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ON UNIMODULAR MATRICES

ISIDORE HELLER AND ALAN JEROME HOFFMAN

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1. Introduction and summary. For the purpose of this note a matrix is called unimodular if every minor determinant equals 0, 1 or -1 .

I. Heller and C. B. Tompkins [1] have considered a set

$$S = \{u_i, v_j, u_i + v_j, u_i - u_{i*}, v_j - v_{j*}\}$$

where the $u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n$ are linearly independent vectors in $m + n = k$ -dimensional space E , and have shown that in the coordinate representation of S with respect to an arbitrary basis in E every nonvanishing determinant of k vectors of S has the same absolute value, and that, with respect to a basis in S , the vectors of S or of any subset of S are the columns of a unimodular matrix. For the purpose of this note the class of unimodular matrices obtained in this fashion shall be denoted as the class T .

A. J. Hoffman and J. B. Kruskal [4] have considered incidence matrices A of vertices versus directed paths of an oriented graph G , and proved that:

(i) if G is alternating, then A is unimodular;

(ii) if the matrix A of *all* directed paths of G is unimodular, then G is alternating. The terms are defined as follows. A graph G is oriented if it has no circular edges, at most one edge between any given two vertices, and each edge is oriented. A path is a sequence of distinct vertices v_1, v_2, \dots, v_k of G such that, for each i from 1 to $k - 1$, G contains an edge connecting v_i with v_{i+1} ; if the orientation of these edges is from v_i to v_{i+1} , the path is directed; if the orientation alternates throughout the sequence, the path is alternating. A loop is a sequence of vertices v_1, v_2, \dots, v_k , which is a path except that $v_k = v_1$. A loop is alternating if successive edges are oppositely oriented and the first and last edges are oppositely oriented. The graph is alternating if every loop is alternating. The incidence matrix $A = (a_{ij})$ of the vertices v_i of G versus a set of directed paths p_1, p_2, \dots, p_k of G is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is in } p_j \\ 0 & \text{otherwise.} \end{cases}$$

The class of unimodular matrices thus associated with alternating graphs shall be denoted by K .

I. Heller [2] and [3] has considered unimodular matrices obtained

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by representing the edges (interpreted as vectors) of an n -simplex in terms of a basis chosen among the edges (in graph theoretical terms: the edges and vertices of the simplex form a complete graph G ; a basis is a maximal tree in G , that is, a tree containing all vertices of G), and has shown that:

(i) the matrix representing all edges of the simplex is unimodular and maximal (i.e., will not remain unimodular when a new column is adjoined);

(ii) the columns of every unimodular matrix of n rows and $n(n + 1)$ columns represent the edges of an n -simplex.

The class of (unimodular) matrices whose columns are among the edges of a simplex shall be denoted by H . H can also be defined as a class of incidence matrices: A matrix A belongs to H if there is some oriented graph F without loops such that A is the incidence matrix of the edges of F versus a set of path in F . That is,

$$a_{ij} = \begin{cases} 1 & \text{if edge } e_i \text{ is in path } p_j \\ -1 & \text{if } -e_i \text{ is in } p_j \\ 0 & \text{otherwise .} \end{cases}$$

In [2] it has further been shown that:

(iii) there exist unimodular matrices which do not belong to H ;

(iv) the classes H and T are identical.

The purpose of the present note is to show that the class K is identical with the set of nonnegative matrices of H .

2. THEOREM. *If a matrix A of n rows and m columns belongs to K (i.e., A is the incidence matrix of the n vertices of some alternating graph G versus a set of m directed paths in G), then A belongs to H (i.e., there is some n -simplex S and a basis B among its edges such that the columns of A represent edges of S in terms of B). Conversely, every non-negative matrix of H belongs to K .*

3. NOTATION. An oriented graph is viewed as a set

$$(3.1) \quad R = V \cup E ,$$

where V is the set of vertices A_1, A_2, \dots, A_n , and E is the set of oriented edges e_v , that is certain ordered pairs (A_i, A_j) with $j \neq i$ of elements of V , such that at most one of the two pairs $(A_i, A_j), (A_j, A_i)$ is in E .

For brevity of notation we define

$$(3.2) \quad [A_i, A_j] = \{(A_i, A_j), (A_j, A_i)\} .$$

The origin and endpoint of an edge e are denoted by ρe and σe :

$$(3.3) \quad \rho(A, B) = A , \quad \sigma(A, B) = B ,$$

If A and B are vertices of R , the relation $A < B$ (A is immediate predecessor of B), also written as $B > A$, is defined by

$$(3.4) \quad A < B \iff (A, B) \in R .$$

Similarly, if a, b are edges of R ,

$$(3.5) \quad a < b \iff \sigma a = \rho b .$$

A subset V' of vertices of R defines a subgraph of R

$$(3.6) \quad R(V') = V' \cup E'$$

where $(A, B) \in E' \iff A \in V', B \in V', (A, B) \in E$.

4. *Proof.* Using the graph-theoretical definition of the class H , the first half of the theorem shall be proved by showing that to each alternating graph G there is an oriented loopless graph F such that the K -matrices associated with G are among the H -matrices associated with F .

A column of a K -matrix is the incidence column K_p of the vertices of G versus a directed path p in G ; a column of an H -matrix is the incidence column H_q of the edges of F versus a path q in F . For given G it will therefore be sufficient to show the existence of an F such that

$$(4.1) \quad \text{to each directed path } p \text{ in } G \text{ there is a path } q = \varphi(p) \text{ in } F \text{ such that } K_p = H_q .$$

This will be shown by constructing an F and a mapping φ of the set of vertices of G onto the set of edges of F in such a way that φ satisfies (4.1), or equivalently, that φ preserves the relation defined in (3.4) and (3.5), that is, for any two distinct vertices A, B of G ,

$$(4.2) \quad A < B \text{ (in } G) \implies \varphi(A) < \varphi(B) \text{ (in } F) .$$

The construction of F and φ shall now be carried out under the assumption that G is connected. If G is not connected, the same construction can be applied to each component of G , yielding an F with an equal number of components.

If G has n vertices, take as the vertices of F a set of $n + 1$ distinct elements P_0, P_1, \dots, P_n .

The n edges e_1, e_2, \dots, e_n of F are defined successively as follows.

First, choose an arbitrary vertex A_1 in G , define

$$(4.3) \quad \varphi(A_1) = e_1 = (P_0, P_1) ,$$

and note that:

(i) the subgraph $G_1 = G(A_1)$, consisting of the one vertex A_1 of G , is, trivially, connected;

- (ii) the graph $F_1 = \{P_0, P_1, (P_0, P_1)\}$ is connected;
- (iii) with respect to G_1 and F_1 , φ trivially satisfies (4.2).

Then, assuming $A_\nu \in G$ already chosen and $e_\nu = \varphi(A_\nu)$ defined for $\nu = 1, 2, \dots, k$ in such a manner that $G_k = G\{A_1, A_2, \dots, A_k\}$ and $F_k = \{P_0, P_1, \dots, P_k, e_1, \dots, e_k\}$ are each connected and φ satisfies (4.2) with respect to G_k and F_k , choose $A_{k+1} \in G$ such that

$$(4.4) \quad [A_i, A_{k+1}] \cap G \neq 0$$

for some $i \leq k$ and define

$$(4.5) \quad \varphi(A_{k+1}) = e_{k+1} = \begin{cases} (\sigma e_i, P_{k+1}) & \text{when } (A_i, A_{k+1}) \in G \\ (P_{k+1}, \rho e_i) & \text{when } (A_{k+1}, A_i) \in G, \end{cases}$$

noting that this definition depends on the choice of i since more than one i may satisfy (4.4).

Obviously, G_{k+1} and F_{k+1} are each connected.

To show that φ satisfies (4.2) with respect to G_{k+1} and F_{k+1} , let $A_r < A_s$ in G_{k+1} .

If $r \leq k$ and $s \leq k$, (4.2) is satisfied according to the induction's hypothesis.

For $\{r, s\} = \{i, k + 1\}$, (4.2) is satisfied by definition (4.5). Namely: for $r = i$, $s = k + 1$, (4.5) defines $e_{k+1} = (\sigma e_i, P_{k+1})$, hence $\sigma e_i = \rho e_{k+1}$, which by (3.5) means $e_i < e_{k+1}$; similarly for $s = i$, $r = k + 1$, (4.5) defines $e_{k+1} = (P_{k+1}, \rho e_i)$, hence $\sigma e_{k+1} = \rho e_i$, which means $e_{k+1} < e_i$.

There remains the case $\{r, s\} = \{j, k + 1\}$, $j \neq i$, $j \leq k$, with

$$(4.6) \quad [A_j, A_{k+1}] \cap G_{k+1} \neq 0,$$

that is either $A_j < A_{k+1}$ or $A_{k+1} < A_j$ in G_{k+1} .

In this case A_{k+1} , which by (4.4) has an edge in common with A_i , now also has an edge in common with $A_j \neq A_i$, thus connecting these two distinct vertices of G_k by the path

$$(4.7) \quad A_i, A_{k+1}, A_j$$

in G_{k+1} but outside G_k .

On the other hand, by the induction's hypothesis, G_k is connected. Hence A_i and A_j are connected by a path in G_k

$$(4.8) \quad A_i, A_{t_1}, A_{t_2}, \dots, A_{t_\lambda}, A_j$$

($\lambda = 0$ not a priori excluded).

The paths (4.7) and (4.8) combine to the loop

$$(4.9) \quad A_{k+1}, A_i, A_{t_1}, A_{t_2}, \dots, A_{t_\lambda}, A_j, A_{k+1}$$

in G_{k+1} , which is obviously also a loop in G .

Since G is alternating, the loop (4.9) must be alternating. This implies that the number of vertices is even, hence $\lambda = 2\nu + 1$, and that the orientation is either

$$(4.10) \quad A_{k+1} < A_i > A_{t_1} < A_{t_2} > \cdots < A_{t_{2\nu}} > A_{t_{2\nu+1}} < A_j > A_{k+1}$$

or the opposite.

Now assume first

$$(4.11) \quad A_{k+1} < A_j ,$$

which implies the orientation (4.10), and consider that part of the loop which is in G_k , namely the path (4.8)

(4.10) and the induction's hypothesis that, relative to G_k and F_k , φ satisfies (4.2), imply

$$(4.12) \quad e_i > e_{i_1} < e_{i_2} > \cdots < e_{i_{2\nu}} > e_{i_{2\nu+1}} < e_j ,$$

hence

$$(4.13) \quad \rho e_i = \sigma e_{i_1} = \rho e_{i_2} = \sigma e_{i_3} = \cdots = \rho e_{i_{2\nu}} = \sigma e_{i_{2\nu+1}} = \rho e_j .$$

The definition (4.5) of e_{k+1} , in conjunction with $A_{k+1} < A_i$ from (4.10), implies

$$(4.14) \quad \sigma e_{k+1} = \rho e_i .$$

This together with (4.13) yields

$$(4.15) \quad \sigma e_{k+1} = \rho e_j , \quad \text{that is } e_{k+1} < e_j ,$$

which proves that assumption (4.11) implies (4.15).

Similarly, the assumption $A_{k+1} > A_j$ yields $e_{k+1} > e_j$, by reversing the relation $<$ and interchanging ρ and σ in the above argument.

This completes the proof that to any connected alternating graph G there exists a connected oriented graph F and a mapping φ satisfying (4.2)

That F has no loops (and hence is a tree) is obvious from the fact that its $n + 1$ vertices are connected by n edges. Hence, the incidence matrices of F certainly belong to class H .

If G consists of k components, the construction will yield an F consisting of k trees.

This completes the proof of the theorem's first half, namely that every K -matrix is an H -matrix.

The second half of the theorem, namely that each nonnegative H -matrix is a K -matrix, is due to J. Edmonds. It will be proved by showing that to each loopless oriented F there is an alternating G and a mapping ψ of the edges of F onto the vertices of G that preserves the relation $<$, that is, for any two edges a, b of F

$$(4.16) \quad a < b \implies \psi(a) < \psi(b) .$$

This is achieved by the following simple construction.

If F has n edges e_1, e_2, \dots, e_n , choose a set of n elements A_1, A_2, \dots, A_n as the vertices of G , define ψ by

$$(4.17) \quad \psi e_i = A_i ,$$

and define the edges of G by

$$(4.18) \quad (A_i, A_j) \in G \iff e_i < e_j ,$$

that is, G shall have an edge oriented from A_i to A_j if and only if $\sigma e_i = \rho e_j$.

Obviously ψ preserves the relation $<$, since (4.18) is equivalent to

$$(4.19) \quad A_i < A_j \iff e_i < e_j .$$

Note that $<$ is also preserved by the inverse of ψ , that is, in the transition from G to F .

Note further that G is oriented (in the sense of the definition given in [4] and cited in §1 of present note), that is:

(a) each edge of G is oriented, since the edges of G have been defined by (4.18) as oriented edges;

(b) G has no circular edge, since $(A_i, A_i) \in G$ for some i would imply $e_i < e_i$, or equivalently $\sigma e_i = \rho e_i$, that is, e_i a circular edge in F , contradicting the assumption on F ;

(c) G has at most one edge between any given two vertices: $(A_i, A_j) \in G$ and $(A_j, A_i) \in G$ for some pair i, j , would imply $e_i < e_j$ and $e_j < e_i$, that is $\sigma e_i = \rho e_j$ and $\sigma e_j = \rho e_i$, hence e_i and e_j would form a 2-loop (with the vertices $\rho e_i, \sigma e_i$), again contradicting the assumption on F .

Finally, to show that G is alternating, note that, by (4.17) and (4.19), G, F and $\varphi = \psi^{-1}$ satisfy the condition (4.1). Thus the incidence matrices (of vertices versus directed paths) associated with G are among the incidence matrices (edges versus paths) associated with F , and hence unimodular. Especially then, the incidence matrix of the vertices versus *all* the directed paths of G is unimodular, which, by the Hoffman-Kruskal Theorem (Theorem 4 in [4], cited in §1 of this note), implies that G is necessarily alternating.

This completes proof of the theorem.

It is worth noting that the last part of the proof (namely that G is alternating) can easily be established without using the result of [4] (which contains more than is needed here).

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