ABELIAN SUBGROUPS OF $p$-GROUPS

CHARLES RAY HOBBY
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Let \( G \) be a finite \( p \)-group where \( p \) is an odd prime. We say that \( G \) has property \( A_n \) if every abelian normal subgroup of \( G \) can be generated by \( n \) elements. Further, if \( G_n \) denotes the \( n \)th element in the descending central series of \( G \), we say that \( G \) has property \( A_n(G_n) \) if every abelian subgroup of \( G_n \) which is normal in \( G \) can be generated by \( n \) elements. If \( G \) has property \( A_1 \), then \( G \) is cyclic. N. Blackburn [1] found all of the groups which have property \( A_2 \). It follows from the work of Blackburn that if \( G \) has property \( A_2 \) then the derived group of \( G \) is abelian and every subgroup of \( G \) has property \( A_2 \). We shall show that if \( G \) has property \( A_2 \) then every subgroup of \( G \) has property \( A_3 \).

We shall use the following notation: \( p \) is an odd prime; \( G = G_1 \supseteq G_2 \supseteq \cdots \) is the descending central series of \( G \); \( Z(G) = Z_1(G) \subset Z_2(G) \subset \cdots \) is the ascending central series of \( G \); \( G^{(k)} \) is the \( k \)th derived group of \( G \); \( (H, K) \) is the subgroup of \( G \) generated by all elements \( (h, k) = h^{-1}k^{-1}hk \) for \( h \in H, k \in K \); \( N \triangleleft G \) means \( N \) is normal in \( G \); \( N \subseteq G \) means \( N \) is properly contained in \( G \); \( C_o(N) \) is the centralizer of \( N \) in \( G \); \( H^p \) is the normal subgroup of \( G \) generated by \( H \); \( \varphi(G) \) is the subgroup generated by \( p \)th powers of elements of \( G \). \( \Omega(G) \) is the subgroup generated by all elements of order \( p \) in \( G \); \( \varphi(G) \) is the Frattini subgroup of \( G \); \( |G| \) is the order of \( G \).

If \( A \triangleleft G \) and \( A \subseteq C_o(A) \), then there is a subgroup \( B \) of \( C_o(A) \) such that \( B \triangleleft G \) and \( [B; A] = p \). It follows that if a normal subgroup \( A \) of \( G \) is properly contained in an abelian subgroup \( C \) of \( G \), then \( A \) is properly contained in some abelian normal subgroup \( B \) of \( G \).

**Lemma 1.** Suppose \( A \triangleleft G \) and \( A \subseteq C \) where \( C \) is an elementary abelian subgroup of \( G \). Then \( G \) contains an elementary abelian normal subgroup \( B \) such that \( A \) is a subgroup of index \( p \) in \( B \).

**Proof.** Suppose \( G \) is a group of minimal order for which the lemma is false. Then \( C \subseteq G \), so there is a subgroup \( M \) of index \( p \) in \( G \) which contains \( C \). It follows that \( M \) contains an elementary abelian normal subgroup \( B_1 \) such that \( [B_1; A] = p \). Set \( D = M \cap C_o(A) \). Then \( B_1 \triangleleft D \triangleleft G \).

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Since \((D, B_λ) \leq A\) and \((A A) = 1\), we have \(B_1 \leq Z_2(D) \triangleleft G\). Therefore \(B_2^g \leq Z_2(D)\). But \(Z_2(D)\) is a regular \(p\)-group for \(p > 2\), so \(B_1^g\) has exponent \(p\). Let \(B\) be a subgroup of \(B_1^g\) which is normal in \(G\) and which contains \(A\) as a subgroup of index \(p\). Clearly \(B\) is elementary abelian, so the lemma is true for \(G\).

**Theorem 1.** If \(G\) has property \(A_3\) then every subgroup of \(G\) has property \(A_3\).

**Proof.** Suppose \(G\) is a group of minimal order for which the theorem is false. Then \(G\) contains an elementary abelian normal subgroup \(A\) of order \(p^3\), and there is a subgroup \(M\) of index \(p\) in \(G\) which does not have property \(A_3\). It follows that \(M\) contains an elementary abelian normal subgroup \(D\) of order \(p^4\). Let \(N\) be a subgroup of order \(p^2\) in \(A\) which is contained in \(M\) and which is normal in \(G\). If we let \(C = C_0(N)\), then \([G: C] \leq p\), hence \([D: D \cap C] \leq p\). Thus we may suppose that \(N \leq D\), since otherwise we could choose a new subgroup \(D_i\) in \((C \cap D)N\) such that \(N \leq D_i \leq M\) and \(D_i\) is elementary abelian of order \(p^4\).

Since \(G\) has property \(A_3\) it follows from Lemma 1 that \(A\) contains the only elements of order \(p\) in \(C_0(A)\). Therefore \(N = D \cap C_0(A)\). It is easy to see that \([C: C_0(A)] \leq p^2\), thus \(C = DC_0(A)\). Therefore, if \(d \in D, g \in G\), then \(g^{-1}dg = d_c\) for some \(d_c \in D, c \in C_0(A)\). We recall that \(D\) is an abelian normal subgroup of \(M\), and that \(M \triangleleft G\). Thus \(D\) and \(g^{-1}Dg\) generate a group of class at most two; hence for \(p > 2\) the group generated by \(D\) and \(g^{-1}Dg\) has exponent \(p\). Thus it follows from \(g^{-1}dg = d_c\) that \(c^p = 1\), whence \(c \in A\). Therefore \(AD \triangleleft G\). But \(A \cap D = N\), so \([AD: D] = p\). Since \(D\) is not normal in \(G\), we must have \(AD = D(g^{-1}Dg)\) for some element \(g \in G\). Therefore \(D \cap g^{-1}Dg\) has order at least \(p^4\) and is contained in \(Z_2(AD)\) which is normal in \(G\). Thus \(AD\) must contain an element of order \(p\) which centralizes \(A\) and which does not belong to \(A\). This is a contradiction.

**Theorem 2.** If \(G\) has property \(A_n\) then \(G_n\) can be generated by \(n\) elements.

**Proof.** Suppose \(G\) is a group of minimal order for which the theorem is false. Then \(G_n\) is not abelian, so \(\phi(G_n) \neq 1\). Let \(Z\) be a group of order \(p\) in \(Z_1(G) \cap \phi(G_n)\). Then \(G_n\) and \((G/Z)_n\) have the same number of generators, so \((G/Z)_n\) must contain an elementary abelian subgroup \(B/Z\) of order \(p^{n+1}\) which is normal in \(G/Z\). Let \(B\) be the preimage of \(B/Z\) in \(G\). Then \(B \triangleleft G\), \(B\) has order \(p^{n+2}\), and \(B^{(1)} \leq Z\). Thus \(B\) has class at most two, hence is regular for \(p > 2\). But \(\gamma(B) \leq Z\), so \(\Omega(B)\) is a group of order at least \(p^{n+1}\) which is normal in \(G\). Thus there is
a subgroup $A$ of $\Omega(B)$ such that $A \triangleleft G$, $\sigma(A) = 1$, and $A$ has order $p^{n+1}$. Let $N$ be a subgroup of index $p$ in $A$ which is normal in $G$. Then $|N| = p^n$ and $N \triangleleft G$ imply $N \subseteq Z_s(G)$, whence $N \subseteq Z_t(G)$. Therefore $A$ is abelian, a contradiction.

**Corollary.** Suppose $G$ has property $A_n(G_n)$, where $G_n$ has exponent $p^m$. Let $k$ be an integer such that $2^k \geq n$. Then $G^{(k+m)} = 1$.

**Proof.** By Theorem 2, $G_n$ can be generated by $n$ elements. Therefore [3, Theorem 2] $\phi(G_n) = \Omega(G_n)$. It follows that $G_n^{(m)} = \langle 1 \rangle$ [4, Theorem 2]. In any $p$-group, $G^{(1)} \subseteq G_n$. Therefore $G^{(k)} \subseteq G_n$, whence $G^{(k+m)} = \langle 1 \rangle$.

**References**


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