

Pacific Journal of Mathematics

ABELIAN SUBGROUPS OF p -GROUPS

CHARLES RAY HOBBY

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Let G be a finite p -group where p is an odd prime. We say that G has *property* A_n if every abelian normal subgroup of G can be generated by n elements. Further, if G_n denotes the n th element in the descending central series of G , we say that G has *property* $A_n(G_n)$ if every abelian subgroup of G_n which is normal in G can be generated by n elements. If G has *property* A_1 , then G is cyclic. N. Blackburn [1] found all of the groups which have *property* A_2 . It follows from the work of Blackburn that if G has *property* A_2 then the derived group of G is abelian and every subgroup of G has *property* A_2 . We shall show that if G has *property* A_3 then every subgroup of G has *property* A_3 . There exist groups which have *property* A_3 in which the derived series is arbitrarily long [2] so no analogue of Blackburn's result on the derived group is possible. We next consider groups G which have *property* $A_n(G_n)$ and show that G_n can be generated by n elements. This leads to the existence of a bound on the derived length of G which depends only on n and the exponent of G_n .

We shall use the following notation: p is an odd prime; $G = G_1 \supset G_2 \supset \dots$ is the descending central series of G ; $Z(G) = Z_1(G) \subset Z_2(G) \subset \dots$ is the ascending central series of G ; $G^{(k)}$ is the k th derived group of G ; $\langle H, K \rangle$ is the subgroup of G generated by all elements $(h, k) = h^{-1}k^{-1}hk$ for $h \in H, k \in K$; $N \triangleleft G$ means N is normal in G ; $N \subset G$ means N is properly contained in G ; $C_a(N)$ is the centralizer of N in G ; H^a is the normal subgroup of G generated by H ; $\mathcal{O}(G)$ is the subgroup generated by p th powers of elements of G . $\Omega(G)$ is the subgroup generated by all elements of order p in G ; $\phi(G)$ is the Frattini subgroup of G ; $|G|$ is the order of G .

If $A \triangleleft G$ and $A \subset C_a(A)$, then there is a subgroup B of $C_a(A)$ such that $B \triangleleft G$ and $[B: A] = p$. It follows that if a normal subgroup A of G is properly contained in an abelian subgroup C of G , then A is properly contained in some abelian normal subgroup B of G .

LEMMA 1. *Suppose $A \triangleleft G$ and $A \subset C$ where C is an elementary abelian subgroup of G . Then G contains an elementary abelian normal subgroup B such that A is a subgroup of index p in B .*

Proof. Suppose G is a group of minimal order for which the lemma is false. Then $C \subset G$, so there is a subgroup M of index p in G which contains C . It follows that M contains an elementary abelian normal subgroup B_1 such that $[B_1: A] = p$. Set $D = M \cap C_a(A)$. Then $B_1 \triangleleft D \triangleleft G$.

Since $(D, B_1) \cong A$ and $(D, A) = 1$, we have $B_1 \cong Z_2(D) \triangleleft G$. Therefore $B_1^g \cong Z_2(D)$. But $Z_2(D)$ is a regular p -group for $p > 2$, so B_1^g has exponent p . Let B be a subgroup of B_1^g which is normal in G and which contains A as a subgroup of index p . Clearly B is elementary abelian, so the lemma is true for G .

THEOREM 1. *If G has property A_3 then every subgroup of G has property A_3 .*

Proof. Suppose G is a group of minimal order for which the theorem is false. Then G contains an elementary abelian normal subgroup A of order p^3 , and there is a subgroup M of index p in G which does not have property A_3 . It follows that M contains an elementary abelian normal subgroup D of order p^4 . Let N be a subgroup of order p^2 in A which is contained in M and which is normal in G . If we let $C = C_G(N)$, then $[G:C] \leq p$, hence $[D:D \cap C] \leq p$. Thus we may suppose that $N \subset D$, since otherwise we could choose a new subgroup D_1 in $(C \cap D)N$ such that $N \subset D_1 \triangleleft M$ and D_1 is elementary abelian of order p^4 .

Since G has property A_3 it follows from Lemma 1 that A contains the only elements of order p in $C_G(A)$. Therefore $N = D \cap C_G(A)$. It is easy to see that $[C:C_G(A)] \leq p^2$, thus $C = DC_G(A)$. Therefore, if $d \in D, g \in G$, then $g^{-1}dg = d_1c$ for some $d_1 \in D, c \in C_G(A)$. We recall that D is an abelian normal subgroup of M , and that $M \triangleleft G$. Thus D and $g^{-1}Dg$ generate a group of class at most two; hence for $p > 2$ the group generated by D and $g^{-1}Dg$ has exponent p . Thus it follows from $g^{-1}dg = d_1c$ that $c^p = 1$, whence $c \in A$. Therefore $AD \triangleleft G$. But $A \cap D = N$, so $[AD:D] = p$. Since D is not normal in G , we must have $AD = D(g^{-1}Dg)$ for some element $g \in G$. Therefore $D \cap g^{-1}Dg$ has order at least p^3 and is contained in $Z_1(AD)$ which is normal in G . Thus AD must contain an element of order p which centralizes A and which does not belong to A . This is a contradiction.

THEOREM 2. *If G has property $A_n(G_n)$ then G_n can be generated by n elements.*

Proof. Suppose G is a group of minimal order for which the theorem is false. Then G_n is not abelian, so $\phi(G_n) \neq 1$. Let Z be a group of order p in $Z_1(G) \cap \phi(G_n)$. Then G_n and $(G/Z)_n$ have the same number of generators, so $(G/Z)_n$ must contain an elementary abelian subgroup B/Z of order p^{n+1} which is normal in G/Z . Let B be the preimage of B/Z in G . Then $B \triangleleft G$, B has order p^{n+2} , and $B^{(1)} \cong Z$. Thus B has class at most two, hence is regular for $p > 2$. But $\mathcal{O}(B) \cong Z$, so $\Omega(B)$ is a group of order at least p^{n+1} which is normal in G . Thus there is

a subgroup A of $\Omega(B)$ such that $A \triangleleft G$, $\mathcal{C}(A) = 1$, and A has order p^{n+1} . Let N be a subgroup of index p in A which is normal in G . Then $|N| = p^n$ and $N \triangleleft G$ imply $N \subseteq Z_n(G)$, whence $N \subseteq Z_1(G_n)$. Therefore A is abelian, a contradiction.

COROLLARY. *Suppose G has property $A_n(G_n)$, where G_n has exponent p^m . Let k be an integer such that $2^k \geq n$. Then $G^{(k+m)} = 1$.*

Proof. By Theorem 2, G_n can be generated by n elements. Therefore [3, Theorem 2] $\phi(G_n) = \Omega(G_n)$. It follows that $G_n^{(m)} = \langle 1 \rangle$ [4, Theorem 2]. In any p -group, $G^{(t)} \subseteq G_{2^t}$. Therefore $G^{(k)} \subseteq G_n$, whence $G^{(k+m)} = \langle 1 \rangle$.

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