

Pacific Journal of Mathematics

RINGS IN WHICH SEMI-PRIMARY IDEALS ARE PRIMARY

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Every ring considered in this paper will be assumed to be commutative and to have a unit element. An ideal A of a ring R will be called semi-primary if its radical \sqrt{A} is prime. That a semi-primary ideal need not be primary is shown by an example in [3; p. 154]. This paper is a study of rings R satisfying the following condition: (*) Every semi-primary ideal of R is primary. The ring Z of integers clearly satisfies (*). More generally, if A is a semi-primary ideal of a ring R such that \sqrt{A} is a maximal ideal of R , then A is primary. [3; p. 153]. Hence, every ring having only maximal nonzero prime ideals satisfies (*).

An ideal A of a ring R is called P -primary if A is primary and $P = \sqrt{A}$. If ring R satisfies (*), then A is \sqrt{A} -primary if and only if \sqrt{A} is prime. Some well-known properties of a ring R satisfying (*) are listed below.

Property 1. If R satisfies (*) and A is an ideal of R , then R/A satisfies (*). [3; p. 148].

Property 2. If R satisfies (*), if A and B are ideals of R such that $A \subseteq B \subseteq \sqrt{A}$, and if A is \sqrt{A} -primary then B is \sqrt{A} -primary. [3; p. 147].

THEOREM 1. *If ring R satisfies (*) and P, A , and Q are ideals of R such that P is prime, $P \subset A$, and Q is P -primary, then $QA = Q$.*

Proof. Since $\sqrt{QA} = P$, QA is P -primary. Thus $Q \cdot A \subseteq QA$ and $A \not\subseteq P$ imply that $Q \subseteq QA \subseteq Q$. Hence $QA = Q$ as asserted.

THEOREM 2. *If P is a nonmaximal prime ideal in a ring R satisfying (*) and if Q is P -primary, then $Q = P$.*

Proof. We let P_1 be a proper maximal ideal properly containing P . If $p_1 \in P_1$ such that $p_1 \notin P$ and if $p \in P$, then $Q \subseteq Q + (pp_1) \subseteq P$. By property 2, $Q + (pp_1)$ is P -primary. Since $pp_1 \in Q + (pp_1)$ and $p_1 \notin P$, $p \in Q + (pp_1)$. Then for some $q \in Q$, $r \in R$, $p(1 - rp_1) = q$. Now $1 - rp_1 \notin P_1$ since $P_1 \subset R$ so that $1 - rp_1 \notin P$. Thus $p \in Q$ and $P \subseteq Q \subseteq P$. Hence $P = Q$ and our proof is complete.

COROLLARY 2.1. *If ring R satisfies (*), if P_1 and P_2 are prime ideals of R with $P_1 \subset P_2$, and if Q is P_2 -primary, then $P_1 \subset Q$.*

Proof. Since $\sqrt{QP_1} = P_1$, QP_1 is P_1 -primary. By Theorem 2, $P_1 = QP_1 \subseteq Q$. Now Q is P_2 -primary so that $P_1 \neq Q$. Hence $P_1 \subset Q$.

COROLLARY 2.2. *If ring R satisfies (*) and P is a nonmaximal prime ideal of R , then P is idempotent.*

Proof. The ideal P^2 has radical P and is therefore P -primary. By Theorem 2, $P^2 = P$.

THEOREM 3. *If R is a ring satisfying (*), if d is not a zero divisor or unit of R , and if P is a minimal prime ideal of (d) , then P is maximal in R .*

Proof. Suppose that P is not maximal in R . Let M denote the complement of P in R . We define A to be the set of all those elements x of R such that there exists $m \in M$ such that $xm \in (d)$. Since P is prime, A is an ideal and $A \subseteq P$. We wish to show that $P = A$. Thus if $p \in P$ and if N is the set of all elements of R of the form $p^k m$ where k is a nonnegative integer and $m \in M$, then N is a multiplicatively closed set containing M and p and hence properly containing M . Because P is a minimal prime ideal of (d) , M is a maximal multiplicatively closed subset of R not meeting (d) . [2; p. 106]. Therefore $N \cap (d) \neq \phi$ so that there exists an integer $k > 0$ and an element m of M such that $p^k m \in (d)$. That is, $p^k \in A$ so that $p \in \sqrt{A}$. Hence $P \subseteq \sqrt{A} \subseteq \sqrt{P} = P$ which implies $P = \sqrt{A}$. This means that A is P -primary. Under the assumption that P is nonmaximal, we conclude that $P = A$ by Theorem 2. Now P is also a minimal prime ideal of (d^2) so that if B is the set of elements y of R such that $ym \in (d^2)$ for some $m \in M$, we likewise have $P = B$. Since $d \in P$, there exist $m \in M$ and $r \in R$ such that $dm = rd^2$. The element d is not a zero divisor so that $m = rd \in (d) \subseteq P$ which is a contradiction to our choice of m . Therefore P is maximal as the theorem asserts.

COROLLARY 3.1. *If ring R satisfies (*) and if P is a proper prime ideal of R containing a nonzero divisor d , then P is maximal in R .*

Proof. There is a minimal prime ideal P_1 of (d) contained in P . [1; p. 9]. By Theorem 3, P_1 is maximal. Hence P is also maximal.

COROLLARY 3.2. *If J is an integral domain satisfying (*), then nonzero proper prime ideals of J are maximal.*

COROLLARY 3.3. *If ring R satisfies (*) and if P is a proper prime ideal of R , then P is either maximal or minimal.*

Proof. Suppose that P is not minimal and let P_1 be a prime ideal properly contained in P . Now R/P_1 is an integral domain satisfying (*) by property 1. By Corollary 3.2, P/P_1 is maximal in R/P_1 . Thus P is maximal in R . [3; p. 151].

THEOREM 4. *If ring R satisfies (*) and P is a finitely generated nonmaximal prime ideal of R then P is a direct summand of R . If P_1 is a prime ideal not containing P , then P and P_1 are relatively prime.*

PROOF. By Corollary 2.2, $P = P^2$. Since P is finitely generated, there exists an element $e \in P$ such that $(1 - e)P = (0)$. [3; p. 215]. Evidently $e^2 = e$, $P = (e)$ and $R = P \oplus (1 - e)$. Now $e(1 - e) \in P_1$ and $e \notin P_1$ so that $1 - e \in P_1$. Therefore $1 = e + (1 - e) \in P + P_1$ so that P and P_1 are relatively prime.

THEOREM 5. *If the Noetherian ring S satisfies (*), S is a finite direct sum of Noetherian primary rings and Noetherian integral domains in which nonzero proper prime ideals are maximal. Conversely if T is a finite direct sum of Noetherian primary rings and Noetherian integral domains in which nonzero proper prime ideals are maximal, then T is a Noetherian ring satisfying (*).*

Proof. Since S is Noetherian, every ideal of S is finitely generated. Let $(0) = Q_1 \cap \cdots \cap Q_s$ be an irredundant representation of (0) as an intersection of greatest primary components where $P_i = \sqrt{Q_i}$. If P_1, P_2, \dots, P_k are the nonmaximal prime ideals of S in this collection, $P_i = Q_i$ for $1 \leq i \leq k$ by Theorem 2. If $1 \leq i < j \leq s$, $P_i + P_j = S$. This follows from Theorem 4 if P_i and P_j are nonmaximal. If P_j , say, is maximal, then $P_j \not\subseteq P_i$ by Corollary 2.1, for $Q_j \not\subseteq P_i$ from the irredundance of the representation. Therefore, $P_i + P_j = S$. Thus the P_i 's, and hence the Q_i 's, are pairwise relatively prime. [3; p. 177]. This means that $S \cong S/P_1 \oplus \cdots \oplus S/P_k \oplus S/Q_{k+1} \oplus \cdots \oplus S/Q_s$. [3; p. 178]. Each S/P_i in this representation is a Noetherian integral domain in which nonzero prime ideals are maximal. Since Q_j for $k + 1 \leq j \leq s$ is P_j -primary with P_j maximal, S/Q_j is a Noetherian primary ring. [3; p. 204].

The converse follows from elementary facts concerning the ideal theory in a finite direct sum since it is apparent that each summand satisfies (*).

We give the following example of ring which is not a finite direct

sum of indecomposable summands and which satisfies (*).

Let $S = \sum_{i=1}^{\infty w} Z_i$, where each Z_i is the ring of integers and $\sum_{i=1}^{\infty w}$ designates the weak direct sum. Let $R = S + Z$ be the usual extension of S to a ring with unit element. [2; p. 87]. Clearly S is a prime ideal of R , as is $I_p = S + pZ$ for every prime p of Z . In fact, each I_p is a maximal ideal of R . It is easy to show that there is no prime ideal P between S and I_p .

Next, assume that P is a prime ideal of R that does not contain all of S . Then some $e_k \notin P$, where e_k is the unity of Z_k . However, since $e_j e_k = 0$ for every $j \neq k$, evidently $Z_k \subset P$ for every $j \neq k$. By the same reasoning, $(1 - e_k)R \subseteq P$. As before, it is easily shown that either $P = (1 - e_k)R$ or $P = (1 - e_k)R + pe_k R$ for some prime p .

Knowing precisely what the prime ideals of R are, it is just a routine matter to check that R satisfies (*).

The author is not able to give necessary and sufficient conditions which he feels are adequate that an arbitrary ring satisfy (*). The condition of Corollary 3.3, while necessary, is not sufficient to imply that a ring satisfy (*) as is shown by the following example.

If S is the ring of polynomials in two indeterminates X and Y over a field K , then every nonzero proper prime ideal of S has height 1 or 2. [4; p. 193]. Therefore if $A = (XY)$ and if $R = S/A$, R is a Noetherian ring in which every prime ideal is maximal as minimal. The nonmaximal prime ideal $(X)/A$ of R , however, is not idempotent so that R does not satisfy (*).

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50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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