

Pacific Journal of Mathematics

**NON-EXISTENCE OF ALMOST-COMPLEX STRUCTURES ON
QUATERNIONIC PROJECTIVE SPACES**

WILLIAM SCHUMACHER MASSEY

NON-EXISTENCE OF ALMOST-COMPLEX STRUCTURES ON QUATERNIONIC PROJECTIVE SPACES

W. S. MASSEY

1. Introduction. In [3] F. Hirzebruch proved that the n -dimensional quaternionic projective space (which we denote by $P_n(\mathbf{H})$) does not admit any almost structure in case $n \neq 2$ or 3 . According to Hirzebruch's lecture at the 1958 International Congress [5], Milnor has since proved that $P_2(\mathbf{H})$ and $P_3(\mathbf{H})$ do not admit almost complex structures. At the time of this writing, Milnor's proof has not yet been published.

It is the purpose of this note to give a short proof of this theorem of Milnor's making use of the theory of the ring $K(X)$ of complex vector bundles over a space X due to Atiyah and Hirzebruch together with certain facts that are readily available in the literature. From the brief description given in Hirzebruch's lecture (loc. cit.) it seems that our method is quite different from Milnor's. Our method may be applicable in other cases to prove the existence or nonexistence of almost complex structures on a manifold.

2. Summary of some known facts. We will make use of the following results:

(a) The cohomology ring $H^*(P_n(\mathbf{H}), \mathbf{Z})$ is a truncated polynomial ring generated by a 4-dimensional cohomology class u and subject to the single relation $u^{n+1} = 0$.

(b) Let τ_n denote the tangent bundle to $P_n(\mathbf{H})$. The total Pontrjagin class of τ_n is given by the formula

$$(1) \quad p(\tau_n) = (1 + u)^{2n+2}(1 + 4u)^{-1}$$

for appropriate choice of the generator u (Borel and Hirzebruch [2], 15.5 or Hirzebruch [3]).

(c) We will use the following notation: If ξ is a real n -plane bundle, then $\xi \otimes C$ denotes its complexification, while if ξ is a complex n -plane bundle, then $\xi_{\mathbf{R}}$ denotes the real $2n$ -plane bundle obtained by "restriction of coefficients" to the reals. Also, ξ^* denotes the complex conjugate bundle. We then have the following relation for any complex vector bundle ξ :

$$(2) \quad \xi_{\mathbf{R}} \otimes C = \xi + \xi^* \quad (\text{Whitney sum}).$$

(see Hirzebruch, [4], p. 68, proof of Theorem 4.5.1). Moreover, for any

Received January 10, 1962. During the preparation of this note the author was partially supported by N. S. F. Grant G 18995.

real vector bundle η ,

$$(3) \quad p_i(\eta) = (-1)^i c_{2i}(\eta \otimes C),$$

(Hirzebruch [4], p. 67).

(d) We will also need to make use of the properties of the ring $K(X)$ as summarized for example in § 2 of Atiyah and Todd [1]. We will use their notation and results without any further comment.

3. **The ring $K(P_n(\mathbf{H}))$.** By proposition 2.3 of [1] the Chern character

$$ch: K(P_n(\mathbf{H})) \rightarrow H^*(P_n(\mathbf{H}), \mathbf{Q})$$

is a monomorphism.

LEMMA 1. *The image of this homomorphism, denoted by $chP_n(\mathbf{H})$, is the subring of $H^*(P_n(\mathbf{H}), \mathbf{Q})$ generated by*

$$2 \cosh \sqrt{v} = 2(1 + v/2! + v^2/4! + \cdots + v^n/(2n)!)$$

where v is an appropriately chosen generator of

$$H^4(P_n(\mathbf{H}), \mathbf{Z}) \subset H^4(\mathbf{H}), \mathbf{Q}.$$

Proof. Consider the well-known principal fibre bundle $\pi_n: S^{4n+3} \rightarrow P_n(\mathbf{H})$ with group $Sp(1)$; let η_n denote the associated bundle with fibre a quaternionic vector space of dimension 1. We assert that the total symplectic Pontrjagin class of this bundle is

$$e(\eta_n) = 1 + e_1(\eta_n) = 1 + v$$

where v is an appropriately chosen generator of $H^4(P_n(\mathbf{H}), \mathbf{Z})$. This follows from the fact that $\pi_n: S^{4n+3} \rightarrow P_n(\mathbf{H})$ is a universal bundle for the group $Sp(1)$ (up to the dimension $4n + 2$), and that the integral cohomology ring of the classifying space for $Sp(1)$ is a polynomial ring generated by the symplectic Pontrjagin class e_1 (see Borel and Hirzebruch, [2], § 9.6).

Let ξ_n denote the complex vector bundle obtained from η_n by "restriction of coefficients" to the complex numbers; the associated principal bundle is a $U(2)$ -bundle which is the extension of $\pi_n: S^{4n+3} \rightarrow P_n(\mathbf{H})$ under the standard inclusion $Sp(1) \subset U(2)$. By § 9.6 of [2],

$$e_1(\eta_n) = -c_2(\xi_n),$$

hence

$$c(\xi_n) = 1 - v.$$

The Chern character (see [2], § 9.1) of ξ_n is

$$ch(\xi_n) = e^{y_1} + e^{y_2}$$

where

$$(1 + y_1)(1 + y_2) = c(\xi_n) = 1 - v .$$

Hence

$$y_1 + y_2 = 0 , \quad y_1 y_2 = -v .$$

From this we conclude that

$$\begin{aligned} y_1 &= -y_2, v = y_1^2, \\ y_1 &= \sqrt{v}, y_2 = -\sqrt{v}, \\ ch(\xi_n) &= \exp(v) + \exp(-v) = 2 \cosh \sqrt{v}. \end{aligned}$$

Hence $ch(P_n(\mathbf{H}))$ contains the subring generated by $2 \cosh \sqrt{v}$; it remains to show that it is exactly equal to this subring. This is done by induction on n exactly as in the proof of Proposition 3.1 of [1]. The details may be left to the reader.

Note that $ch P_n(\mathbf{H})$ may be equivalently described as the subring of $H^*(P_n(\mathbf{H}), \mathbf{Q})$ generated by

$$w = 2 \cosh \sqrt{v} - 2 = v + 2v^3/4! + \dots + 2v^n/(2n)! .$$

For many purposes this description of $ch P_n(\mathbf{H})$ is more convenient; note that $w^{n+1} = 0$, and $\{1, w, w^2, \dots, w^n\}$ is a basis of $ch P_n(\mathbf{H})$ over the integers.

Lemma 1 and equation (1) above are both stated in terms of "appropriately" chosen generators, v and u respectively, of the infinite cyclic group $H^*(P_n(\mathbf{H}), \mathbf{Z})$. Therefore $u = \pm v$. We assert that $u = +v$. To prove this, it obviously suffices to show that $ch(\tau_n \otimes \mathbf{C})$ belongs to the subring of $H^*(P_n(\mathbf{H}), \mathbf{Q})$ generated by $2 \cosh \sqrt{u}$, but that it does not belong to the subring of $H^*(P_n(\mathbf{H}), \mathbf{Q})$ generated by $2 \cosh \sqrt{-u}$. This we will now do by an essentially straightforward, but rather lengthy, computation.

LEMMA 2. *The Chern character of $\tau_n \otimes \mathbf{C}$ is given by*

$$ch(\tau_n \otimes \mathbf{C}) = (4n + 4) \cosh \sqrt{u} - 4 \cosh^2 \sqrt{u} .$$

Proof. It follows from equations (1) and (3) that the total Chern class of $\tau_n \otimes \mathbf{C}$ is given by

$$c(\tau_n \otimes \mathbf{C}) = (1 - u)^{2n+2} (1 - 4u)^{-1} .$$

To compute the Chern character of $\tau_n \otimes \mathbf{C}$, we may proceed as follows: Write the total Chern class as a formal product

$$c(\tau_n \otimes \mathbf{C}) = \prod_{i=1}^{4n} (1 + x_i) ,$$

where the x_i 's have degree 2. Then

$$ch(\tau_n \otimes C) = \sum_{i=1}^{4n} \exp(x_i) ,$$

To actually carry out the computation, take the logarithm of both sides of the equation

$$\prod (1 + x_i) = (1 - u)^{2n+2}(1 - 4u)^{-1} ,$$

and use the MacLaurin series expansion of $\log(1 + z)$ and $\log(1 - z)$. The result is

$$(4) \quad \sum_{k>0} (-1)^{k+1} s_k/k = - \sum_{k>0} (2n + 2 - 4^k)u^k/k ,$$

where

$$s_k = \sum_{i=1}^{4n} x_i^k .$$

Since each x_i is of degree 2, while u is of degree 4, we conclude from equation (4) that

$$s_k = 0 \quad \text{for } k \text{ odd} ,$$

$$s_{2k} = 2(2n + 2 - 4^k)u^k .$$

Therefore

$$\begin{aligned} ch(\tau_n \otimes C) &= \sum_{i=1}^{4n} \exp(x_i) = \sum_{i=1}^{4n} \sum_k x_i^k/k ! \\ &= 4n + \sum_{k>0} s_k/k ! \\ &= 4n + \sum_{k>0} 2(2n + 2 - 4^k)u^k/(2k) ! \\ &= (4n + 4) \cosh \sqrt{u} - 2 \cosh \sqrt{4u} - 2 \\ &= (4n + 4) \cosh \sqrt{u} - 4 \cosh^2 \sqrt{u} , \end{aligned}$$

as was to be proved.

It is obvious from this formula that $ch(\tau_n \otimes C)$ belongs to the subring of $H^*(P_n(H), \mathbb{Q})$ generated by $2 \cosh \sqrt{u}$; we must now prove that $ch(\tau_n \otimes C)$ does *not* belong to the subring generated by

$$2 \cosh \sqrt{-u} = 2 \cos \sqrt{u} .$$

Assume the contrary; then there exist integers a_0, a_1, \dots, a_n such that

$$ch(\tau_n \otimes C) = \sum_{k=0}^n a_k (2 \cos \sqrt{u} - 2)^k ,$$

that is,

$$\begin{aligned}
& 2 \sum_{k=1}^n (2n + 2 - 4^k) u^k / (2k)! \\
&= \sum_{k=1}^n a_k (-u + 2u^2/4! - 2u^3/6! + \cdots \pm 2u^n/(2n)!)^k.
\end{aligned}$$

If we compare coefficients of u , u^2 , u^3 , and u^4 in this equation, we obtain

$$\begin{aligned}
a_1 &= -2n + 2, \\
a_2 &= (n - 4)/3, \\
a_3 &= (7 - n)/18, \\
a_4 &= (5n - 47)/504.
\end{aligned}$$

For $n = 2$ or 3 , a_2 is not an integer; for $n < 7$, a_3 is not an integer; and for any value of n , it is impossible that both a_3 and a_4 are integers. For, if a_3 is an integer, then

$$n \equiv 7 \pmod{18} \text{ or } 5n \equiv 35 \pmod{18},$$

while if a_4 is an integer, then

$$5n \equiv 47 \pmod{18}$$

which is a contradiction.

This completes the proof that $u = +v$.

4. Proof of the theorem. Assume τ_n admits an almost complex structure θ_n ; we will show that this leads to a contradiction.

θ_n is a complex $2n$ -plane bundle over $P_n(\mathbf{H})$ such that $\tau_n = \theta_{n\mathbf{R}}$. Then by equation (2)

$$\tau_n \otimes \mathbf{C} = \theta_{n\mathbf{R}} \otimes \mathbf{C} = \theta_n + \theta_n^*.$$

Next, recall that

$$ch_i(\theta_n^*) = (-1)^i ch_i(\theta_n)$$

where ch_i denotes the component of ch of degree $2i$. However, since the base space of the bundle θ_n is $P_n(\mathbf{H})$,

$$ch(\theta_n^*) = ch(\theta_n).$$

Therefore

$$ch(\tau_n \otimes \mathbf{C}) = 2ch(\theta_n).$$

It follows from Lemma 2 that

$$\begin{aligned}
ch(\theta_n) &= (2n + 2) \cosh \sqrt{u} - 2 \cosh^2 \sqrt{u} \\
&= -\frac{1}{2} (2 \cosh \sqrt{u} - 2)^2 + (n - 1) (2 \cosh \sqrt{u} - 2) + 2n.
\end{aligned}$$

This is the desired contradiction, since the Chern character of any complex

vector bundle is an *integral* linear combination of the powers of $w = (2 \cosh \sqrt{u} - 2)$.

REMARK. We have actually proved a slightly stronger theorem, in that we have shown that for any integer $n > 1$, $P_n(\mathbf{H})$ does not admit a "generalized almost complex structure" as defined by Hirzebruch in his lecture [5]. As Hirzebruch remarks, this can be proved easily by induction on n , once the case $n = 2$ is taken care of. However, the above computations of $ch(P_n(\mathbf{H}))$ and $ch(\tau_n \otimes C)$ may be of some independent interest.

BIBLIOGRAPHY

1. M. F. Atiyah and J. A. Todd, *On complex Stiefel manifolds*, Proc. Cambridge Phil. Soc., **56** (1960), 343-353.
2. A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces*, I. Amer. J. Math., **80** (1958), 458-538.
3. F. Hirzebruch, *Über die quaternionalen projektiven Räume* S.—B. Math. Nat. Kl. Bayer. Akad. Wiss. 1953, (1954), 301-312.
4. ———, *Neue Topologische Methoden in der Algebraischen Geometrie*, Ergeb. der Math. u. Grenzgebiete (N. F.) 9, Berlin, 1956.
5. ———, *Komplexe Mannigfaltigkeiten*, Proc. Int. Congress. Math., 14-21 August 1958, Edinburgh, pp. 119-136.

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RALPH S. PHILLIPS

Stanford University
Stanford, California

M. G. ARSOVE

University of Washington
Seattle 5, Washington

A. L. WHITEMAN

University of Southern California
Los Angeles 7, California

LOWELL J. PAIGE

University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH

T. M. CHERRY

D. DERRY

M. OHTSUKA

H. L. ROYDEN

E. SPANIER

E. G. STRAUS

F. WOLF

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Tsuyoshi Andô, <i>On fundamental properties of a Banach space with a cone</i>	1163
Sterling K. Berberian, <i>A note on hyponormal operators</i>	1171
Errett Albert Bishop, <i>Analytic functions with values in a Frechet space</i>	1177
(Sherman) Elwood Bohn, <i>Equicontinuity of solutions of a quasi-linear equation</i>	1193
Andrew Michael Bruckner and E. Ostrow, <i>Some function classes related to the class of convex functions</i>	1203
J. H. Curtiss, <i>Limits and bounds for divided differences on a Jordan curve in the complex domain</i>	1217
P. H. Doyle, III and John Gilbert Hocking, <i>Dimensional invertibility</i>	1235
David G. Feingold and Richard Steven Varga, <i>Block diagonally dominant matrices and generalizations of the Gerschgorin circle theorem</i>	1241
Leonard Dubois Fountain and Lloyd Kenneth Jackson, <i>A generalized solution of the boundary value problem for $y'' = f(x, y, y')$</i>	1251
Robert William Gilmer, Jr., <i>Rings in which semi-primary ideals are primary</i>	1273
Ruth Goodman, <i>K-polar polynomials</i>	1277
Israel Halperin and Maria Wonenburger, <i>On the additivity of lattice completeness</i>	1289
Robert Winship Heath, <i>Arc-wise connectedness in semi-metric spaces</i>	1301
Isidore Heller and Alan Jerome Hoffman, <i>On unimodular matrices</i>	1321
Robert G. Heyneman, <i>Duality in general ergodic theory</i>	1329
Charles Ray Hobby, <i>Abelian subgroups of p-groups</i>	1343
Kenneth Myron Hoffman and Hugo Rossi, <i>The minimum boundary for an analytic polyhedron</i>	1347
Adam Koranyi, <i>The Bergman kernel function for tubes over convex cones</i>	1355
Pesi Rustom Masani and Jack Max Robertson, <i>The time-domain analysis of a continuous parameter weakly stationary stochastic process</i>	1361
William Schumacher Massey, <i>Non-existence of almost-complex structures on quaternionic projective spaces</i>	1379
Deane Montgomery and Chung-Tao Yang, <i>A theorem on the action of SO(3)</i>	1385
Ronald John Nunke, <i>A note on Abelian group extensions</i>	1401
Carl Mark Pearcy, <i>A complete set of unitary invariants for operators generating finite W^*-algebras of type I</i>	1405
Edward C. Posner, <i>Integral closure of rings of solutions of linear differential equations</i>	1417
Duane Sather, <i>Asymptotics. III. Stationary phase for two parameters with an application to Bessel functions</i>	1423
J. Śladkowska, <i>Bounds of analytic functions of two complex variables in domains with the Bergman-Shilov boundary</i>	1435
Joseph Gail Stampfli, <i>Hyponormal operators</i>	1453
George Gustave Weill, <i>Some extremal properties of linear combinations of kernels on Riemann surfaces</i>	1459
Edward Takashi Kobayashi, <i>Errata: "A remark on the Nijenhuis tensor"</i>	1467