

Pacific Journal of Mathematics

ABELIAN SUBGROUPS OF p -GROUPS

CHARLES RAY HOBBY

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Let G be a finite p -group where p is an odd prime. We say that G has *property* A_n if every abelian normal subgroup of G can be generated by n elements. Further, if G_n denotes the n th element in the descending central series of G , we say that G has *property* $A_n(G_n)$ if every abelian subgroup of G_n which is normal in G can be generated by n elements. If G has property A_1 , then G is cyclic. N. Blackburn [1] found all of the groups which have property A_2 . It follows from the work of Blackburn that if G has property A_2 then the derived group of G is abelian and every subgroup of G has property A_2 . We shall show that if G has property A_3 then every subgroup of G has property A_3 . There exist groups which have property A_3 in which the derived series is arbitrarily long [2] so no analogue of Blackburn's result on the derived group is possible. We next consider groups G which have property $A_n(G_n)$ and show that G_n can be generated by n elements. This leads to the existence of a bound on the derived length of G which depends only on n and the exponent of G_n .

We shall use the following notation: p is an odd prime; $G = G_1 \supset G_2 \supset \dots$ is the descending central series of G ; $Z(G) = Z_1(G) \subset Z_2(G) \subset \dots$ is the ascending central series of G ; $G^{(k)}$ is the k th derived group of G ; (H, K) is the subgroup of G generated by all elements $(h, k) = h^{-1}k^{-1}hk$ for $h \in H, k \in K$; $N \triangleleft G$ means N is normal in G ; $N \subset G$ means N is properly contained in G ; $C_G(N)$ is the centralizer of N in G ; H^G is the normal subgroup of G generated by H ; $\mathcal{O}(G)$ is the subgroup generated by p th powers of elements of G . $\Omega(G)$ is the subgroup generated by all elements of order p in G ; $\phi(G)$ is the Frattini subgroup of G ; $|G|$ is the order of G .

If $A \triangleleft G$ and $A \subset C_G(A)$, then there is a subgroup B of $C_G(A)$ such that $B \triangleleft G$ and $[B: A] = p$. It follows that if a normal subgroup A of G is properly contained in an abelian subgroup C of G , then A is properly contained in some abelian normal subgroup B of G .

LEMMA 1. *Suppose $A \triangleleft G$ and $A \subset C$ where C is an elementary abelian subgroup of G . Then G contains an elementary abelian normal subgroup B such that A is a subgroup of index p in B .*

Proof. Suppose G is a group of minimal order for which the lemma is false. Then $C \subset G$, so there is a subgroup M of index p in G which contains C . It follows that M contains an elementary abelian normal subgroup B_1 such that $[B_1: A] = p$. Set $D = M \cap C_G(A)$. Then $B_1 \triangleleft D \triangleleft G$.

Received May 29, 1962.

Since $(D, B_1) \cong A$ and $(D, A) = 1$, we have $B_1 \cong Z_2(D) \triangleleft G$. Therefore $B_1^g \cong Z_2(D)$. But $Z_2(D)$ is a regular p -group for $p > 2$, so B_1^g has exponent p . Let B be a subgroup of B_1^g which is normal in G and which contains A as a subgroup of index p . Clearly B is elementary abelian, so the lemma is true for G .

THEOREM 1. *If G has property A_3 then every subgroup of G has property A_3 .*

Proof. Suppose G is a group of minimal order for which the theorem is false. Then G contains an elementary abelian normal subgroup A of order p^3 , and there is a subgroup M of index p in G which does not have property A_3 . It follows that M contains an elementary abelian normal subgroup D of order p^4 . Let N be a subgroup of order p^2 in A which is contained in M and which is normal in G . If we let $C = C_G(N)$, then $[G:C] \leq p$, hence $[D:D \cap C] \leq p$. Thus we may suppose that $N \subset D$, since otherwise we could choose a new subgroup D_1 in $(C \cap D)N$ such that $N \subset D_1 \triangleleft M$ and D_1 is elementary abelian of order p^4 .

Since G has property A_3 it follows from Lemma 1 that A contains the only elements of order p in $C_G(A)$. Therefore $N = D \cap C_G(A)$. It is easy to see that $[C:C_G(A)] \leq p^2$, thus $C = DC_G(A)$. Therefore, if $d \in D, g \in G$, then $g^{-1}dg = d_1c$ for some $d_1 \in D, c \in C_G(A)$. We recall that D is an abelian normal subgroup of M , and that $M \triangleleft G$. Thus D and $g^{-1}Dg$ generate a group of class at most two; hence for $p > 2$ the group generated by D and $g^{-1}Dg$ has exponent p . Thus it follows from $g^{-1}dg = d_1c$ that $c^p = 1$, whence $c \in A$. Therefore $AD \triangleleft G$. But $A \cap D = N$, so $[AD:D] = p$. Since D is not normal in G , we must have $AD = D(g^{-1}Dg)$ for some element $g \in G$. Therefore $D \cap g^{-1}Dg$ has order at least p^3 and is contained in $Z_1(AD)$ which is normal in G . Thus AD must contain an element of order p which centralizes A and which does not belong to A . This is a contradiction.

THEOREM 2. *If G has property $A_n(G_n)$ then G_n can be generated by n elements.*

Proof. Suppose G is a group of minimal order for which the theorem is false. Then G_n is not abelian, so $\phi(G_n) \neq 1$. Let Z be a group of order p in $Z_1(G) \cap \phi(G_n)$. Then G_n and $(G/Z)_n$ have the same number of generators, so $(G/Z)_n$ must contain an elementary abelian subgroup B/Z of order p^{n+1} which is normal in G/Z . Let B be the preimage of B/Z in G . Then $B \triangleleft G$, B has order p^{n+2} , and $B^{(1)} \cong Z$. Thus B has class at most two, hence is regular for $p > 2$. But $\mathcal{C}(B) \cong Z$, so $\Omega(B)$ is a group of order at least p^{n+1} which is normal in G . Thus there is

a subgroup A of $\Omega(B)$ such that $A \triangleleft G$, $\mathcal{O}(A) = 1$, and A has order p^{n+1} . Let N be a subgroup of index p in A which is normal in G . Then $|N| = p^n$ and $N \triangleleft G$ imply $N \subseteq Z_n(G)$, whence $N \subseteq Z_1(G_n)$. Therefore A is abelian, a contradiction.

COROLLARY. *Suppose G has property $A_n(G_n)$, where G_n has exponent p^n . Let k be an integer such that $2^k \geq n$. Then $G^{(k+m)} = 1$.*

Proof. By Theorem 2, G_n can be generated by n elements. Therefore [3, Theorem 2] $\phi(G_n) = \Omega(G_n)$. It follows that $G_n^{(m)} = \langle 1 \rangle$ [4, Theorem 2]. In any p -group, $G^{(t)} \subseteq G_{2^t}$. Therefore $G^{(k)} \subseteq G_n$, whence $G^{(k+m)} = \langle 1 \rangle$.

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50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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