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Let $G$ be a finite $p$-group where $p$ is an odd prime. We say that $G$ has *property* $A_n$ if every abelian normal subgroup of $G$ can be generated by $n$ elements. Further, if $G_n$ denotes the $n$th element in the descending central series of $G$, we say that $G$ has property $A_n(G_n)$ if every abelian subgroup of $G_n$ which is normal in $G$ can be generated by $n$ elements. If $G$ has property $A_1$, then $G$ is cyclic. N. Blackburn [1] found all of the groups which have property $A_2$. It follows from the work of Blackburn that if $G$ has property $A_2$ then the derived group of $G$ is abelian and every subgroup of $G$ has property $A_2$. We shall show that if $G$ has property $A_3$ then every subgroup of $G$ has property $A_3$. There exist groups which have property $A_3$ in which the derived series is arbitrarily long [2] so no analogue of Blackburn's result on the derived group is possible. We next consider groups $G$ which have property $A_n(G_n)$ and show that $G_n$ can be generated by $n$ elements. This leads to the existence of a bound on the derived length of $G$ which depends only on $n$ and the exponent of $G_n$.

We shall use the following notation: $p$ is an odd prime; $G=G_1\supset G_2\supset\cdots$ is the descending central series of $G$; $Z(G)=Z_1(G)\subset Z_2(G)\subset\cdots$ is the ascending central series of $G$; $G^{(k)}$ is the $k$th derived group of $G$; $(H, K)$ is the subgroup of $G$ generated by all elements $(h, k)=h^{-1}k^{-1}hk$ for $h\in H$, $k\in K$; $N\lhd G$ means $N$ is normal in $G$; $N\subset G$ means $N$ is properly contained in $G$; $C_G(N)$ is the centralizer of $N$ in $G$; $H^G$ is the normal subgroup of $G$ generated by $H$; $\mho(G)$ is the subgroup generated by $p$th powers of elements of $G$. $\Omega(G)$ is the subgroup generated by all elements of order $p$ in $G$; $\phi(G)$ is the Frattini subgroup of $G$; $|G|$ is the order of $G$.

If $A\lhd G$ and $A\subset C_G(A)$, then there is a subgroup $B$ of $C_G(A)$ such that $B\lhd G$ and $[B\colon A]=p$. It follows that if a normal subgroup $A$ of $G$ is properly contained in an abelian subgroup $C$ of $G$, then $A$ is properly contained in some abelian normal subgroup $B$ of $G$.

LEMMA 1. *Suppose $A\lhd G$ and $A\subset C$ where $C$ is an elementary abelian subgroup of $G$. Then $G$ contains an elementary abelian normal subgroup $B$ such that $A$ is a subgroup of index $p$ in $B$.*

*Proof.* Suppose $G$ is a group of minimal order for which the lemma is false. Then $C\subset G$, so there is a subgroup $M$ of index $p$ in $G$ which contains $C$. It follows that $M$ contains an elementary abelian normal subgroup $B_1$ such that $[B_1\colon A]=p$. Set $D=M\cap C_G(A)$. Then $B_1\lhd D\lhd G$.

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Since $(D, B_1) \subseteqq A$ and $(D, A) = 1$, we have $B_1 \subseteqq Z_2(D) \lhd G$. Therefore $B_1^G \subseteqq Z_2(D)$. But $Z_2(D)$ is a regular $p$-group for $p > 2$, so $B_1^G$ has exponent $p$. Let $B$ be a subgroup of $B_1^G$ which is normal in $G$ and which contains $A$ as a subgroup of index $p$. Clearly $B$ is elementary abelian, so the lemma is true for $G$.

THEOREM 1. *If $G$ has property $A_3$ then every subgroup of $G$ has property $A_3$.*

*Proof.* Suppose $G$ is a group of minimal order for which the theorem is false. Then $G$ contains an elementary abelian normal subgroup $A$ of order $p^3$, and there is a subgroup $M$ of index $p$ in $G$ which does not have property $A_3$. It follows that $M$ contains an elementary abelian normal subgroup $D$ of order $p^4$. Let $N$ be a subgroup of order $p^2$ in $A$ which is contained in $M$ and which is normal in $G$. If we let $C = C_G(N)$, then $[G: C] \leqq p$, hence $[D: D \cap C] \leqq p$. Thus we may suppose that $N \subset D$, since otherwise we could choose a new subgroup $D_1$ in $(C \cap D)N$ such that $N \subset D_1 \lhd M$ and $D_1$ is elementary abelian of order $p^4$.

Since $G$ has property $A_3$ it follows from Lemma 1 that $A$ contains the only elements of order $p$ in $C_G(A)$. Therefore $N = D \cap C_G(A)$. It is easy to see that $[C: C_G(A)] \leqq p^2$, thus $C = DC_G(A)$. Therefore, if $d \in D$, $g \in G$, then $g^{-1}dg = d_1c$ for some $d_1 \in D$, $c \in C_G(A)$. We recall that $D$ is an abelian normal subgroup of $M$, and that $M \lhd G$. Thus $D$ and $g^{-1}Dg$ generate a group of class at most two; hence for $p > 2$ the group generated by $D$ and $g^{-1}Dg$ has exponent $p$. Thus it follows from $g^{-1}dg = d_1c$ that $c^p = 1$, whence $c \in A$. Therefore $AD \lhd G$. But $A \cap D = N$, so $[AD: D] = p$. Since $D$ is not normal in $G$, we must have $AD = D(g^{-1}Dg)$ for some element $g \in G$. Therefore $D \cap g^{-1}Dg$ has order at least $p^3$ and is contained in $Z_1(AD)$ which is normal in $G$. Thus $AD$ must contain an element of order $p$ which centralizes $A$ and which does not belong to $A$. This is a contradiction.

THEOREM 2. *If $G$ has property $A_n(G_n)$ then $G_n$ can be generated by $n$ elements.*

*Proof.* Suppose $G$ is a group of minimal order for which the theorem is false. Then $G_n$ is not abelian, so $\phi(G_n) \neq 1$. Let $Z$ be a group of order $p$ in $Z_1(G) \cap \phi(G_n)$. Then $G_n$ and $(G/Z)_n$ have the same number of generators, so $(G/Z)_n$ must contain an elementary abelian subgroup $B/Z$ of order $p^{n+1}$ which is normal in $G/Z$. Let $B$ be the preimage of $B/Z$ in $G$. Then $B \lhd G$, $B$ has order $p^{n+2}$, and $B^{(1)} \subseteqq Z$. Thus $B$ has class at most two, hence is regular for $p > 2$. But $\mho(B) \subseteqq Z$, so $\Omega(B)$ is a group of order at least $p^{n+1}$ which is normal in $G$. Thus there is a subgroup $A$ of $\Omega(B)$ such that $A \lhd G$, $\mho(A) = 1$, and $A$ has order $p^{n+1}$. Let $N$ be a subgroup of index $p$ in $A$ which is normal in $G$. Then $|N| = p^n$ and $N \lhd G$ imply $N \subseteqq Z_n(G)$, whence $N \subseteqq Z_1(G_n)$. Therefore $A$ is abelian, a contradiction.

COROLLARY. *Suppose $G$ has property $A_n(G_n)$, where $G_n$ has exponent $p^m$. Let $k$ be an integer such that $2^k \geqq n$. Then $G^{(k+m)} = 1$.*

*Proof.* By Theorem 2, $G_n$ can be generated by $n$ elements. Therefore [3, Theorem 2] $\phi(G_n) = \Omega(G_n)$. It follows that $G_n^{(m)} = \langle 1 \rangle$ [4, Theorem 2]. In any $p$-group, $G^{(t)} \subseteqq G_{2^t}$. Therefore $G^{(k)} \subseteqq G_n$, whence $G^{(k+m)} = \langle 1 \rangle$.

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