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ON THE ADDITIVITY OF LATTICE COMPLETENESS

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ON THE ADDITIVITY OF LATTICE COMPLETENESS

to the memory of Maurice Audin ISRAEL HALPERIN AND MARIA WONENBURGER

1. Introduction. It was shown in [1, Theorem 4.3] that upper \aleph -continuity¹ is additive in the following sense:

(1.1) Suppose that [0, a], [0, b] are upper \aleph -continuous in a relatively complemented modular lattice. Then $[0, a \cup b]$ is upper \aleph -continuous provided that $[0, a \cup b]$ is upper \aleph -complete.

But it may happen that [0, a], [0, b] are both upper \aleph -complete (both may even be von Neumann geometries with a perspective to b) and yet $[0, a \cup b]$ is not upper \aleph -complete. In fact there are von Neumann rings \mathscr{R} for which the lattice $\overline{R}_{\mathscr{G}}$, with $\mathscr{S} = \mathscr{R}_2$, is not even upper \aleph_0 -complete (see the Remark preceding Definition 3.1)

With a modest supplementary condition however, additivity of upper \aleph -completeness does hold, as we show in this paper.

2. Terminology and notation. We shall use the notation of [1], [2], and [4].

I will denote a set of indices α and \overline{I} will denote the cardinal power of I.

 \aleph will denote an infinite cardinal, Ω will denote the least ordinal number whose corresponding cardinal power is \aleph .

A lattice is called upper \aleph -complete if the union $a = \bigcup (a_{\alpha} | \alpha \in I)$ exists whenever $\overline{I} \leq \aleph$, and is called upper \aleph -continuous if for every b: $b \cap a = \bigcup ((b \cap \bigcup (a_{\alpha} | \alpha \in F)))$ all finite $F \subset I$, with dual definitions for lower \aleph -completeness and lower \aleph -continuity. The lattice is called \aleph complete, respectively \aleph -continuous if it is both upper and lower \aleph -continuous.

A complemented modular lattice L is called an \aleph -von Neumanngeometry if it is \aleph -complete and \aleph -continuous (irreducibility is not assumed).

If we omit the \aleph in any of these designations, this implies that the lattice L has the corresponding \aleph -property for all \aleph .

If \mathscr{R} is an associative regular ring (not necessarily with unit element) then $\overline{R}_{\mathscr{R}}$ denotes the relatively complemented modular lattice of its principal right ideals, ordered by inclusion. \mathscr{R} is called an \aleph -von Neumannring, respectively a von Neumann ring, according as $\overline{R}_{\mathscr{R}}$ is an \aleph -von

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¹ Terminology and notation are explained in section 2 below.

Neumann-geometry, respectively a von Neumann geometry.

In any relatively complemented modular lattice, if $a \ge b$ then [a - b]will denote an arbitrary (but fixed) element such that $[a - b] \dot{\bigcup} b = a$ (the dot indicates that the summands in the union are independent). We write $a \sim b$ to denote: a is perspective to b, and $a \le b$ to denote: $a \sim b_1$ for some $b_1 \le b$. Elements a, b are called *completely disjoint*, (notation: (a, b)P) if: $a_1 \sim b_1$, $a_1 \le a$, $b_1 \le b$ together imply $a_1 = 0$.

3. The additivity of completeness theorem.

In this section a, b, c, $\cdots x_{\alpha}$, \cdots will denote elements in a given relatively complemented modular lattice L.

If $[0, a \cup c]$ is upper \aleph -complete we shall write $u(a, c, \aleph)$ to mean:

(3.1) Whenever $x_{\alpha} \leq a \cup c$ for all $\alpha \in I$ (with $\overline{I} \leq \aleph$) and

 $a \cap (\bigcup (x_{\beta} | \beta \in F)) = 0$

for all finite $F \subset I$, then $a \cap (\bigcup (x_{\alpha} | \alpha \in I)) = 0$.

It is important to note: if $u(a, c, \aleph)$ holds then $u(a', c', \aleph)$ holds for all $a' \leq a, c' \leq c$.

Clearly, if $[0, a \cup c]$ is upper \aleph -complete and upper \aleph -continuous then $u(a, c, \aleph)$ does hold.

Similarly, if $[0, a \cup c]$ is lower \aleph -complete we shall write $l(a, c, \aleph)$ to denote:

 $(\overline{3.1}) \quad Whenever \ x_{\alpha} \leq a \cup c \ for \ all \ \alpha \in I \ (with \ \overline{I} \leq \aleph) \ and$

 $a \cup (\bigcap (x_{\beta} | \beta \in F)) = a \cup c$

for all finite $F \subset I$, then $a \cup (\bigcap (x_{\alpha} | \alpha \in I)) = a \cup c$.

It is important to note: if $l(a, c, \aleph)$ holds then $l(a', c', \aleph)$ holds for all $a' \leq a, c' \leq c$.

Clearly, if $[0, a \cup c]$ is lower \aleph -complete and lower \aleph -continuous then $l(a, c, \aleph)$ does hold.

LEMMA 3.1. Suppose that each of $[0, a \cup b]$, $[0, b \cup c]$, $[0, a \cup c]$ is upper \aleph -complete and suppose that $u(a, c, \aleph)$ holds. Then $[0, a \cup b \cup c]$ is upper \aleph -complete.

Proof. We may suppose that $\{a, b, c\}$ is an independent set, for if c, b are replaced by $[c - (a \cap c)]$ and $[b - (b \cap (a \cup c))]$ respectively the hypotheses of Lemma 3.1 continue to hold and the conclusion is not changed.

Using transfinite induction, we may suppose that Lemma 3.1 holds

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for all $\aleph' < \aleph$. We may therefore assume that x_{α} is given, $\leq a \cup b \cup c$ for all $0 < \alpha < \Omega$, that $\bigcup (x_{\alpha} | \alpha \leq \beta)$ exists for all $\beta < \Omega$ and we need only show that $\bigcup (x_{\alpha} | \alpha < \Omega)$ exists.

We may suppose $x_{\alpha} \leq x_{\beta}$ for $\alpha \leq \beta < \Omega$ (by replacing the original x_{α} by $\bigcup (x_{\beta} | \beta \leq \alpha)$ for all $(\alpha < \Omega)$.

Set $\overline{x}_0 = \bigcup((x_{\alpha} \cap (a \cap b)) | \alpha < \Omega)$ (this union exists since, by hypothesis, [0, $a \cup b$] is upper \aleph -complete). Set $\overline{x}_{\alpha} = \overline{x}_0 \cup x_{\alpha}$ for $0 < \alpha < \Omega$ and observe that $\overline{x}_{\beta} \leq \overline{x}_{\alpha}$ for all $0 \leq \beta \leq \alpha < \Omega$.

Set $y_0 = \bar{x}_0$ and $y_{\alpha} = [\bar{x}_{\alpha} - \bigcup(\bar{x}_{\beta}|0 \leq \beta < \alpha)]$ for $0 < \alpha < \Omega$. Then $\bigcup(y_{\beta}|0 \leq \beta < \alpha) = \bigcup(\bar{x}_{\beta}|0 \leq \beta < \alpha)$ for all $0 < \alpha < \Omega$, as may be verified easily by transfinite induction.

Clearly, we need only show that $\bigcup (y_{\alpha}|0 \leq \alpha < \Omega)$ exists. Hence it is sufficient to show that $\bigcup_{\alpha} y_{\alpha}$ exists, where (for the rest of this proof) we write \bigcup_{α} to mean $\bigcup_{0 < \alpha < \Omega}$ (note: $0 \leq \alpha < \Omega$ has been replaced by $0 < \alpha < \Omega$).

Set $u = (a \cup (\bigcup_{\alpha}((a \cup y_{\alpha}) \cap (b \cup c)))) \cap (b \cup (\bigcup_{\alpha}((b \cup y_{\alpha}) \cap (a \cup c))))$ (this union exists since, by hypothesis, $[0, b \cup c]$ and $[0, a \cup c]$ are upper \clubsuit -complete). We observe that $u \ge y_{\beta}$ for all $0 < \beta < \Omega$ since each factor of u has this property: for fixed β , $a \cup (\bigcup_{\alpha}((a \cup y_{\alpha}) \cap (b \cup c))) \ge$ $a \cup ((a \cup y_{\beta}) \cap (b \cup c)) = (a \cup y_{\beta}) \cap (a \cup b \cup c) = a \cup y_{\beta} \ge y_{\beta}$.

We shall show that u is the desired union $\bigcup_{\alpha} y_{\alpha}$. It is clearly sufficient to show for every w: if $u \ge w \ge y_{\alpha}$ for all $0 < \alpha < \Omega$ then $u \le w$. Since $a \cup y_{\alpha} \le a \cup w$ and $b \cup y_{\alpha} \le b \cup w$ for all $0 < \alpha < \Omega$,

$$u \leq (a \cup ((a \cup w) \cap (b \cup c))) \cap (b \cup ((b \cup w) \cap (a \cup c)))$$

= $(a \cup w) \cap (b \cup w) = w \cup (a \cap (b \cup w))$.

It is therefore sufficient to show that $a \cap (b \cup w) \leq w$. We shall show that $a \cap (b \cup u) = 0$; this will imply:

$$a \cap (b \cup w) \leq a \cap (b \cup u) = 0 \leq w$$
.

Now $b \cup u = (a \cup b \cup (\bigcup_{\alpha} (a \cup y_{\alpha}) \cap (b \cup c)))) \cap (b \cup (\bigcup_{\alpha} ((b \cup y_{\alpha}) \cap (a \cup c)))),$

$$a \cap (b \cup u) = a \cap (b \cup (\bigcup_{\alpha} ((b \cup y_{\alpha}) \cap (a \cup c))))$$

= $a \cap ((b \cap (a \cup c)) \cup (\bigcup_{\alpha} ((b \cup y_{\alpha}) \cap (a \cup c))))$
= $a \cap (\bigcup_{\alpha} ((b \cup y_{\alpha}) \cap (a \cup c)))$.

Since $u(a, c, \aleph)$ is assumed to hold we need only show:

$$a \cap (\bigcup(((b \cup y_{\alpha}) \cap (a \cup c)) | \alpha = \alpha_1, \cdots, \alpha_m)) = 0$$

for every finite set of indices $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < \Omega$. Hence it is sufficient to show that

$$a \cap (b \cup (\bigcup(y_{\alpha} | \alpha = \alpha_1, \cdots, \alpha_m))) = 0$$
,

and so it is sufficient to show that

$$(3.2) (a \cup b) \cap (\bigcup (y_{\alpha} | \alpha = \alpha_1, \cdots, \alpha_m)) = 0.$$

For this purpose, we note: $y_{\alpha} \cap (\bigcup(y_{\beta}|0 \leq \beta < \alpha) = 0 \text{ for all } 0 < \alpha < \Omega$. This implies that $\{y_{\alpha}|\alpha = 0, \alpha_1, \dots, \alpha_m\}$ is an independent set and hence $y_0 \cap (\bigcup(y_{\alpha}|\alpha = \alpha_1, \dots, \alpha_m)) = 0$. This implies (3.2) since the left side of (3.2) is $\leq y_0$. Thus Lemma 3.1 is proved.

COROLLARY 1. Suppose that $[0, a_i \cup a_j]$ is upper \aleph -complete for $i, j = 1, \dots, m$ for some finite integer m and suppose that $u(a_i, a_j, \aleph)$ holds whenever i < j. Then $[0, a_1 \cup \cdots \cup a_m]$ is upper \aleph -complete.

Proof. If $m \leq 2$ the conclusion is part of the hypotheses. Suppose that m > 2 and that the Corollary is known to hold with m - 1 in place of m; then Lemma 3.1 can be applied (with $a = a_1$, $b = a_3 \cup \cdots \cup a_m$ and $c = a_2$) to show that the Corollary holds for m itself. By induction on m, the Corollary is established.

COROLLARY 2. Suppose that $[0, a_i \cup a_j]$ is upper \aleph -complete and upper \aleph -continuous for $i, j = 1, \dots, m$ for some finite integer m. Then $[0, a_1 \cup \dots \cup a_m]$ is upper \aleph -complete and upper \aleph -continuous.

Proof. Since upper \aleph -continuity of $[0, a_i \cup a_j]$ implies that $u(a_i, a_j, \aleph)$ holds, Corollary 1 shows that $[0, a_1 \cup \cdots \cup a_m]$ is upper \aleph -complete. The upper \aleph -continuity then follows from [1, Theorem 4.3].

LEMMA 3.2. Suppose that $a = a_1 \cup a_2 \cup \cdots \cup a_m$ and $a_i \leq a_1 \cup \cdots \cup a_{i-1}$ for $1 < i \leq m$. Then a can be expressed in the form:

(3.3) $a_1 \stackrel{.}{\cup} \overline{a_2} \stackrel{.}{\cup} \cdots \stackrel{.}{\cup} \overline{a_n}$ for some $n \ge m$ and elements $\overline{a_2}, \cdots, \overline{a_n}$ such that $\overline{a_i} \le a_1$ for all $1 < i \le n$.

Moreover \bar{a}_2 may be taken to coincide with a_2 if $a_1 \cap a_2 = 0$.

Proof. Lemma 3.2 holds trivially if m = 1 and also if m = 2 and $a_1 \cap a_2 = 0$. We may therefore suppose (by induction) that m > 1 and that $b = a_1 \cup \cdots \cup a_{m-1}$ has the form (3.3).

We can replace a_m by $[a_m - (a_m \cap b)]$ since the hypotheses of Lemma 3.2 continue to hold and the conclusion is not changed. After this change,

 $a_m \cap b = a_m \cap (a_1 \stackrel{.}{\cup} \overline{a}_2 \stackrel{.}{\cup} \cdots \stackrel{.}{\cup} \overline{a}_n) = 0$.

Since $a_m \leq a_1 \stackrel{.}{\cup} \overline{a}_2 \stackrel{.}{\cup} \cdots \stackrel{.}{\cup} \overline{a}_n$ there is a perspectivity mapping φ of $[0, a_m]$ with $\varphi(a_m) \leq b$. Then

$$a_m = a_{m,1} \stackrel{\cdot}{\cup} a_{m,2} \stackrel{\cdot}{\cup} \cdots \stackrel{\cdot}{\cup} a_{m,n}$$

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where

$$\varphi(a_{m,1}) = \varphi(a_m) \cap a_1$$
,

and for $1 < i \leq n$,

$$arphi(a_{m,i}) = [(arphi(a_m) \cap (a_1 \stackrel{.}{\cup} \overline{a}_2 \stackrel{.}{\cup} \cdots \stackrel{.}{\cup} \overline{a}_i)) \ - (arphi(a_m) \cap (a_1 \stackrel{.}{\cup} \overline{a}_2 \stackrel{.}{\cup} \cdots \stackrel{.}{\cup} \overline{a}_{i-1})] \;.$$

Obviously, $a_{m,1} \leq a_1$. If i > 1 then $a_{m,i} \sim \varphi(a_{m,i})$; $\varphi(a_{m,i}) \leq \overline{a}_i$; $\overline{a}_i \leq a_1$; and $a_{m,i} \cap (\varphi(a_{m,i}) \cup \overline{a}_i \cup a_1) = 0$; these facts imply that $a_{m,i} \leq a_1$ (use (2.2) of [1]). The conclusion of Lemma 3.2 now follows at once.

LEMMA 3.3. Suppose that

(i) $a = a_1 \cup a_2 \cup \cdots \cup a_m$ for some finite $m \ge 2$, (ii) $a_2 \sim a_1$, (iii) $a_i \le a_1 \cup \cdots \cup a_{i-1}$ for $2 < i \le m$, (iv) $[0, a_1 \cup a_2]$ is upper \mathbf{K} -complete, (v) $u(a_1, a_2, \mathbf{K})$ holds.

Then [0, a] is upper \mathbf{K} -complete.

Proof. Applying Lemma 3.2, and using a new m and new elements a_3, \dots, a_m we may suppose that (i), (iii) hold in the strengthened form: $a = a_1 \stackrel{.}{\cup} a_2 \stackrel{.}{\cup} \cdots \stackrel{.}{\cup} a_m$ and $a_i \leq a_1$ for $2 < i \leq m$.

Suppose that $1 \leq i < j \leq m$. If $i \neq 2$ then $a_j \leq a_2$ (because of (ii)) and there is a perspectivity mapping φ of $[0, a_i \cup a_j]$ with $\varphi(a_i) \leq a_1$ and $\varphi(a_j) \leq a_2$. Hence $[0, a_i \cup a_j]$ is upper \aleph -complete and $u(a_i, a_j, \aleph)$ holds in this case.

If i = 2 there is a perspectivity mapping φ of $[0, a_2 \cup a_j]$ with $\varphi(a_2) = a_1, \varphi(a_j) = a_j$; the result for $[0, a_1 \cup a_j]$ obtained previously now implies: $[0, a_2 \cup a_j]$ is upper \aleph -complete and $u(a_2, a_j, \aleph)$ holds.

Corollary 1 to Lemma 3.1 now applies to these elements a_1, \dots, a_m and this completes the proof of Lemma 3.3.

COROLLARY. Suppose that the hypotheses (i), (ii), (iii), of Lemma 3.3 hold and suppose also that

(vi) $[0, a_1 \cup a_2]$ is upper \aleph -complete and upper \aleph -continuous.

Then [0, a] is upper \aleph -complete and upper \aleph -continuous.

Proof. (vi) implies (iv), (v). Hence [0, a] is upper X-complete by Lemma 3.3. Upper X-continuity then follows from [1, Theorem 4.3].

LEMMA 3.4. (Additivity of lower \aleph -continuity). Suppose that $[0, a_1 \cup \cdots \cup a_m]$ is lower \aleph -complete and that $[0, a_i]$ is lower \aleph -

continuous for $i=1, \dots, m$. Then $[0, a_1 \cup \dots \cup a_m]$ is lower \aleph -continuous.

Proof. We may assume that $\{a_1, \dots, a_m\}$ is an independent set (replace a_i by $[a_i - (a_i \cap (a_1 \cup \dots \cup a_{i-1}))]$ for $2 \leq i \leq m$).

Then $[a_1, a_1 \cup a_2]$ is lower \aleph -continuous since it is lattice isomorphic to $[0, a_2]$ under the mapping: $x \to x \cap a_2$. Similarly $[a_2, a_1 \cup a_2]$ is lower \aleph -continuous. By the dual of [1, Theorem 4.3], $[0, a_1 \cup a_2] = ([a_1 \cap a_2, a_1 \cup a_2])$ is lower \aleph -continuous. Lemma 3.4 follows by induction on m.

LEMMA 3.5. Suppose that each of $[0, a \cup b]$, $[0, b \cup c)$, $[0, a \cup c]$ is lower \aleph -complete and suppose that $l(a, c, \aleph)$ holds. Then $[0, a \cup b \cup c]$ is lower \aleph -complete.

Proof. We may suppose that $\{a, b, c\}$ is an independent set, for if c, b are replaced by $[c - (a \cap c)]$ and $[b - (b \cap (a \cup c))]$ respectively the hypotheses of Lemma 3.5 continue to hold $(l(a, c_1, \aleph))$ is equivalent to $l(a, c, \aleph)$ if $a \cup c_1 = a \cup c$ and the conclusion is not changed.

Now set $B = a \cup c$, $C = b \cup a$, $A = b \cup c$, and $1 = a \cup b \cup c$. We have: $[A \cap B, 1] (= [c, a \cup b \cup c])$ is lower \mathbb{K} -complete since it is lattice isomorphic to $[0, a \cup b]$ under the mapping $x \to x \cap (a \cup b)$. Similarly each of $[B \cap C, 1]$, $[C \cap A, 1]$ is lower \mathbb{K} -complete.

We can now show that $[0, a \cup b \cup c] (= [A \cap B \cap C, 1])$ is lower \aleph -complete (by applying the dual of Lemma 3.1) if we can show:

(3.4) Whenever $X_{\alpha} \ge C \cap A$ for $\alpha \in I$ (with $\overline{I} \le \aleph$) and $C \cup (\bigcap (X_{\beta} | \beta \in F)) = 1$ for all finite $F \subset I$, then $C \cup (\bigcap (X_{\alpha} | \alpha \in I)) = 1$.

Since $C \cap A = b$ and $C = a \cup b$, (3.4) can be rewritten:

(3.4)' Whenever $X_{\alpha} \geq b$ for $\alpha \in I$ (with $\overline{I} \leq \mathfrak{R}$) and $a \cup (\bigcap (X_{\beta} | \beta \in F)) = a \cup b \cup c$ for all finite $F \subset I$ then $a \cup (\bigcap (X_{\alpha} | \alpha \in I)) = a \cup b \cup c$.

Suppose that the hypotheses of (3.4)' hold and set $x_{\alpha} = X_{\alpha} \cap (\alpha \cup c)$. Then $x_{\alpha} \leq a \cup c$ for all α and

$$a \cup (\bigcap (x_{\beta}|\beta \in F))$$

= $a \cup ((\bigcap (X_{\beta}|\beta \in F)) \cap (a \cup c)) = (a \cup (\bigcap (X_{\beta}|\beta \in F))) \cap (a \cup c)$
= $(a \cup b \cup c) \cap (a \cup c) = a \cup c$.

Since $l(a, c, \aleph)$ holds, it follows that

$$a \cup (\bigcap (x_{\alpha} | \alpha \in I)) = a \cup c ; a \cup (\bigcap (X_{\alpha} | \alpha \in I) \cap (a \cup c)) = a \cup c ;$$

 $a \cup (\bigcap (X_{\alpha} | \alpha \in I)) \ge a \cup c \text{ (hence } = a \cup b \cup c) .$

This means: (3.4)' does hold. This completes the proof of Lemma 3.5.

COROLLARY 1. Suppose that $[0, a_i \cup a_j]$ is lower \aleph -complete for $i, j = 1, \dots, m$.

Suppose also that $l(a_i, a_j, \aleph)$ holds for all i < j. Then $[0, a_1 \cup \cdots \cup a_m]$ is lower \aleph -complete.

Proof. This follows from Lemma 3.5 by induction on m, just as Corollary 1 to Lemma 3.1 followed from Lemma 3.1.

COROLLARY 2. Suppose that $[0, a_i \cup a_j]$ is lower \aleph -complete and lower \aleph -continuous for $i, j = 1, \dots, m$. Then $[0, a_1 \cup \dots \cup a_m]$ is lower \aleph -continuous.

Proof. Since lower \aleph -continuity of $[0, a_i \cup a_j]$ implies that $l(a_i, a_j, \aleph)$ holds, Corollary 1 shows that $[0, a_1 \cup \cdots \cup a_m]$ is lower \aleph -complete. The lower \aleph -continuity of $[0, a_1 \cup \cdots \cup a_m]$ then follows from Lemma 3.4.

LEMMA 3.6. Suppose that

(i) $a = a_1 \cup a_2 \cup \cdots \cup a_m$ for some finite $m \ge 2$, (ii) $a_2 \sim a_1$, (iii) $a_i \le a_1 \cup \cdots \cup a_{i-1}$ for $2 < i \le m$, (iv) $[0, a_1 \cup a_2]$ is lower \Join -complete, (v) $l(a_1, a_2, \bigstar)$ holds.

Then [0, a] is lower \aleph -complete.

COROLLARY. Suppose that (i), (ii), (iii) hold and also

(vi) $[0, a_1 \cup a_2]$ is lower \aleph -complete and lower \aleph -continuous.

Then [0, a] is lower \aleph -complete and lower \aleph -continuous.

Proof. Lemma 3.6 and its Corollary follow from Lemma 3.5 and Lemma 3.4 just as Lemma 3.3 and its Corollary followed from Corollary 1 to Lemma 3.1 and [1, Theorem 4.3].

THEOREM 3.1. Suppose that each of $[0, a_i \cup a_j]$ is an \aleph -von Neumanngeometry (respectively a von Neumann-geometry) for $i, j = 1, \dots, m$. Then $[0, a_1 \cup \dots \cup a_m]$ is an \aleph -von Neumann-geometry (respectively a von Neumann geometry).

Proof. This follows from Corollary 2 to Lemma 3.1 and Corollary 2 to Lemma 3.5.

COROLLARY 1. Suppose that (i) $a = a_1 \cup a_2 \cup \cdots \cup a_m$ for some finite $m \ge 2$, (ii) $a_2 \sim a_1$,

(iii) $a_i \lesssim a_1 \cup \cdots \cup a_{i-1}$ for $2 < i \leq m$,

(iv) $[0, a_1 \cup a_2]$ is an \aleph -von Neumann-geometry (respectively a von Neumann-geometry).

Then [0, a] is an \aleph -von Neumann-geometry, respectively a von Neumann-geometry.

Proof. This follows from the Corollary to Lemma 3.3 and the Corollary to Lemma 3.6.

COROLLARY 2. Suppose that \mathscr{R} is an \aleph -von Neumann-ring (respectively a von Neumann-ring). If $\overline{R}_{\mathscr{R}}$ has a basis x_1, x_2, \dots, x_m such that $x_2 \sim x_1$ and $x_i \leq x_1$ for $2 < i \leq m$, then \mathscr{R}_2 is an \aleph -von Neumann-ring (respectively, a von Neumann-ring).

Proof. By hypothesis, the unit element of the lattice $\overline{R}_{\mathscr{R}}$ is the union $x_1 \stackrel{.}{\cup} \cdots \stackrel{.}{\cup} x_m$. The unit element of $\overline{R}_{\mathscr{G}}$, with $\mathscr{G} = \mathscr{R}_2$, can be represented as a union $x_1 \stackrel{.}{\cup} \cdots \stackrel{.}{\cup} x_m \stackrel{.}{\cup} y_1 \stackrel{.}{\cup} \cdots \stackrel{.}{\cup} y_m$ with $y_i \sim x_i$ and hence $y_i \leq x_1$ for $1 \leq i \leq m$. Since $[0, x_1 \stackrel{.}{\cup} x_2]$ is an \aleph -von Neumann geometry (respectively a von Neumann geometry) along with $\overline{R}_{\mathscr{R}}$, Corollary 1 applies and this completes the proof of Corollary 2.

COROLLARY 3. Suppose that \mathscr{R} and \mathscr{R}_2 are both \aleph -von Neumannrings (respectively von Neumann-rings). Then \mathscr{R}_n is an \aleph -von Neumann-ring (respectively a von Neumann-ring) for all finite n.

Proof. If n > 2 the unit element of $\overline{R}_{\mathscr{G}}$, with $\mathscr{G} = \mathscr{R}_n$, can be expressed as $x_1 \stackrel{.}{\cup} x_2 \stackrel{.}{\cup} \cdots \stackrel{.}{\cup} x_n$ where x_1 is the unit element of $\overline{R}_{\mathscr{R}}$, $x_i \sim x_1$ for all *i*, and $[0, x_1 \stackrel{.}{\cup} x_2] = \overline{R}_{\mathscr{R}_2}$. Theorem 3.1 applies and this completes the proof of Corollary 3.

REMARK. Let \mathscr{R} be the ring of sequences $x = (x^n)$ with all x^n complex numbers and all but a finite number of x^n real, with componentwise addition and multiplication; this example was given by Kaplansky [3, page 526]. This \mathscr{R} is a von Neumann-ring but \mathscr{R}_2 is not even upper \aleph_0 -complete.

DEFINITION 3.1. If L is a relatively complemented modular lattice, then an element a is called Boolean (with respect to L) if $b_1 \sim b_2$, $b_1 \leq a$ together imply $b_1 = b_2$; a is called the *Boolean part* of L (necessarily unique if it exists)² if a is Boolean and $a_1 \leq a$ for every Boolean a_1 .

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² This is an abuse of language: properly, [0, a] should be called the Boolean part of L_{\star}

LEMMA 3.7. Suppose that L is a relatively complemented modular lattice. If (a, b)P holds then for every c in L, $c \cap (a \cup b) = (c \cap a) \cup (c \cap b)$ and $[0, a \cup b]$ is the direct sum of [0, a] and [0, b]. On the other hand if a is Boolean then

- (i) $b \leq a$ implies that b is Boolean,
- (ii) $b \cap a = 0$ implies that (b, a)P holds,
- (iii) $b \ge a$ implies that the relative complement [b-a] is unique,
- (iv) $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$ for all b, c in L,
- (v) [0, a] is a Boolean algebra.

Proof. Suppose that (a, b)P holds and set $d = [(c \cap (a \cup b)) - ((c \cap a) \cup (c \cap b))], d_a = (d \cup b) \cap a, d_b = (d \cup a) \cap b$. Then $d \leq a \cup b, d \cap a = d \cap b = 0, d_a \cup d = (d \cup b) \cap (d \cup a) = d_b \cup d$, so $d_a \sim d_b$. Since $d_a \leq a, d_b \leq b$ and (a, b)P holds, we must have: $d_a = 0; b = d_a \cup b = d \cup b; d \leq b;$ hence $d = 0, c \cap (a \cup b) = (c \cap a) \cup (c \cap b)$. If $c \leq a \cup b$ then $c = (c \cap a) \cup (c \cap b);$ and if $c = c_1 \cup c_2$ with $c_1 \leq a, c_2 \leq b$ then $c \cap a = c_1 \cup (c_2 \cap b \cap a) = c_1 \cup 0 = c_1, c \cap b = c_2$. This proves that $[0, a \cup b]$ is the direct sum of [0, a] and [0, b].

(i) and (ii) are obvious from the definition of Boolean element.

(ii) asserts that a is in the centre of L as defined in [1, (2.5)]. But if a is in the centre of L and b is any element in L with $b \ge a$ then a is in the centre of [0, b], hence [b - a] is uniquely determined (use [1, (2.6)]). This proves (iii).

If b, c are arbitrary elements in L, set $b_1 = [b - (a \cap b)]$, $c_1 = [c - (a \cap c)]$. Since $a \cap b_1 = a \cap c_1 = 0$ and a is in the centre of L, it follows that $(a, b_1)P$, $(a, c_1)P$, hence $(a, b_1 \cup c_1)P$ (use [1, (2,6)]); therefore $a \cap (b_1 \cup c_1) = 0$. By the modular law

$$a \cap (b \cup c) = a \cap (b_1 \cup c_1 \cup (a \cap b) \cup (a \cap c))$$

= $(a \cap b) \cup (a \cap c) \cup (a \cap (b_1 \cup c_1))$
= $(a \cap b) \cup (a \cap c)$

and hence (iv) holds.

Thus [0, a] is a distributive complemented lattice, equivalently: a Boolean algebra. This proves (v).

LEMMA 3.8. Suppose that L has a unit element $1=a_1 \cup a_2 \cup \cdots \cup a_m$ with $m \ge 2$, $a_2 \sim a_1$, $a_i \le a_1$ for $2 < i \le m$ and $a_1 \cap a_2 = 0$. Then the Boolean part of L exists and is 0.

Proof. By Lemma 3.2 we may assume that $1 = a_1 \stackrel{.}{\cup} \cdots \stackrel{.}{\cup} a_m$ with $m \ge 2, a_2 \sim a_1$ and $a_i \le a_1$ for $2 < i \le m$.

To prove Lemma 3.8 we may suppose that $a \neq 0$ and we need only exhibit elements b_1 , b_2 such that $b_1 \leq a$, $b_1 \sim b_2$, and $b_1 \neq b_2$.

If $a_i \cap a \neq 0$ for any *i* it suffices to choose this element as b_1 since the relations $a_1 \sim a_2$ and $a_i \leq a_1$ if $i \neq 1$ imply $b_1 \sim b_2$ for some $b_2 \neq b_1$ (even $b_1 \cap b_2 = 0$).

On the other hand, if $a_i \cap a = 0$ for all i, set $b_1 = (a_1 \cup \cdots \cup a_i) \cap a$ where i is the smallest integer for which this element is different from 0 (necessarily $1 < i \leq m$) and set $b_2 = ((a_1 \cup \cdots \cup a_{i-1}) \cup b_1) \cap a_i$. Then $b_1 \sim b_2$ since $(a_1 \cup \cdots \cup a_{i-1}) \cup b_1 = (a_1 \cup \cdots \cup a_{i-1}) \cup b_2$; and $b_1 \neq b_2$ since $b_2 \leq a_i$ and $b_1 \cap a_i \leq a \cap a_i = 0$. This completes the proof of Lemma 3.8.

LEMMA 3.9. Suppose that L is an upper complete complemented modular lattice and let a be the union of all Boolean elements in L. Then a is the Boolean part of L.

Proof. We need only show that a is Boolean, that is, we may suppose that $b \leq a$, that φ is a perspective mapping of [0, b], that $b \neq \varphi(b)$ and we need only derive a contradiction. By replacing b by $[b - (b \cap \varphi(b))]$ we may suppose $b \neq 0$ and $b \cap \varphi(b) = 0$.

Now for every $c: (\varphi(b \cap c)) \sim (b \cap c)$ and $(\varphi(b \cap c)) \cap (b \cap c) = 0$. If c is Boolean this implies: $b \cap c = 0$, and hence (since c is Boolean) (b, c)P holds. It follows from [1, formula (2.6)] that (b, a)P holds, contradicting the fact that $b \neq 0$ and $b \leq a$. This contradiction proves Lemma 3.9.

THEOREM 3.2. Suppose that L is a relatively complemented modular lattice and

(i) $a = a_0 \cup a_1 \cup a_2 \cup \cdots \cup a_m$ for some finite $m \ge 2$,

(ii) $(a_0, a_1 \cup \cdots \cup a_m)P$ holds,

(iii) $a_2 \sim a_1, a_2 \cap a_1 = 0$,

(iv) $a_i \leq a_1 \cup \cdots \cup a_{i-1}$ for $2 < i \leq m$,

(v) φ is a perspective mapping of [0, b] with $\varphi(b) \leq a$.

Let π denote one of the properties: to be upper \aleph -complete and upper \aleph -continuous, or to be lower \aleph -complete and lower \aleph -continuous. Then $[0, a \cup b]$ has property π if both of $[0, a_1 \cup a_2]$ and $[0, a_0 \cup \varphi^{-1}(a_0 \cap \varphi(b))]$ have property π ; if a_0 is the Boolean part of [0, a] and [0, b] has a Boolean part b_0 , it is sufficient that $[0, a_1 \cup a_2]$ and $[0, a_0 \cup b_0]$ should both have property π .

Proof. Since $(a_0, a_1 \cup \cdots \cup a_m)P$ holds, Lemma 3.7 shows that $\varphi(b) = \varphi(b_1) \cup \varphi(b_2)$ where $b_1 = \varphi^{-1}(a_0 \cap \varphi(b))$ and $b_2 = \varphi^{-1}((a_1 \cup \cdots \cup a_m) \cap \varphi(b))$. Then $(a_0 \cup b_1, a_1 \cup \cdots \cup a_m \cup b_2)P$ holds (use [1, (2.6)]).

By Lemma 3.7, $[0, a \cup b]$ is the direct sum of $[0, a_0 \cup b_1]$ and $[0, a_1 \cup \cdots \cup a_m \cup b_2]$ and has property π if each of the summands has it.

Since $b_2 \leq a_1 \cup \cdots \cup a_m$, $[0, a_1 \cup \cdots \cup a_m \cup b_2]$ has property π if $[0, a_1 \cup a_2]$ has it, by Lemma 3.3 and its Corollary and Lemma 3.6 and its Corollary.

If a_0 is the Boolean part of [0, a] then $\varphi(b) \cap a_0$ is Boolean with respect to [0, a], a fortiori Boolean with respect to $[0, \varphi(b)]$. Thus, b_1 is Boolean with respect to [0, b]. If [0, b] has a Boolean part b_0 then $b_1 \leq b_0$ and $a_0 \cup b_1 \leq a_0 \cup b_0$, hence $[0, a_0 \cup b_1]$ has property π if $[0, a_0 \cup b_0]$ has it.

This proves all parts of Theorem 3.2.

REMARK. If \mathscr{R} is a von Neumann ring then \mathscr{R} has a unique decomposition as a direct sum $\mathscr{R} = \mathscr{R} \bigoplus \mathscr{R}$ such that $\overline{R}_{\mathscr{R}}$ is the Boolean part of $\overline{R}_{\mathscr{R}}$ and $\overline{R}_{\mathscr{R}}$ has a basis x_1, x_2, x_3 with $x_2 \sim x_1$ and $x_3 \leq x_1$. Then Theorem 3.2 and Corollary 2 to Theorem 3.1 apply and show that \mathscr{R}_2 is a von Neumann ring if and only if \mathscr{R}_2 is a von Neumann ring (for details see [2]).

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