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ON THE ADDITIVITY OF LATTICE COMPLETENESS

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ON THE ADDITIVITY OF LATTICE COMPLETENESS

to the memory of Maurice Audin

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1. Introduction. It was shown in [1, Theorem 4.3] that upper $\aleph$-continuity[1] is additive in the following sense:

(1.1) *Suppose that* $[0, a]$, $[0, b]$ *are upper* $\aleph$*-continuous in a relatively complemented modular lattice. Then* $[0, a \cup b]$ *is upper* $\aleph$*-continuous provided that* $[0, a \cup b]$ *is upper* $\aleph$*-complete.*

But it may happen that $[0, a]$, $[0, b]$ are both upper $\aleph$-complete (both may even be von Neumann geometries with a perspective to b) and yet $[0, a \cup b]$ is *not* upper $\aleph$-complete. In fact there are von Neumann rings $\mathscr{R}$ for which the lattice $\bar{R}_{\mathscr{S}}$, with $\mathscr{S} = \mathscr{R}_2$, is not even upper $\aleph_0$-complete (see the Remark preceding Definition 3.1)

With a modest supplementary condition however, additivity of upper $\aleph$-completeness does hold, as we show in this paper.

2. Terminology and notation. We shall use the notation of [1], [2], and [4].

I will denote a set of indices α and $\bar{I}$ will denote the cardinal power of I.

$\aleph$ will denote an infinite cardinal, Ω will denote the least ordinal number whose corresponding cardinal power is $\aleph$.

A lattice is called *upper* $\aleph$*-complete* if the union $a = \bigcup (a_\alpha | \alpha \in I)$ exists whenever $\bar{I} \leqq \aleph$, and is called *upper* $\aleph$*-continuous* if for every b: $b \cap a = \bigcup ((b \cap \bigcup (a_\alpha | \alpha \in F)) | \text{all finite } F \subset I)$, with dual definitions for *lower* $\aleph$*-completeness* and *lower* $\aleph$-continuity. The lattice is called $\aleph$*-complete*, respectively $\aleph$*-continuous* if it is both upper and lower $\aleph$-continuous.

A complemented modular lattice L is called an $\aleph$*-von Neumann-geometry* if it is $\aleph$-complete and $\aleph$-continuous (irreducibility is *not* assumed).

If we omit the $\aleph$ in any of these designations, this implies that the lattice L has the corresponding $\aleph$-property for all $\aleph$.

If $\mathscr{R}$ is an associative regular ring (not necessarily with unit element) then $\bar{R}_{\mathscr{R}}$ denotes the relatively complemented modular lattice of its principal right ideals, ordered by inclusion. $\mathscr{R}$ is called an $\aleph$*-von Neumann-ring*, respectively a *von Neumann ring*, according as $\bar{R}_{\mathscr{R}}$ is an $\aleph$-von

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[1] Terminology and notation are explained in section 2 below.

Neumann-geometry, respectively a von Neumann geometry.

In any relatively complemented modular lattice, if $a \geqq b$ then $[a - b]$ will denote an arbitrary (but fixed) element such that $[a - b] \dot{\bigcup} b = a$ (the dot indicates that the summands in the union are independent). We write $a \sim b$ to denote: a is perspective to b, and $a \lesssim b$ to denote: $a \sim b_1$ for some $b_1 \leqq b$. Elements a, b are called *completely disjoint*, (notation: $(a, b)P$) if: $a_1 \sim b_1$, $a_1 \leqq a$, $b_1 \leqq b$ together imply $a_1 = 0$.

3. The additivity of completeness theorem.

In this section $a, b, c, \cdots x_\alpha, \cdots$ will denote elements in a given relatively complemented modular lattice L.

If $[0, a \cup c]$ is upper $\aleph$-complete we shall write $u(a, c, \aleph)$ to mean:

(3.1) *Whenever* $x_\alpha \leqq a \cup c$ *for all* $\alpha \in I$ (*with* $\bar{I} \leqq \aleph$) *and*

$$a \cap (\bigcup(x_\beta | \beta \in F)) = 0$$

for all finite $F \subset I$, *then* $a \cap (\bigcup(x_\alpha | \alpha \in I)) = 0$.

It is important to note: if $u(a, c, \aleph)$ holds then $u(a', c', \aleph)$ holds for all $a' \leqq a, c' \leqq c$.

Clearly, if $[0, a \cup c]$ is upper $\aleph$-complete *and* upper $\aleph$-continuous then $u(a, c, \aleph)$ does hold.

Similarly, if $[0, a \cup c]$ is lower $\aleph$-complete we shall write $l(a, c, \aleph)$ to denote:

$\overline{(3.1)}$ *Whenever* $x_\alpha \leqq a \cup c$ *for all* $\alpha \in I$ (*with* $\bar{I} \leqq \aleph$) *and*

$$a \cup (\bigcap(x_\beta | \beta \in F)) = a \cup c$$

for all finite $F \subset I$, *then* $a \cup (\bigcap(x_\alpha | \alpha \in I)) = a \cup c$.

It is important to note: if $l(a, c, \aleph)$ holds then $l(a', c', \aleph)$ holds for all $a' \leqq a, c' \leqq c$.

Clearly, if $[0, a \cup c]$ is lower $\aleph$-complete *and* lower $\aleph$-continuous then $l(a, c, \aleph)$ does hold.

LEMMA 3.1. *Suppose that each of* $[0, a \cup b]$, $[0, b \cup c]$, $[0, a \cup c]$ *is upper* $\aleph$-*complete and suppose that* $u(a, c, \aleph)$ *holds. Then* $[0, a \cup b \cup c]$ *is upper* $\aleph$-*complete.*

Proof. We may suppose that $\{a, b, c\}$ is an independent set, for if c, b are replaced by $[c - (a \cap c)]$ and $[b - (b \cap (a \cup c))]$ respectively the hypotheses of Lemma 3.1 continue to hold and the conclusion is not changed.

Using transfinite induction, we may suppose that Lemma 3.1 holds for all $\aleph' < \aleph$. We may therefore assume that x_α is given, $\leqq a \cup b \cup c$ for all $0 < \alpha < \Omega$, that $\bigcup(x_\alpha | \alpha \leqq \beta)$ exists for all $\beta < \Omega$ and we need only show that $\bigcup(x_\alpha | \alpha < \Omega)$ exists.

We may suppose $x_\alpha \leqq x_\beta$ for $\alpha \leqq \beta < \Omega$ (by replacing the original x_α by $\bigcup(x_\beta | \beta \leqq \alpha)$ for all $(\alpha < \Omega)$.

Set $\bar{x}_0 = \bigcup((x_\alpha \cap (a \cap b)) | \alpha < \Omega)$ (this union exists since, by hypothesis, $[0, a \cup b]$ is upper $\aleph$-complete). Set $\bar{x}_\alpha = \bar{x}_0 \cup x_\alpha$ for $0 < \alpha < \Omega$ and observe that $\bar{x}_\beta \leqq \bar{x}_\alpha$ for all $0 \leqq \beta \leqq \alpha < \Omega$.

Set $y_0 = \bar{x}_0$ and $y_\alpha = [\bar{x}_\alpha - \bigcup(\bar{x}_\beta | 0 \leqq \beta < \alpha)]$ for $0 < \alpha < \Omega$. Then $\bigcup(y_\beta | 0 \leqq \beta < \alpha) = \bigcup(\bar{x}_\beta | 0 \leqq \beta < \alpha)$ for all $0 < \alpha < \Omega$, as may be verified easily by transfinite induction.

Clearly, we need only show that $\bigcup(y_\alpha | 0 \leqq \alpha < \Omega)$ exists. Hence it is sufficient to show that $\bigcup_\alpha y_\alpha$ exists, where (for the rest of this proof) we write $\bigcup_\alpha$ to mean $\bigcup_{0<\alpha<\Omega}$ (note: $0 \leqq \alpha < \Omega$ has been replaced by $0 < \alpha < \Omega$).

Set $u = \big(a \cup (\bigcup_\alpha((a \cup y_\alpha) \cap (b \cup c)))\big) \cap \big(b \cup (\bigcup_\alpha((b \cup y_\alpha) \cap (a \cup c)))\big)$ (this union exists since, by hypothesis, $[0, b \cup c]$ and $[0, a \cup c]$ are upper $\aleph$-complete). We observe that $u \geqq y_\beta$ for all $0 < \beta < \Omega$ since each factor of u has this property: for fixed β, $a \cup (\bigcup_\alpha((a \cup y_\alpha) \cap (b \cup c))) \geqq a \cup ((a \cup y_\beta) \cap (b \cup c)) = (a \cup y_\beta) \cap (a \cup b \cup c) = a \cup y_\beta \geqq y_\beta$.

We shall show that u is the desired union $\bigcup_\alpha y_\alpha$. It is clearly sufficient to show for every w: if $u \geqq w \geqq y_\alpha$ for all $0 < \alpha < \Omega$ then $u \leqq w$.

Since $a \cup y_\alpha \leqq a \cup w$ and $b \cup y_\alpha \leqq b \cup w$ for all $0 < \alpha < \Omega$,

$$\begin{aligned} u &\leqq \big(a \cup ((a \cup w) \cap (b \cup c))\big) \cap \big(b \cup ((b \cup w) \cap (a \cup c))\big) \\ &= (a \cup w) \cap (b \cup w) = w \cup (a \cap (b \cup w)) . \end{aligned}$$

It is therefore sufficient to show that $a \cap (b \cup w) \leqq w$. We shall show that $a \cap (b \cup u) = 0$; this will imply:

$$a \cap (b \cup w) \leqq a \cap (b \cup u) = 0 \leqq w .$$

Now $b \cup u = \big(a \cup b \cup (\bigcup_\alpha(a \cup y_\alpha) \cap (b \cup c)))\big) \cap \big(b \cup (\bigcup_\alpha((b \cup y_\alpha) \cap (a \cup c)))\big)$,

$$\begin{aligned} a \cap (b \cup u) &= a \cap \big(b \cup (\bigcup_\alpha((b \cup y_\alpha) \cap (a \cup c)))\big) \\ &= a \cap ((b \cap (a \cup c)) \cup \big(\bigcup_\alpha((b \cup y_\alpha) \cap (a \cup c))\big)) \\ &= a \cap \big(\bigcup_\alpha((b \cup y_\alpha) \cap (a \cup c))\big) . \end{aligned}$$

Since $u(a, c, \aleph)$ is assumed to hold we need only show:

$$a \cap (\bigcup(((b \cup y_\alpha) \cap (a \cup c)) | \alpha = \alpha_1, \cdots, \alpha_m)) = 0$$

for every finite set of indices $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < \Omega$.

Hence it is sufficient to show that

$$a \cap (b \cup (\bigcup(y_\alpha | \alpha = \alpha_1, \cdots, \alpha_m))) = 0 ,$$

and so it is sufficient to show that

$$(a \cup b) \cap (\bigcup(y_\alpha | \alpha = \alpha_1, \cdots, \alpha_m)) = 0 . \tag{3.2}$$

For this purpose, we note: $y_\alpha \cap (\bigcup(y_\beta | 0 \leqq \beta < \alpha) = 0$ for all $0 < \alpha < \Omega$. This implies that $\{y_\alpha | \alpha = 0, \alpha_1, \cdots, \alpha_m\}$ is an independent set and hence $y_0 \cap (\bigcup(y_\alpha | \alpha = \alpha_1, \cdots, \alpha_m)) = 0$. This implies (3.2) since the left side of (3.2) is $\leqq y_0$. Thus Lemma 3.1 is proved.

COROLLARY 1. *Suppose that* $[0, a_i \cup a_j]$ *is upper* $\aleph$*-complete for* $i, j = 1, \cdots, m$ *for some finite integer* m *and suppose that* $u(a_i, a_j, \aleph)$ *holds whenever* $i < j$. *Then* $[0, a_1 \cup \cdots \cup a_m]$ *is upper* $\aleph$*-complete.*

Proof. If $m \leqq 2$ the conclusion is part of the hypotheses. Suppose that $m > 2$ and that the Corollary is known to hold with $m - 1$ in place of m; then Lemma 3.1 can be applied (with $a = a_1$, $b = a_3 \cup \cdots \cup a_m$ and $c = a_2$) to show that the Corollary holds for m itself. By induction on m, the Corollary is established.

COROLLARY 2. *Suppose that* $[0, a_i \cup a_j]$ *is upper* $\aleph$*-complete and upper* $\aleph$*-continuous for* $i, j = 1, \cdots, m$ *for some finite integer* m. *Then* $[0, a_1 \cup \cdots \cup a_m]$ *is upper* $\aleph$*-complete and upper* $\aleph$*-continuous.*

Proof. Since upper $\aleph$-continuity of $[0, a_i \cup a_j]$ implies that $u(a_i, a_j, \aleph)$ holds, Corollary 1 shows that $[0, a_1 \cup \cdots \cup a_m]$ is upper $\aleph$-complete. The upper $\aleph$-continuity then follows from [1, Theorem 4.3].

LEMMA 3.2. *Suppose that* $a = a_1 \cup a_2 \cup \cdots \cup a_m$ *and* $a_i \lesssim a_1 \cup \cdots \cup a_{i-1}$ *for* $1 < i \leqq m$. *Then* a *can be expressed in the form*:

(3.3) $a_1 \dot{\cup} \bar{a}_2 \dot{\cup} \cdots \dot{\cup} \bar{a}_n$ *for some* $n \geqq m$ *and elements* $\bar{a}_2, \cdots, \bar{a}_n$ *such that* $\bar{a}_i \lesssim a_1$ *for all* $1 < i \leqq n$.

Moreover $\bar{a}_2$ *may be taken to coincide with* a_2 *if* $a_1 \cap a_2 = 0$.

Proof. Lemma 3.2 holds trivially if $m = 1$ and also if $m = 2$ and $a_1 \cap a_2 = 0$. We may therefore suppose (by induction) that $m > 1$ and that $b = a_1 \cup \cdots \cup a_{m-1}$ has the form (3.3).

We can replace a_m by $[a_m - (a_m \cap b)]$ since the hypotheses of Lemma 3.2 continue to hold and the conclusion is not changed. After this change,

$$a_m \cap b = a_m \cap (a_1 \dot{\cup} \bar{a}_2 \dot{\cup} \cdots \dot{\cup} \bar{a}_n) = 0 .$$

Since $a_m \lesssim a_1 \dot{\cup} \bar{a}_2 \dot{\cup} \cdots \dot{\cup} \bar{a}_n$ there is a perspectivity mapping φ of $[0, a_m]$ with $\varphi(a_m) \leqq b$. Then

$$a_m = a_{m,1} \dot{\cup} a_{m,2} \dot{\cup} \cdots \dot{\cup} a_{m,n}$$

where

$$\varphi(a_{m,1}) = \varphi(a_m) \cap a_1 ,$$

and for $1 < i \leqq n$,

$$\varphi(a_{m,i}) = [(\varphi(a_m) \cap (a_1 \dot{\cup} \bar{a}_2 \dot{\cup} \cdots \dot{\cup} \bar{a}_i)) - (\varphi(a_m) \cap (a_1 \dot{\cup} \bar{a}_2 \dot{\cup} \cdots \dot{\cup} \bar{a}_{i-1})] .$$

Obviously, $a_{m,1} \lesssim a_1$. If $i > 1$ then $a_{m,i} \sim \varphi(a_{m,i})$; $\varphi(a_{m,i}) \lesssim \bar{a}_i$; $\bar{a}_i \lesssim a_1$; and $a_{m,i} \cap (\varphi(a_{m,i}) \cup \bar{a}_i \cup a_1) = 0$; these facts imply that $a_{m,i} \lesssim a_1$ (use (2.2) of [1]). The conclusion of Lemma 3.2 now follows at once.

LEMMA 3.3. *Suppose that*

(i) $a = a_1 \cup a_2 \cup \cdots \cup a_m$ *for some finite* $m \geqq 2$,
(ii) $a_2 \sim a_1$,
(iii) $a_i \lesssim a_1 \cup \cdots \cup a_{i-1}$ *for* $2 < i \leqq m$,
(iv) $[0, a_1 \dot{\cup} a_2]$ *is upper* $\aleph$*-complete*,
(v) $u(a_1, a_2, \aleph)$ *holds*.

Then $[0, a]$ *is upper* $\aleph$*-complete*.

Proof. Applying Lemma 3.2, and using a new m and new elements $a_3, \cdots, a_m$ we may suppose that (i), (iii) hold in the strengthened form: $a = a_1 \dot{\cup} a_2 \dot{\cup} \cdots \dot{\cup} a_m$ and $a_i \lesssim a_1$ for $2 < i \leqq m$.

Suppose that $1 \leqq i < j \leqq m$. If $i \neq 2$ then $a_j \lesssim a_2$ (because of (ii)) and there is a perspectivity mapping φ of $[0, a_i \cup a_j]$ with $\varphi(a_i) \leqq a_1$ and $\varphi(a_j) \leqq a_2$. Hence $[0, a_i \cup a_j]$ is upper $\aleph$-complete and $u(a_i, a_j, \aleph)$ holds in this case.

If $i = 2$ there is a perspectivity mapping φ of $[0, a_2 \cup a_j]$ with $\varphi(a_2) = a_1$, $\varphi(a_j) = a_j$; the result for $[0, a_1 \cup a_j]$ obtained previously now implies: $[0, a_2 \cup a_j]$ is upper $\aleph$-complete and $u(a_2, a_j, \aleph)$ holds.

Corollary 1 to Lemma 3.1 now applies to these elements $a_1, \cdots, a_m$ and this completes the proof of Lemma 3.3.

COROLLARY. *Suppose that the hypotheses* (i), (ii), (iii), *of Lemma* 3.3 *hold and suppose also that*

(vi) $[0, a_1 \dot{\cup} a_2]$ *is upper* $\aleph$*-complete and upper* $\aleph$*-continuous*.

Then $[0, a]$ *is upper* $\aleph$*-complete and upper* $\aleph$*-continuous*.

Proof. (vi) implies (iv), (v). Hence $[0, a]$ is upper $\aleph$-complete by Lemma 3.3. Upper $\aleph$-continuity then follows from [1, Theorem 4.3].

LEMMA 3.4. (*Additivity of lower* $\aleph$*-continuity*). *Suppose that* $[0, a_1 \cup \cdots \cup a_m]$ *is lower* $\aleph$*-complete and that* $[0, a_i]$ *is lower* $\aleph$*-*

continuous for $i=1, \cdots, m$. *Then* $[0, a_1 \cup \cdots \cup a_m]$ *is lower* $\aleph$*-continuous.*

Proof. We may assume that $\{a_1, \cdots, a_m\}$ is an independent set (replace a_i by $[a_i - (a_i \cap (a_1 \cup \cdots \cup a_{i-1}))]$ for $2 \leqq i \leqq m$).

Then $[a_1, a_1 \cup a_2]$ is lower $\aleph$-continuous since it is lattice isomorphic to $[0, a_2]$ under the mapping: $x \to x \cap a_2$. Similarly $[a_2, a_1 \cup a_2]$ is lower $\aleph$-continuous. By the dual of [1, Theorem 4.3], $[0, a_1 \cup a_2] = ([a_1 \cap a_2, a_1 \cup a_2])$ is lower $\aleph$-continuous. Lemma 3.4 follows by induction on m.

LEMMA 3.5. *Suppose that each of* $[0, a \cup b]$, $[0, b \cup c)$, $[0, a \cup c]$ *is lower* $\aleph$*-complete and suppose that* $l(a, c, \aleph)$ *holds. Then* $[0, a \cup b \cup c]$ *is lower* $\aleph$*-complete.*

Proof. We may suppose that $\{a, b, c\}$ is an independent set, for if c, b are replaced by $[c - (a \cap c)]$ and $[b - (b \cap (a \cup c))]$ respectively the hypotheses of Lemma 3.5 continue to hold ($l(a, c_1, \aleph)$ is equivalent to $l(a, c, \aleph)$ if $a \cup c_1 = a \cup c$) and the conclusion is not changed.

Now set $B = a \cup c$, $C = b \cup a$, $A = b \cup c$, and $1 = a \cup b \cup c$. We have: $[A \cap B, 1] (= [c, a \cup b \cup c])$ is lower $\aleph$-complete since it is lattice isomorphic to $[0, a \cup b]$ under the mapping $x \to x \cap (a \cup b)$. Similarly each of $[B \cap C, 1]$, $[C \cap A, 1]$ is lower $\aleph$-complete.

We can now show that $[0, a \cup b \cup c] (= [A \cap B \cap C, 1])$ is lower $\aleph$-complete (by applying the dual of Lemma 3.1) if we can show:

(3.4) *Whenever* $X_\alpha \geqq C \cap A$ *for* $\alpha \in I$ (*with* $\bar{I} \leqq \aleph$) *and* $C \cup (\bigcap(X_\beta | \beta \in F)) = 1$ *for all finite* $F \subset I$, *then* $C \cup (\bigcap(X_\alpha | \alpha \in I)) = 1$.

Since $C \cap A = b$ and $C = a \cup b$, (3.4) can be rewritten:

(3.4)′ *Whenever* $X_\alpha \geqq b$ *for* $\alpha \in I$ (*with* $\bar{I} \leqq \aleph$) *and* $a \cup (\bigcap(X_\beta | \beta \in F)) = a \cup b \cup c$ *for all finite* $F \subset I$ *then* $a \cup (\bigcap (X_\alpha | \alpha \in I)) = a \cup b \cup c$.

Suppose that the hypotheses of (3.4)′ hold and set $x_\alpha = X_\alpha \cap (a \cup c)$. Then $x_\alpha \leqq a \cup c$ for all α and

$$\begin{aligned} &a \cup (\bigcap (x_\beta | \beta \in F)) \\ &= a \cup ((\bigcap(X_\beta | \beta \in F)) \cap (a \cup c)) = (a \cup (\bigcap (X_\beta | \beta \in F))) \cap (a \cup c) \\ &= (a \cup b \cup c) \cap (a \cup c) = a \cup c\,. \end{aligned}$$

Since $l(a, c, \aleph)$ holds, it follows that

$$\begin{aligned} &a \cup (\bigcap (x_\alpha | \alpha \in I)) = a \cup c\,;\ a \cup (\bigcap (X_\alpha | \alpha \in I) \cap (a \cup c)) = a \cup c\,; \\ &a \cup (\bigcap (X_\alpha | \alpha \in I)) \geqq a \cup c \ \text{(hence} = a \cup b \cup c)\,. \end{aligned}$$

This means: (3.4)′ does hold. This completes the proof of Lemma 3.5.

COROLLARY 1. *Suppose that* $[0, a_i \cup a_j]$ *is lower* $\aleph$*-complete for* $i, j = 1, \cdots, m$.

Suppose also that $l(a_i, a_j, \aleph)$ *holds for all* $i<j$. *Then* $[0, a_1 \cup \cdots \cup a_m]$ *is lower* $\aleph$*-complete.*

Proof. This follows from Lemma 3.5 by induction on m, just as Corollary 1 to Lemma 3.1 followed from Lemma 3.1.

COROLLARY 2. *Suppose that* $[0, a_i \cup a_j]$ *is lower* $\aleph$*-complete and lower* $\aleph$*-continuous for* $i, j = 1, \cdots, m$. *Then* $[0, a_1 \cup \cdots \cup a_m]$ *is lower* $\aleph$*-continuous.*

Proof. Since lower $\aleph$-continuity of $[0, a_i \cup a_j]$ implies that $l(a_i, a_j, \aleph)$ holds, Corollary 1 shows that $[0, a_1 \cup \cdots \cup a_m]$ is lower $\aleph$-complete. The lower $\aleph$-continuity of $[0, a_1 \cup \cdots \cup a_m]$ then follows from Lemma 3.4.

LEMMA 3.6. *Suppose that*

(i) $a = a_1 \cup a_2 \cup \cdots \cup a_m$ *for some finite* $m \geqq 2$,
(ii) $a_2 \sim a_1$,
(iii) $a_i \lesssim a_1 \cup \cdots \cup a_{i-1}$ *for* $2 < i \leqq m$,
(iv) $[0, a_1 \dot{\cup} a_2]$ *is lower* $\aleph$*-complete,*
(v) $l(a_1, a_2, \aleph)$ *holds.*

Then $[0, a]$ *is lower* $\aleph$*-complete.*

COROLLARY. *Suppose that* (i), (ii), (iii) *hold and also*

(vi) $[0, a_1 \dot{\cup} a_2]$ *is lower* $\aleph$*-complete and lower* $\aleph$*-continuous.*

Then $[0, a]$ *is lower* $\aleph$*-complete and lower* $\aleph$*-continuous.*

Proof. Lemma 3.6 and its Corollary follow from Lemma 3.5 and Lemma 3.4 just as Lemma 3.3 and its Corollary followed from Corollary 1 to Lemma 3.1 and [1, Theorem 4.3].

THEOREM 3.1. *Suppose that each of* $[0, a_i \cup a_j]$ *is an* $\aleph$*-von Neumann-geometry (respectively a von Neumann-geometry) for* $i, j = 1, \cdots, m$. *Then* $[0, a_1 \cup \cdots \cup a_m]$ *is an* $\aleph$*-von Neumann-geometry (respectively a von Neumann geometry).*

Proof. This follows from Corollary 2 to Lemma 3.1 and Corollary 2 to Lemma 3.5.

COROLLARY 1. *Suppose that*

(i) $a = a_1 \cup a_2 \cup \cdots \cup a_m$ *for some finite* $m \geqq 2$,

(ii) $a_2 \sim a_1$,
(iii) $a_i \lesssim a_1 \cup \cdots \cup a_{i-1}$ *for* $2 < i \leqq m$,
(iv) $[0, a_1 \dot{\cup} a_2]$ *is an* $\aleph$*-von Neumann-geometry (respectively a von Neumann-geometry).*

Then $[0, a]$ *is an* $\aleph$*-von Neumann-geometry, respectively a von Neumann-geometry.*

Proof. This follows from the Corollary to Lemma 3.3 and the Corollary to Lemma 3.6.

COROLLARY 2. *Suppose that* $\mathscr{R}$ *is an* $\aleph$*-von Neumann-ring (respectively a von Neumann-ring). If* $\bar{R}_{\mathscr{R}}$ *has a basis* $x_1, x_2, \cdots, x_m$ *such that* $x_2 \sim x_1$ *and* $x_i \lesssim x_1$ *for* $2 < i \leqq m$, *then* $\mathscr{R}_2$ *is an* $\aleph$*-von Neumann-ring (respectively, a von Neumann-ring).*

Proof. By hypothesis, the unit element of the lattice $\bar{R}_{\mathscr{R}}$ is the union $x_1 \dot{\cup} \cdots \dot{\cup} x_m$. The unit element of $\bar{R}_{\mathscr{S}}$, with $\mathscr{S} = \mathscr{R}_2$, can be represented as a union $x_1 \dot{\cup} \cdots \dot{\cup} x_m \dot{\cup} y_1 \dot{\cup} \cdots \dot{\cup} y_m$ with $y_i \sim x_i$ and hence $y_i \lesssim x_1$ for $1 \leqq i \leqq m$. Since $[0, x_1 \dot{\cup} x_2]$ is an $\aleph$-von Neumann geometry (respectively a von Neumann geometry) along with $\bar{R}_{\mathscr{R}}$, Corollary 1 applies and this completes the proof of Corollary 2.

COROLLARY 3. *Suppose that* $\mathscr{R}$ *and* $\mathscr{R}_2$ *are both* $\aleph$*-von Neumann-rings (respectively von Neumann-rings). Then* $\mathscr{R}_n$ *is an* $\aleph$*-von Neumann-ring (respectively a von Neumann-ring) for all finite* n.

Proof. If $n > 2$ the unit element of $\bar{R}_{\mathscr{S}}$, with $\mathscr{S} = \mathscr{R}_n$, can be expressed as $x_1 \dot{\cup} x_2 \dot{\cup} \cdots \dot{\cup} x_n$ where x_1 is the unit element of $\bar{R}_{\mathscr{R}}$, $x_i \sim x_1$ for all i, and $[0, x_1 \dot{\cup} x_2] = \bar{R}_{\mathscr{R}_2}$. Theorem 3.1 applies and this completes the proof of Corollary 3.

REMARK. Let $\mathscr{R}$ be the ring of sequences $x = (x^n)$ with all x^n complex numbers and all but a finite number of x^n real, with componentwise addition and multiplication; this example was given by Kaplansky [3, page 526]. This $\mathscr{R}$ is a von Neumann-ring but $\mathscr{R}_2$ is not even upper $\aleph_0$-complete.

DEFINITION 3.1. If L is a relatively complemented modular lattice, then an element a is called Boolean (with respect to L) if $b_1 \sim b_2$, $b_1 \leqq a$ together imply $b_1 = b_2$; a is called the *Boolean part* of L (necessarily unique if it exists)[2] if a is Boolean and $a_1 \leqq a$ for every Boolean a_1.

[2] This is an abuse of language: properly, $[0, a]$ should be called the Boolean part of L.

LEMMA 3.7. *Suppose that L is a relatively complemented modular lattice. If $(a, b)P$ holds then for every c in L, $c \cap (a \cup b) = (c \cap a) \cup (c \cap b)$ and $[0, a \cup b]$ is the direct sum of $[0, a]$ and $[0, b]$. On the other hand if a is Boolean then*

(i) *$b \leqq a$ implies that b is Boolean,*
(ii) *$b \cap a = 0$ implies that $(b, a)P$ holds,*
(iii) *$b \geqq a$ implies that the relative complement $[b - a]$ is unique,*
(iv) *$a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$ for all b, c in L,*
(v) *$[0, a]$ is a Boolean algebra.*

Proof. Suppose that $(a, b)P$ holds and set $d = [(c \cap (a \cup b)) - ((c \cap a) \cup (c \cap b))]$, $d_a = (d \cup b) \cap a$, $d_b = (d \cup a) \cap b$. Then $d \leqq a \cup b$, $d \cap a = d \cap b = 0$, $d_a \dot{\cup} d = (d \cup b) \cap (d \cup a) = d_b \dot{\cup} d$, so $d_a \sim d_b$. Since $d_a \leqq a$, $d_b \leqq b$ and $(a, b)P$ holds, we must have: $d_a = 0$; $b = d_a \cup b = d \cup b$; $d \leqq b$; hence $d = 0$, $c \cap (a \cup b) = (c \cap a) \cup (c \cap b)$. If $c \leqq a \cup b$ then $c = (c \cap a) \cup (c \cap b)$; and if $c = c_1 \cup c_2$ with $c_1 \leqq a$, $c_2 \leqq b$ then $c \cap a = c_1 \cup (c_2 \cap b \cap a) = c_1 \cup 0 = c_1$, $c \cap b = c_2$. This proves that $[0, a \cup b]$ is the direct sum of $[0, a]$ and $[0, b]$.

(i) and (ii) are obvious from the definition of Boolean element.

(ii) asserts that a is in the centre of L as defined in [1, (2.5)]. But if a is in the centre of L and b is any element in L with $b \geqq a$ then a is in the centre of $[0, b]$, hence $[b - a]$ is uniquely determined (use [1, (2.6)]). This proves (iii).

If b, c are arbitrary elements in L, set $b_1 = [b - (a \cap b)]$, $c_1 = [c - (a \cap c)]$. Since $a \cap b_1 = a \cap c_1 = 0$ and a is in the centre of L, it follows that $(a, b_1)P$, $(a, c_1)P$, hence $(a, b_1 \cup c_1)P$ (use [1, (2,6)]); therefore $a \cap (b_1 \cup c_1) = 0$. By the modular law

$$\begin{aligned} a \cap (b \cup c) &= a \cap (b_1 \cup c_1 \cup (a \cap b) \cup (a \cap c)) \\ &= (a \cap b) \cup (a \cap c) \cup (a \cap (b_1 \cup c_1)) \\ &= (a \cap b) \cup (a \cap c) \end{aligned}$$

and hence (iv) holds.

Thus $[0, a]$ is a distributive complemented lattice, equivalently: a Boolean algebra. This proves (v).

LEMMA 3.8. *Suppose that L has a unit element $1 = a_1 \cup a_2 \cup \cdots \cup a_m$ with $m \geqq 2$, $a_2 \sim a_1$, $a_i \lesssim a_1$ for $2 < i \leqq m$ and $a_1 \cap a_2 = 0$. Then the Boolean part of L exists and is 0.*

Proof. By Lemma 3.2 we may assume that $1 = a_1 \dot{\cup} \cdots \dot{\cup} a_m$ with $m \geqq 2$, $a_2 \sim a_1$ and $a_i \lesssim a_1$ for $2 < i \leqq m$.

To prove Lemma 3.8 we may suppose that $a \neq 0$ and we need only exhibit elements b_1, b_2 such that $b_1 \leqq a$, $b_1 \sim b_2$, and $b_1 \neq b_2$.

If $a_i \cap a \neq 0$ for any i it suffices to choose this element as b_1 since the relations $a_1 \sim a_2$ and $a_i \lesssim a_1$ if $i \neq 1$ imply $b_1 \sim b_2$ for some $b_2 \neq b_1$ (even $b_1 \cap b_2 = 0$).

On the other hand, if $a_i \cap a = 0$ for all i, set $b_1 = (a_1 \cup \cdots \cup a_i) \cap a$ where i is the smallest integer for which this element is different from 0 (necessarily $1 < i \leqq m$) and set $b_2 = ((a_1 \cup \cdots \cup a_{i-1}) \dot{\cup} b_1) \cap a_i$. Then $b_1 \sim b_2$ since $(a_1 \cup \cdots \cup a_{i-1}) \dot{\cup} b_1 = (a_1 \cup \cdots \cup a_{i-1}) \dot{\cup} b_2$; and $b_1 \neq b_2$ since $b_2 \leqq a_i$ and $b_1 \cap a_i \leqq a \cap a_i = 0$. This completes the proof of Lemma 3.8.

LEMMA 3.9. *Suppose that L is an upper complete complemented modular lattice and let a be the union of all Boolean elements in L. Then a is the Boolean part of L.*

Proof. We need only show that a is Boolean, that is, we may suppose that $b \leqq a$, that φ is a perspective mapping of $[0, b]$, that $b \neq \varphi(b)$ and we need only derive a contradiction. By replacing b by $[b - (b \cap \varphi(b))]$ we may suppose $b \neq 0$ and $b \cap \varphi(b) = 0$.

Now for every c: $(\varphi(b \cap c)) \sim (b \cap c)$ and $(\varphi(b \cap c)) \cap (b \cap c) = 0$. If c is Boolean this implies: $b \cap c = 0$, and hence (since c is Boolean) $(b, c)P$ holds. It follows from [1, formula (2.6)] that $(b, a)P$ holds, contradicting the fact that $b \neq 0$ and $b \leqq a$. This contradiction proves Lemma 3.9.

THEOREM 3.2. *Suppose that L is a relatively complemented modular lattice and*

(i) $a = a_0 \cup a_1 \cup a_2 \cup \cdots \cup a_m$ *for some finite* $m \geqq 2$,
(ii) $(a_0, a_1 \cup \cdots \cup a_m)P$ *holds,*
(iii) $a_2 \sim a_1, a_2 \cap a_1 = 0$,
(iv) $a_i \lesssim a_1 \cup \cdots \cup a_{i-1}$ *for* $2 < i \leqq m$,
(v) φ *is a perspective mapping of* $[0, b]$ *with* $\varphi(b) \leqq a$.

Let π denote one of the properties: to be upper $\aleph$-complete and upper $\aleph$-continuous, or to be lower $\aleph$-complete and lower $\aleph$-continuous. Then $[0, a \cup b]$ has property π if both of $[0, a_1 \cup a_2]$ and $[0, a_0 \cup \varphi^{-1}(a_0 \cap \varphi(b))]$ have property π; if a_0 is the Boolean part of $[0, a]$ and $[0, b]$ has a Boolean part b_0, it is sufficient that $[0, a_1 \cup a_2]$ and $[0, a_0 \cup b_0]$ should both have property π.

Proof. Since $(a_0, a_1 \cup \cdots \cup a_m)P$ holds, Lemma 3.7 shows that $\varphi(b) = \varphi(b_1) \dot{\cup} \varphi(b_2)$ where $b_1 = \varphi^{-1}(a_0 \cap \varphi(b))$ and $b_2 = \varphi^{-1}((a_1 \cup \cdots \cup a_m) \cap \varphi(b))$. Then $(a_0 \cup b_1, a_1 \cup \cdots \cup a_m \cup b_2)P$ holds (use [1, (2.6)]).

By Lemma 3.7, $[0, a \cup b]$ is the direct sum of $[0, a_0 \cup b_1]$ and $[0, a_1 \cup \cdots \cup a_m \cup b_2]$ and has property π if each of the summands has it.

Since $b_2 \lesssim a_1 \cup \cdots \cup a_m$, $[0, a_1 \cup \cdots \cup a_m \cup b_2]$ has property π if $[0, a_1 \cup a_2]$ has it, by Lemma 3.3 and its Corollary and Lemma 3.6 and its Corollary.

If a_0 is the Boolean part of $[0, a]$ then $\varphi(b) \cap a_0$ is Boolean with respect to $[0, a]$, a fortiori Boolean with respect to $[0, \varphi(b)]$. Thus, b_1 is Boolean with respect to $[0, b]$. If $[0, b]$ has a Boolean part b_0 then $b_1 \leqq b_0$ and $a_0 \cup b_1 \leqq a_0 \cup b_0$, hence $[0, a_0 \cup b_1]$ has property π if $[0, a_0 \cup b_0]$ has it.

This proves all parts of Theorem 3.2.

REMARK. If $\mathscr{R}$ is a von Neumann ring then $\mathscr{R}$ has a unique decomposition as a direct sum $\mathscr{R} = \mathscr{B} \oplus \mathscr{R}$ such that $\bar{R}_{\mathscr{B}}$ is the Boolean part of $\bar{R}_{\mathscr{R}}$ and $\bar{R}_{\mathscr{R}}$ has a basis x_1, x_2, x_3 with $x_2 \sim x_1$ and $x_3 \lesssim x_1$. Then Theorem 3.2 and Corollary 2 to Theorem 3.1 apply and show that $\mathscr{R}_2$ is a von Neumann ring if and only if $\mathscr{B}_2$ is a von Neumann ring (for details see [2]).

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