gh) = (u \circ g)h \in S(U) .$$

Thus $\tau(U)(h \otimes g) \in S'_F(U)$. It follows that the range of $\tau(U)$ is a subset of $S'_F(U)$. It is now clear that the family of mappings $\tau(U)$ induces an 0-homomorphism $\tau$ of $S_F$ into $S'_F$. To show that $\tau$ is one-to-one we must prove

(a) If $\tau(U)(\sum_{i=1}^N h_i \otimes g_i) = 0$ then each $p$ in $U$ has a neighborhood $V$ such that $r_{VU}(\sum_{i=1}^N h_i \otimes g_i) = 0$.

To show that $\tau$ is onto we must prove

(b) If $f \in S'_F(U)$ then each p in $U$ has a neighborhood $V$ such that $r_{VU}f = \tau(V)(\sum_{i=1}^N h_i \otimes g_i)$ for some elements $h_i$ in $S(V)$ and $g_i$ in $0_F(V)$.

We first prove (a). If we let $R$ be the sheaf of relations on $U$ of $h_1, \cdots, h_N$ by the coherence of $R$ there exists a neighborhood $V_0$ of $p$ and elements $r_1 = (r_1^1, \cdots, r_1^N), \cdots, r_n = (r_n^1, \cdots, r_n^N)$ of $R(V_0)$ which generate $R$ at each point of $V_0$. Now

$$\sum_{i=1}^{N} g_i h_i = \tau(U)\Big(\sum_{i=1}^{N} h_i \otimes g_i\Big) = 0 .$$

Thus for each $u$ in $F^*$ we have

$$\sum_{i=1}^{N} (u \circ g_i) h_i = 0$$

so that $(u \circ g_1, \cdots, u \circ g_N) \in R(U)$ for all $u$ in $F^*$. By definition this means that $(g_1, \cdots, g_N) \in R'_F(U)$. Therefore by Theorem 2 we see that there exists a neighborhood $V$ of $p$ and $G = (G_1, \cdots, G_n)$ in $(0_F(V))^n$ such that $(g_1, \cdots, g_N) = G_1 r_1 + \cdots + G_n r_n$. Thus on $V$ we have

$$\begin{aligned} \sum_{i=1}^{N} h_i \otimes g_i &= \sum_{i=1}^{N} h_i \otimes \left(\sum_{j=1}^{n} G_j r_j^i\right) \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{N} (r_j^i h_i)\right) \otimes G_j = 0 \end{aligned}$$

since $r_j \in R(V)$ for each $j$. This proves (a).

To prove (b) notice by Theorem 2 that there exists a neighborhood $V$ of $p$, elements $h_1, \cdots, h_N$ in $S(V)$, and elements $g_1, \cdots, g_N$ in $0_F(V)$ such that on $V$ we have

$$f = \sum_{i=1}^{N} g_i h_i = \tau(V)\left(\sum_{i=1}^{N} h_i \otimes g_i\right) .$$

This completes the proof of Theorem 3.

We state for future reference a version of a theorem of Banach, first giving a definition.

DEFINITION 3. If $\{g_n\}$ is a sequence of vectors in a Frechet space $F_\infty$ the series $\sum_{n=1}^{\infty} g_n$ is called *absolutely convergent* if the series $\sum_{n=1}^{\infty} \| g_n \|$ converges for each continuous semi-norm $\| \quad \|$ on $F$.

Notice that a continuous linear transformation from a Frechet space $F$ to a Frechet space $G$ takes absolutely convergent sequences into absolutely convergent sequences.

LEMMA 5. *Let $\sigma$ be a continuous linear map of a Frechet space $F$ onto a Frechet space $G$. Let $\{g_i\}$ be an absolutely convergent sequence from $G$. Then there exists an absolutely convergent sequence $\{f_i\}$ in $F$ such that $\sigma(f_i) = g_i$ for all $i$.*

*Proof*. Let $\{\| \quad \|_k\}$ be a defining sequence of semi-norms on $F$. Since the map $\sigma$ is continuous, we see ([1] p. 40) that for each $k$ the set $\sigma\{f : \| f \|_k \leqq 1\}$ contains a neighborhood $\{g : \| g \|'_k \leqq 1\}$ of 0 in $G$, where $\| \quad \|'_k$ is some continuous semi-norm on $G$. Thus for each $g$ in

$G$ and each $k$ there exists $f$ in $F$ with $\sigma(f) = g$ and $\|f\|_k \leqq \|g\|'_k$. Now for each $k$ choose $j = j(k)$ such that

$$\sum_{n=j}^{\infty} \|g_n\|'_k < 2^{-k},$$

so that

$$\sum_{k=1}^{\infty} \sum_{n=j(k)}^{\infty} \|g_n\|'_k < \infty .$$

We may assume that $j(1) < j(2) < \cdots$. For each $n$ with $j(k) \leqq n < j(k+1)$ choose $f_n$ in $F$ with $\sigma(f_n) = g_n$ and $\|f_n\|_k \leqq \|g_n\|'_k$. If for each $n$ we let $k(n)$ be the smallest value of $k$ for which $n < j(k+1)$, it follows that

$$\sum_{n=1}^{\infty} \|f_n\|_{k(n)} < \infty .$$

Since for each $t$ we have $\|f_n\|_t \leqq \|f_n\|_k$ for all $k \geqq t$ it follows that

$$\sum_{n=1}^{\infty} \|f_n\|_t$$

is finite for all $t$. This proves the lemma.

THEOREM 4. *If $S$ is a coherent analytic sheaf on a Stein manifold $M$ and if $F$ is a Frechet space then $H^N(M, S_F) = 0$ for all $N \geqq 1$.*

*Proof.* Let $f$ be an element of $H^N(M, S_F)$. Consider a locally finite covering $\{U_i\}$ of $M$ by holomorphically convex open sets $U_i$, so fine that $f$ is represented by an element of $H^N(\{U_i\}, S_F)$. For each finite sequence $K = (i_1, \cdots, i_k)$ of positive integers let $U_K = U_{i_1} \cap \cdots \cap U_{i_k}$. The element $f$ of $H^N(M, S_F)$ can be considered to belong to $H^N(\{U_i\}, S_F)$ and therefore can be represented by a cocycle $f = \{f_I\}$ of $Z^N(\{U_i\}, S_F)$. Here $I$ is any sequence of $N+1$ positive integers, and, for each $I$, $f_I$ is an element of $S_F(U_I)$. Also $\delta f = 0$, where $\delta$ is the coboundary operator from $C^N(\{U_i\}, S_F)$ into $C^{N+1}(\{U_i\}, S_F)$ and $Z^N(\{U_i\}, S_F)$ is the kernel of $\delta$. By choosing the covering $\{U_i\}$ fine enough we may assume that for each $K$ there exist elements $h_{1K}, \cdots, h_{\alpha K}$, with $\alpha$ depending on $K$, in $S(U_K)$ which generate $S$ at each point of $U_K$. This implies ([3], expose XVIII, p. 9) that every $h$ in $S(U_K)$ has a representation of the form $h = \sum_{i=1}^{\alpha} g_i h_{iK}$, with $g_i \in 0(U_K)$. We may also choose the covering $\{U_i\}$ so fine that, for each $I$, $f_I$ can be represented in the form

$$f_I = \sum_{i=1}^{\alpha} h_{iI} \otimes g_{iI}$$

with $h_{iI}$ as above and with $g_{iI}$ in $0_F(U_I)$.

By Theorem 1 there exists a sequence $\{P_n\}$ of continuous mutually annihilating projections on $F$ whose ranges are one dimensional and a sequence $\{b_n\}$ of vectors in $F$ bounded in each continuous semi-norm on $F$ having the following properties. For each $I$ and $i$ the series $\sum_{n=1}^{\infty} P_n \circ g_{iI}$ converges to $g_{iI}$ on $U_I$. For each $I$ and $i$ we have $P_n \circ g_{iI} = g_{iI}^n b_n$, where $g_{iI}^n \in 0(U_I)$. For each $I$ and $i$ the series $\sum_{n=1}^{\infty} g_{iI}^n$ converges absolutely in the Frechet space $0(U_I)$. Now since for each $n$ the projection $P_n$ induces a homomorphism of the sheaf $S_F$ onto itself, the element $\{P_n f_I\}$ of $C^N(\{U_i\}, S_F)$ is in $Z^N(\{U_i\}, S_F)$. Also

$$\begin{aligned} P_n f_I &= \sum_{i=1}^{\alpha} h_{iI} \otimes P_n g_{iI} \\ &= \sum_{i=1}^{\alpha} h_{iI} \otimes g_{iI}^n b_n = \left(\sum_{i=1}^{\alpha} g_{iI}^n h_{iI}\right) \otimes b_n \,. \end{aligned}$$

If for each $n$ and $I$ we let $f_I^n$ be the element $\sum_{i=1}^{\alpha} g_{iI}^n h_{iI}$ of $S(U_I)$ it follows that for each $n$ the element $f^n = \{f_I^n\}^{i=1}$ of $C^N(\{U_i\}, S)$ belongs to $Z^N(\{U_i\}, S)$. It is also clear that $f^n b_n = P_n f$.

Now there exists a natural Frechet space topology on each $S(U)$, described in [4], expose XVII. This topology has the property that for each $h$ in $S(U)$ the map $g \to gh$ of $0(U)$ into $S(U)$ is continuous. We therefore see that for each $I$ the series

$$\sum_{n=1}^{\infty} f_I^n = \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\alpha} g_{iI}^n h_{iI}\right)$$

converges absolutely in $S(U_I)$ because for each $I$ and $i$ the series $\sum_{n=1}^{\infty} g_{iI}^n$ converges absolutely in $0(U_I)$. Now the space $C^N(\{U_i\}, S)$ is the Cartesian product of the Frechet spaces $S(U_I)$, and therefore possesses a Frechet space structure. Moreover $Z^N(\{U_i\}, S)$ is closed in $C^N(\{U_i\}, S)$ and is therefore also a Frechet space. Since for each $I$ the series $\sum_{n=1}^{\infty} f_I^n$ converges absolutely in $S(U_I)$ it follows that $\sum_{n=1}^{\infty} f^n$ converges absolutely in $Z^N(\{U_i\}, S)$. By Theorem B of [3] and Leray's theorem (see [5] p. 213) we see that the coboundary map $\delta$ of the Frechet space $C^{N-1}(\{U_i\}, S)$ into $Z^N(\{U_i\}, S)$ is onto. From [4] we also see that $\delta$ is continuous.

Let $J$ stand for an arbitrary $N$-tuple of positive integers. Thus for each $J$, by the above, there is a continuous homomorphism.

$$\tau_J : (G_1, \cdots, G_\alpha) \to \sum_{i=1}^{\alpha} G_i h_{iJ}$$

of the Frechet space $(0(U_J))^\alpha$ onto the Frechet space $S(U_J)$. These maps induce a continuous homomorphism $\tau$ of the Frechet space $\Phi$ onto the Frechet space $C^{N-1}(\{U_i\}, S)$, where $\Phi$ is defined to be the product $\prod_J (0(U_J))^\alpha$, with $\alpha$ depending as above on $J$, of the Frechet spaces $(0(U_J))^\alpha$. Thus

$$\sigma = \delta \circ \tau$$

is a continuous homomorphism of $\Phi$ onto $Z^N(\{U_i\}, S)$. Since $\sum_{n=1}^{\infty} f^n$ converges absolutely in $Z^N(\{U_i\}, S)$ it follows from Lemma 5 that there exists an absolutely convergent sequence $\{G^n\}$ in $\Phi$ with $\sigma(G^n) = f^n$ for all $n$. For each $n$ write $G^n = \{G^n_J\}$, where

$$G^n_J = (G^n_{1J}, \cdots, G^n_{\alpha J}) \in (0(U_J))^\alpha .$$

Thus for each $J$ we see that the series $\sum_{n=1}^{\infty} G^n_J$ converges absolutely and uniformly on every compact subset of $U_J$, so that the series $\sum_{n=1}^{\infty} G^n_J b_n$ converges absolutely in $(0_F(U_J))^\alpha$ to an element

$$G_J = (G_{1J}, \cdots, G_{\alpha J})$$

in $(0_F(U_J))^\alpha$. Thus for each $i$ and $J$ we have $G_{iJ} = \sum_{n=1}^{\infty} G^n_{iJ} b_n$.

For each $J$ let $e_J$ be the element

$$e_J = \sum_{i=1}^{\alpha} h_{iJ} \otimes G_{iJ}$$

of $S_F(U_J)$. Thus $e = \{e_J\} \in C^{N-1}(\{U_i\}, S_F)$. We shall finish the proof by showing that $\delta e = f$. To this end it is sufficient by Lemma 4 to show $u'(\delta e) = u'(f)$ for all $u$ in $F^*$. We compute:

$$\begin{aligned} u'(e_J) &= \sum_{i=1}^{\alpha} \langle G_{iJ}, u\rangle h_{iJ} \\ &= \sum_{i=1}^{\alpha} \left\langle \sum_{n=1}^{\infty} G^n_{iJ} b_n, u \right\rangle h_{iJ} \\ &= \sum_{n=1}^{\infty} \left( \sum_{i=1}^{\alpha} G^n_{iJ} h_{iJ} \right) \langle b_n, u\rangle \\ &= \sum_{n=1}^{\infty} (\tau_J(G^n_J)) \langle b_n, u\rangle \end{aligned}$$

absolutely in $S(U_J)$. Thus

$$u'(e) = \sum_{n=1}^{\infty} (\tau(G^n)) \langle b_n, u\rangle$$

absolutely in $C^{N-1}(\{U_i\}, S)$. Thus

$$\begin{aligned} u'(\delta e) = \delta(u'(e)) &= \sum_{n=1}^{\infty} (\delta \circ \tau)(G^n) \langle b_n, u\rangle \\ &= \sum_{n=1}^{\infty} \sigma(G^n) \langle b_n, u\rangle = \sum_{n=1}^{\infty} f^n \langle b_n, u\rangle . \end{aligned}$$

Also for each $I$ we have

$$u'(f_I) = \sum_{i=1}^{\alpha} \langle g_{iI}, u\rangle h_{iI}$$

$$= \sum_{i=1}^{\alpha} \left\langle \sum_{n=1}^{\infty} g_{iI}^{n} b_n, u \right\rangle h_{iI}$$
$$= \sum_{n=1}^{\infty} \left( \sum_{i=1}^{\alpha} g_{iI}^{n} h_{iI} \right) \langle b_n, u \rangle = \sum_{n=1}^{\infty} f_I^n \langle b_n, u \rangle .$$

Therefore $u'(f) = \sum_{n=1}^{\infty} f^n \langle b_n, u \rangle$. It follows that $u'(f) = u'(\delta e)$ for all $u$ in $F^*$, so that $f = \delta e$. This completes the proof of Theorem 4.

## References

1. S. Banach, *Theorie des operations lineaires*, Warsaw, 1932.
2. E. Bishop, *Some global problems in the theory of functions of several complex variables* Amer. J. of Math., **83** (1961), 479-498.
3. H. Cartan, *Seminaire Ecole Normale Superieure*, 1951-1952.
4. ———, *Seminaire Ecole Normale Superieure*, 1952-1953.
5. R. Godement, *Theorie des faisceaux*, Paris, 1958.
6. A. Grothendieck, *Sur certains espoces de fonctions holomorphes. II.*, Jour. für die reine und angewandte Math., **192** (1953), 77-95.
7. A. E. Taylor, *A geometric theorem and its applications to biorthogonal systems*, Bull. Amer. Math. Soc., **53** (1947), 614-616.

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