A THEOREM ON THE ACTION OF SO(3)

Deane Montgomery and Chung-Tao Yang
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1. Introduction. We shall use notions given in [1]. Let $G$ be a compact Lie group acting on a locally compact Hausdorff space $X$. We denote by $F(G, X)$ the set of stationary points of $G$ in $X$, that is, $F(G, X) = \{x \in X | Gx = x\}$. If $G$ is a cyclic group generated by $g \in G$, $F(G, X)$ is also written $F(g, X)$.

Whenever $x \in X$, we call $Gx = \{gx | g \in G\}$ the orbit of $x$ and $G_x = \{g \in G | gx = x\}$ the isotropy group at $x$. By a principal orbit we mean an orbit $Gx$ such that $G_x$ is minimal. By an exceptional orbit we mean an orbit of maximal dimension which is not a principal orbit. By a singular orbit we mean an orbit not of maximal dimension. Denote by $U$ the union of all the principal orbits, by $D$ the union of all the exceptional orbits and by $B$ the union of all the singular orbits. Then $U$, $D$ and $B$ are all $G$-invariant and they are mutually disjoint. Moreover, $X = U \cup D \cup B$ and both $B$ and $D \cup B$ are closed in $X$.

Denote by $X^*$ the orbit space $X/G$ and by $\pi$ the natural projection of $X$ onto $X^*$. Whenever $A \subset X$, $A^*$ denotes the image $\pi A$. If $X$ is a connected cohomology $n$-manifold over $Z$ [1; p. 9], where $Z$ denotes the ring of integers, then the following results are known.

(1.1) $U^*$ is connected [1; p. 122] so that whenever $x, y \in U$, $G_x$ and $G_y$ are conjugate.

(1.2) $\dim_\mathbb{R} B^* \leq \dim_\mathbb{R} U^* - 1$ so that if $r$ is the dimension of principal orbits and $B_k$ is the union of all the $k$-dimensional singular orbits ($k < r$), then $\dim_\mathbb{R} B_k \leq n - r + k - 1$ [1; p. 118]. Hence $\dim_\mathbb{R} B \leq n - 2$.

Denote by $E^{n+1}$ the euclidean $(n + 1)$-space, by $S^n$ the unit $n$-sphere in $E^{n+1}$ and by SO(3) the rotation group of $E^3$. In this note $G$ is to be SO(3) and $X$ is to be a compact cohomology $n$-manifold over $Z$ with $H^*(X; Z) = H^*(S^n; Z)$.

Let us first observe the following examples.

1. Let $G = \text{SO}(3)$ act trivially on $X = S^n$. (Here we have $n = 1$.)

2. Let $G = \text{SO}(3)$ act on $E^{n+1} = E^5 \times E^{n-4}$ ($n \geq 4$) by the definition $g(x, y) = (gx, y)$,

where the action of $G$ on $E^5$ is an irreducible orthogonal action. Then $G$ acts on $X = S^n$ and in this action, the 2-dimensional orbits are all

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1385
protective planes, $F(G, X)$ is an $(n - 5)$-sphere and for every $x \in U$, $G_x$ is a dihedral group of order 4.

3. Let $G = \text{SO}(3)$ act on $E^{n+1} = E^3 \times E^3 \times E^{n-5}(n \geq 5)$ by the definition
   
   $$g(x, y, z) = (gx, gy, z),$$

   where the action on $E^3$ is the familiar one. Then $G$ acts on $X = S^n$ and in this action, the 2-dimensional orbits are all 2-spheres, $F(G, X)$ is an $(n - 6)$-sphere and for every $x \in U$, $G_x$ is the identity group.

   In all three examples, $D = \phi$ and $\text{dim} \; B = n - 2$. The orbit space $X^*$ is $X$ itself in the first example and it is a closed $(n - 3)$-cell with boundary $B^*$ in the other two examples.

   The purpose of this note is to prove that if $X$ is a compact cohomology $n$-manifold over $Z$ with $H^*(X; Z) = \mathbb{Z}^*(S^n; Z)$, then every action of $G = \text{SO}(3)$ on $X$ with $\text{dim}_Z B = n - 2$ strongly resembles one of these examples. In fact, we shall prove the following:

   **THEOREM.** Let $X$ be a compact cohomology $n$-manifold over $Z$ with $H^*(X; Z) = H^*(S^n; Z)$ and let $G = \text{SO}(3)$ act on $X$ with $\text{dim}_Z B = n - 2$. Then $D = \phi$ and one of the following occurs.

1. $n = 1$ and $G$ acts trivially on $X$.

2. $n \geq 4$ and for every $x \in U$, $G_x$ is a dihedral group of order 4. Moreover, the 2-dimensional orbits are all projective planes and $F(G, X)$ is a compact cohomology $(n - 5)$-manifold over $Z$ with $H^*(F(G, X); Z) = H^*(S^{n-5}; Z)$, where $Z$ denotes the prime field of characteristic 2.

3. $n \geq 5$ and for every $x \in U$, $G_x$ is the identity group. Moreover, the 2-dimensional orbits are all 2-spheres and $F(G, X)$ is a compact cohomology $(n - 6)$-manifold over $Z$ with $H^*(F(G, X); Z) = H^*(S^{n-6}; Z)$.

   In the last two cases, $B^*$ is a compact cohomology $(n - 4)$-manifold over $Z$ with $H^*(B^*; Z) = H^*(S^{n-4}; Z)$ and $X^*$ is a compact Hausdorff space which is cohomologically trivial over $Z$ and such that $X^* - B^*$ is a cohomology $(n - 3)$-manifold over $Z$.

   The proof of this theorem is given in the next three sections.

2. The set $D$. Let $X$ be a connected cohomology $n$-manifold over $Z$ and let $G = \text{SO}(3)$ act on $X$ with $\text{dim}_Z B = n - 2$. If $G$ acts trivially on $X$, it is clear that $n = 1$ and that $D = \phi$. Hence we shall assume that the action of $G$ on $X$ is nontrivial.

   Since $G$ is a 3-dimensional simple group which has no 2-dimensional
subgroup, it follows that

(2.1) $G$ acts effectively on $X$ and no orbit is 1-dimensional.

(2.2) Principal orbits are 3-dimensional so that for every $x \in U \cup D$, $G_x$ is finite.

By (2.1), principal orbits are either 2-dimensional or 3-dimensional. If principal orbits are 2-dimensional, then $B = F(G, X)$ so that, by (1.2), $\dim B < n - 2$, contrary to our assumption.

(2.3) Denote by $B_2$ the union of all the 2-dimensional orbits. Then $\dim B_2 = n - 2$ so that $B_2 \neq \emptyset$ and $n \geq 4$. Moreover, whenever $G_z$ is a 2-dimensional orbit, $G_z$ is either a circle group or the normalizer of a circle group and accordingly $G_z$ is either a 2-sphere or a projective plane.

By (2.2), $n = \dim X \geq \dim U \geq 3$. We infer that $B_2 \neq \emptyset$ so that $n - 2 = \dim B_2 \geq 2$. Hence $n \geq 4$.

(2.4) Let $x \in U$. Whenever $y \in D$, there is a $g \in G$ such that $G_x$ is a normal subgroup of $G_y$.

Let $S$ be a connected slice at $y$ [1; p. 105]. Then $S$ is a connected cohomology $(n - 3)$-manifold over $Z$ and $G_y$ acts on $S$. As seen in [7], $S$ is also a connected cohomology $(n - 3)$-manifold over $Z_p$ for every prime $p$, where $Z_p$ denotes the prime field of characteristic $p$.

Let $x' \in S \cap U$. We claim that $G_{x'}$ is a normal subgroup of $G_y$. Since $G_y$ is a finite group (see (2.2)) and $G_{x'}$ is a subgroup of $G_y$, there exists a neighborhood $N$ of the identity in $G$ such that $N^{-1}G_{x'}N \cap G_y = G_{x'}$. Let $V$ be a neighborhood of $x'$ such that whenever $x'' \in V$, $hG_{x'}h^{-1} \subset G_{x'}$ for some $h \in N$. (For the existence of $V$, see [4; p. 216]).

Then for every $x'' \in V \cap S$, $G_{x''} \subset N^{-1}G_yN \cap G_y = G_{x'}$ so that $G_{x''} = G_{x'}$. Therefore $G_{x'}$ leaves every point of $V \cap S$ fixed. Since $S$ is a connected cohomology $(n - 3)$-manifold over $Z_p$ for every prime $p$, it follows from Newman’s theorem [6] that $G_{x'}$ leaves every point of $S$ fixed. Hence $G_{x'} = \{g \in G_y | gx'' = x'' \text{ for all } x'' \in S\}$, which is clearly a normal subgroup of $G_y$. By (1.1), $G_x$ and $G_{x'}$ are conjugate so that our assertion follows.

(2.5) Let $x \in U$. Whenever $G_x$ is 2-dimensional, there is a $g \in G$ such that $G_x \subset G_x$. Hence $G_x$ is either cyclic or dihedral and it is cyclic if there is a 2-dimensional orbit which is a 2-sphere.

For the rest of this section, we assume that

$$H^*_s(X; Z) = H^*(S^n; Z).$$

Under this assumption, $\tilde{H}^*_s(X; Z) = H^*(S^n; Z) = Z$. Hence $X$ is compact.

(2.6) Let $T$ be a circle group in $G$. Then $F(T, X)$ is a compact cohomology $(n - 4)$-manifold over $Z$ with $H^*(F(T, X); Z) = H^*(S^{n-4}; Z)$. 


Since \( F(T, X) \) intersects every singular orbit at one or two points, 
\[
\dim_x F(T, X) = \dim_x B^* = n - 4.
\]
Hence our assertion follows [1; Chapters IV and V].

(2.7) Let \( g \in G \) be of order \( p^\alpha \), where \( p \) is a prime and \( \alpha \) is a positive integer. If \( g \in G_x \) for some \( x \in U \cup D \), then \( F(g, X) \) is a compact cohomology \( (n - 2) \)-manifold over \( Z_p \) with \( H^*(F(g, X); Z_p) = H^*(S^{n-2}; Z_p) \). Hence \( F(g, X) \) intersects every principal orbit.

It is known that \( X \) is also a compact cohomology \( n \)-manifold over \( Z_p \) with \( H^*(X; Z_p) = H^*(S^n); Z_p) \). Since \( G \) is connected, \( g \) preserves the orientation of \( X \). It follows that for some \( r < n \) of the same parity, \( F(g, X) \) is a compact cohomology \( r \)-manifold over \( Z_p \) with \( H^*(F(g, X); Z_p) = H^*(S^r; Z_p) \) [1; Chapters IV and V].

Let \( T \) be the circle group in \( G \) containing \( g \). By (2.6), \( F(g, X) \cap B = F(T, X) \) is a compact cohomology \( (n - 4) \)-manifold over \( Z_p \). Since, by hypothesis, there exists a point of \( U \cup D \) contained in \( F(g, X) \), \( F(g, X) \cap B \) is properly contained in \( F(g, X) \) so that \( r = n - 2 \). Hence \( F(g, X) \) is a compact cohomology \( (n - 2) \)-manifold over \( Z_p \) with \( H^*(F(g, X); Z_p) = H^*(S^{n-2}; Z_p) \).

Since \( \dim_x D^* < n - 3 \) [1; p. 121] and since \( F(g, X) \) intersects every exceptional orbit at a set of dimension \( \leq 1 \), it follows that \( \dim_x(F(g, X) \cap D) \leq \dim_x(F(g, X) \cap D) < n - 2 \). But we have \( \dim_x F(g, X) = n - 2 \) and \( \dim_x (F(g, X) \cap B) = n - 4 \). Therefore \( F(g, X) \cap U \neq \emptyset \). Hence, by (1.1), \( F(g, X) \) intersects every principal orbit.

(2.8) Let \( x \in U \) and \( y \in D \). Let \( p \) be a prime and let \( \alpha \) be a positive integer. If \( G_y \) has an element of order \( p^\alpha \), so does \( G_x \).

Let \( g \in G_x \) be of order \( p^\alpha \). By (2.7), \( F(g, X) \cap Gx \neq \emptyset \) so that for some \( h \in G \), \( hx \in F(g, X) \). Hence \( h^{-1}gh \) is an element of \( G_x \) of order \( p^\alpha \).

(2.9) \( D = \emptyset \).

Suppose that \( D \neq \emptyset \). Let \( x \in U \) and \( y \in D \) be such that \( G_x \) is a proper normal subgroup of \( G_y \) (see (2.4)). We first claim that \( G_y \) is dihedral.

It is well known that a finite subgroup of \( SO(3) \) is either cyclic or dihedral or tetrahedral or octahedral or icosahedral. If \( G_y \) is cyclic, so is \( G_x \). Let the order of \( G_y \) be \( p_1^{s_1} \cdots p_k^{s_k} \), where \( p_1, \cdots, p_k \) are distinct primes and \( s_1, \cdots, s_k \) are positive integers. Then for every \( i = 1, \cdots, k \), \( G_y \) contains an element of order \( p_i^{s_i} \) so that, by (2.8), \( G_x \) also contains an element of order \( p_i^{s_i} \). Hence \( G_x \) is of order \( \geq p_1^{s_1} \cdots p_k^{s_k} \) and consequently \( G_x = G_y \), contrary to the fact that \( G_x \) is a proper subgroup of \( G_y \). If \( G_y \) is either tetrahedral or octahedral or icosahedral, then
by (2.8), $G_x$ contains a subgroup of order 2 and a subgroup of order 3. In case $G_x$ is octahedral, it also contains a subgroup of order 4. Hence $G_x$, as a normal subgroup of $G_y$, is equal to $G_y$, contrary to our hypothesis. This proves that $G_y$ is dihedral.

Now the order of $G_y$ is even. It follows from (2.7) that whenever $g \in G$ is of order 2, $F(g, X)$ is a compact cohomology $(n - 2)$-manifold over $Z_2$ with $H^*(F(g, X); Z_2) = H^*(S^{n-2}; Z_2)$. Let $H$ be a dihedral subgroup of $G$ of order 4. By Borel's theorem [1; p. 175], $F(H, X)$ is a compact cohomology $(n - 3)$-manifold over $Z_2$ with $H^*(F(H, X); Z_2) = H^*(S^{n-3}; Z_2)$. Since $\dim Z_2(F(H, X) \cap (D \cup B)) \leq \dim Z_2(F(H, X) \cap (D \cup B)) < n - 3$, it follows that $F(H, X) \cap U$ is not null. Hence we may assume that $H \subset G_x \subset G_y$.

Let $T$ be the circle group in $G$ such that its normalizer contains $G_x$. Then $H \cap T \subset G_x \cap T \subset G_y \cap T$ so that $G_y \cap T$ is a cyclic group and $G_x \cap T$ is a proper subgroup of $G_y \cap T$ of even order. Let the order of $G_y \cap T$ be $2^{s_0}p_1 \cdots p_k$, where $p_1, \ldots, p_k$ are distinct odd primes and $s_0, s_1, \ldots, s_k$ are positive integers. By (2.8), there are $k + 1$ elements $g_0, g_1, \ldots, g_k$ of $G_x$ of order $2^{s_0}p_1^i \cdots p_k^j$, where $p_1^i, \ldots, p_k^j$ respectively. Since $p_1, \ldots, p_k$ are odd, $g_1, g_2, \ldots, g_k$ are in $G_x \cap T$. Therefore no element of $G_x \cap T$ is of order $2^{s_0}$. But this implies that $s_0 > 1$ so that $g_0 \in G_x \cap T$. Hence we have arrived at a contradiction.

3. Case that the 2-dimensional orbits are all projective planes.

Let $X$ be a compact cohomology $n$-manifold over $Z$ with $H^*(X; Z) = H^*(S^n; Z)$ and let $G = SO(3)$ act nontrivially on $X$ with $\dim Z \geq n - 2$. Throughout this section, we assume that for some $x \in U$, $G_x$ is of even order.

(3.1) Let $H$ be a dihedral subgroup of $G$ of order 4 and let $M$ be the normalizer of $H$ that is the octahedral group containing $H$. Then $F(H, X)$ is a compact cohomology $(n - 3)$-manifold over $Z_2$ with $H^*(F(H, X); Z_2) = H^*(S^{n-3}; Z_2)$ and $K = M/H$ is isomorphic to the symmetric group of three elements and acts on $F(H, X)$. Moreover, the natural map of $F(H, X)/K$ into $X^*$ is onto.

By (2.7), for every $g \in G$ of order 2, $F(g, X)$ is a compact cohomology $(n - 2)$-manifold over $Z_2$ with $H^*(F(g, X); Z_2) = H^*(S^{n-2}; Z_2)$. It follows from Borel's theorem [1; p. 175] that $F(H, X)$ is a compact cohomology $(n - 3)$-manifold over $Z_2$ with $H^*(F(H, X); Z_2) = H^*(S^{n-3}; Z_2)$.

Clearly $K = M/H$ is isomorphic to the symmetric group of three elements and the action of $M$ on $F(H, X)$ induces an action of $K$ on $F(H, X)$. Moreover, there is a natural map $f : F(H, X)/K \to X^*$. Let $z \in F(H, X) \cap B$. If $Gz = z$, then $F(H, X) \cap Gz = z$. If $Gz$ is 2-dimensional, then $G_x$ contains $H$ so that by (2.3) it is the normalizer of a circle group. Therefore any two isomorphic dihedral subgroups of
$G_z$ are conjugate in $G_z$. Let $g$ be an element of $G$ with $gz \in F(H, X)$. It is clear that $g^{-1}hg \subseteq g^{-1}G_zg = G_z$ so that for some $h \in G_z$, $h^{-1}g^{-1}Hgh = H$ or $gh \in M$. Hence $gz = ghz \in Mz$. This proves that $F(H, X) \cap Gz \subset Mz$.

From these results it follows that $F(H, X)$ intersects every singular orbit at a finite set. [This and one or two facts mentioned below can be seen by examining the standard action of SO(3) on $S^2$ or on $P^2$ (viewed as the acts of lines through the region in $E^3$).] Therefore, by (1.2),
\[
\dim_z (F(H, X) \cap B) \leq \dim_z B^* < n - 3.
\]
As a consequence of this result and that $D = \phi$ (see (2.9)), we have $F(H, X) \cap U \neq \phi$. Hence $F(H, X)$ intersects every principal orbit and consequently it intersects every orbit. This proves that the natural map $f: F(H, X)/K \to X^*$ is onto.

(3.2) Every 2-dimensional orbit is a projective plane and intersects $F(H, X)$ at exactly three points.

Let $Gz$ be a 2-dimensional orbit. By (3.1), $F(H, X)$ intersects $Gz$ so that we may assume that $z \in F(H, X)$. Since $G_z$ contains $H$, it follows from (2.3) that $G_z$ is the normalizer of a circle group. Hence $G_z$ is a projective plane.

In the proof of (3.1) we have shown that $F(H, X) \cap Gz \subset Mz$. But it is clear that $Mz \subset F(H, X) \cap Gz$. Hence
\[
F(H, X) \cap Gz = Mz = M/(M \cap G_z).
\]
Since $M$ is of order 24 and $M \cap G_z$ is of order 8, it follows that $F(H, X) \cap Gz$ contains exactly three points.

(3.3) $B^*$ is a compact cohomology $(n - 4)$-manifold over $Z$ with $H^*(B^*; Z) = H^*(S^{n-4}; Z)$.

Let $T$ be a circle group in $G$. It is clear that $F(T, X) \subset B$. Since, by (2.1) and (3.2), every singular orbit is either a point or a projective plane, it follows that $F(T, X)$ intersects every singular orbit at exactly one point. Therefore the natural projection $\pi$ maps $F(T, X)$ homeomorphically onto $B^*$ and hence our assertion follows from (2.6).

(3.4) Let $Y = F(H, X) - F(G, X)$. Then $\tilde{Y} = F(H, X)$ and every point of $Y$ has a neighborhood $V$ in $Y$ which is a cohomology $(n - 3)$-manifold over $Z$ and such that the isotropy group is constant on $V - B$.

Let $T$ be a circle group whose normalizer $N$ contains $H$. Then $F(H, X) \supset F(N, X) = F(T, X) \supset F(G, X)$. Since $F(H, X)$ is a compact $(n - 3)$-manifold over $Z_t$ (see (3.1)) and since $F(T, X)$ is a compact $(n-4)$-manifold over $Z_t$ (see (2.6)), it follows that the closure of $F(H, X) - F(T, X)$ is $F(H, X)$. Hence $\tilde{Y} = F(H, X)$. 

Let \( x \in Y \cap U \) and let \( S \) be a slice at \( x \). Then \( S \) is a cohomology \((n - 3)\)-manifold over \( Z \). Moreover, \( G_x = G_z \) for all \( y \in S \) so that \( S \subset Y \). Since both \( S \) and \( Y \) are cohomology \((n - 3)\)-manifolds over \( Z \), it follows that \( S \) is open in \( Y \). Hence our assertion follows by taking \( S \) as \( V \).

Let \( z \in Y \cap B \) and let \( S \) be a slice at \( z \). Then \( S \) is a cohomology \((n - 2)\)-manifold over \( Z \) and \( G_z \) is the normalizer of a circle group \( T \) acting on \( S \). Whenever \( x \in S \cap U \), \( G_x \cap T \) is a finite cyclic group in \( T \) and the index of \( G_x \cap T \) in \( G_x \) is 2 because \( G_x \) is in a dihedral subgroup of \( G_z \). Since the order of \( G_z \) is independent of \( x \in S \cap U \), so is the order of \( G_x \cap T \). Hence \( G_x \cap T \) is independent of \( x \in S \cap U \) so that for \( x \in F(H, S) \cap U \).

\[
G_zS = H(G_z \cap T)S = HS = S
\]

and

\[
F(G_x, S) = F(G_z(G_z \cap T), S) = F(H(H \cap T), S) = F(H, S).
\]

Let \( Q \) be a neighborhood of the identity of \( G \) such that \( Q^{-1}TQ \cap G_z = T \). If \( y \in F(H, X) \) with \( g \in Q \) and \( y \in S \), then \( g^{-1}Hg \subset g^{-1}G_yg = G_y \subset G_z \) so that \( g^{-1}(H \cap T)g \subset Q^{-1}TQ \cap G_z = T \). Therefore \( g^{-1}Tg = T \) or \( g \in G_z \). Hence \( g \in G_y \subset S \). This proves that \( F(H, S) = F(H, X) \cap S = F(H, X) \cap QS \) is open in \( F(H, X) \) so that it is a cohomology \((n - 3)\)-manifold over \( Z \).

Since \( S \) is a cohomology \((n - 2)\)-manifold over \( Z \) with

\[
F(H(H \cap T), S) = F(H, S),
\]

it follows that \( F(H, S) \) is also a cohomology \((n - 3)\)-manifold over \( Z \). (If \( Z \) acts on a cohomology \( m \)-manifold over \( Z \) with \( F(Z) \) being a cohomology \((m - 1)\)-manifold over \( Z \), then \( F(Z) \) is also a cohomology \((m - 1)\)-manifold over \( Z \).) That \( G_z \) is constant on \( F(H, S) \cap U \) is a direct consequence of the fact that \( F(G_x, S) = F(H, S) \) for all \( x \in F(H, S) \cap U \).

(3.5) \( Y \) is a connected cohomology \((n - 3)\)-manifold over \( Z \) and the isotropy group is constant on \( Y - B \).

By (3.4), \( Y \) is a cohomology \((n - 3)\)-manifold over \( Z \). Let \( T \) be a circle group in \( G \) whose normalizer \( N \) contains \( H \). Then \( F(H, X) \supset F(N, X) = F(T, X) \supset F(G, X) \). From (2.6) and (3.1), it is easily seen that \( F(H, X) - F(T, X) \) has exactly two components with \( F(T, X) \) as their common boundary. By (2.3), there exists a point \( z \) of \( F(T, X) \) such that \( G_z \) is a projective plane so that \( z \in F(T, X) - F(G, X) \). Hence \( Y \) is connected.

Let \( x \in Y \cap U \). Then \( F(G_x, X) \cap Y \) is clearly closed in \( Y \). But, by (3.4), it is also open in \( Y \). Hence, by the connectedness of \( Y \), \( F(G_x, X) \cap Y = Y \).
(3.6) Whenever \( x \in F(H, X) \cap U \), \( G_x = H \). Hence for every \( x \in U \), \( G_x \) is a dihedral group of order 4.

Let \( x \) be a point of \( F(H, X) \cap U \). Since \( H \subset G_x \), \( F(H, X) \supset F(G_x, X) \). But, by (3.4) and (3.5), \( F(H, X) \subset F(G_x, X) \). Hence \( F(H, X) = F(G_x, X) \).

It is clear that \( G' = \{ g \in G \mid gF(H, X) = F(H, X) \} \) is a closed subgroup of \( G \) containing \( M \). Since \( F(H, X) = F(G_x, X), G_x \) is a normal subgroup of \( G' \) so that \( G' \) is contained in the normalizer of \( G_x \). But, by (2.5), \( G_x \) is dihedral and \( H \) is the only dihedral group whose normalizer contains \( M \). It follows that \( G_x = H \). Hence, by (1.1), the isotropy group at any point of \( U \) is a dihedral group of order 4.

(3.7) Whenever \( x \in F(H, X), F(H, X) \cap G_x = Kx \) which contains one point or three points or six points according as \( G_x \) is 0-dimensional or 2-dimensional or 3-dimensional.

If \( G_x \) is 0-dimensional, it is clear that \( F(H, X) \cap G_x = x = Kx \). If \( G_x \) is 2-dimensional, we have shown in the proof of (3.2) that \( F(H, X) \cap G_x = Mx = Kx \) which contains exactly three points.

Now let \( G_x \) be 3-dimensional. If \( g \) is an element of \( G \) with \( gx \in F(H, X), \) then, by (3.6), \( gHg^{-1} = gG_xg^{-1} = G_x = H \) so that \( g \in M \). Therefore \( F(H, X) \cap G_x \subset Mx \). But it is obvious that \( Mx \subset F(H, X) \cap G_x \). Hence

\[
F(H, X) \cap G_x = Mx = Kx
\]

which clearly contains six points.

From this result, it is easily seen that the natural map \( f: F(H, X)/K \to X^* \) is a homeomorphism onto.

(3.8) Whenever \( a \in K \) is of order 2, we abbreviate \( F(a, F(H, X)) \) by \( F(a) \). Then \( F(a) \subset B \) and \( F(a) \) is a compact cohomology \((n-4)\)-manifold over \( Z \) with \( H^*(F(a); Z) = H^*(S^{n-4}; Z) \). Moreover, \( F(H, X) - F(a) \) contains exactly two components \( V \) and \( V' \) with \( aV = V' \).

Whenever \( x \in F(H, X) \cap U \), \( G_x = H \) (see (3.6)) so that \( x \notin F(a) \). Hence \( F(a) \subset B \). Let \( a = a'\mathcal{H} \) with \( a' \) being of order 4 and let \( T \) be the circle group containing \( a' \). Then \( F(a) = F(T, X) \) and hence the first part follows from (2.6). Now \( F(H, X) \) is a compact cohomology \((n-3)\)-manifold over \( Z_3 \) with \( H^*(F(H, X); Z_3) = H^*(S^{n-3}; Z_3) \) and \( F(a) = F(a, F(H, X)) \) is a compact cohomology \((n-4)\)-manifold over \( Z_4 \). The second part follows.

(3.9) \( F(H, X) - B \) contains exactly six components and whenever \( P \) is a component of \( F(H, X) - B \), \( KP = F(H, X) - B \) and the natural
projection $\pi$ maps $P$ homeomorphically onto $U^*$.

Let $P$ be a component of $F(H, X) - B$. Since the isotropy group is constant on $P$ (see (3.5)), the natural projection $\pi$ defines a local homeomorphism $\pi': P \to U^*$. By (3.7), for every $x^* \in U^*$, $\pi'^{-1}x^*$ contains no more than six points. We infer that $\pi'$ is closed so that $\pi'P$ is both open and closed in $U^*$. Hence, by the connectedness of $U^*$, $\pi'P = U^*$.

Let $Q$ be a second component of $F(H, X) - B$ and let $y \in Q$. Then there is a point $x \in P$ such that $\pi x = \pi y$. Therefore, by (3.7), for some $k \in K$, $y = kx$ so that $Q = kP$. Hence $KP = F(H, X) - B$.

Let $x \in P$. By (3.8), $x$ and $ax$ belong to different components of $F(H, X) - F(a) \supset F(H, X) - B$. Therefore $aP$ is a component of $F(H, X) - B$ different from $P$. Similarly, $bP$ and $cP$ are components of $F(H, X) - B$ different from $P$.

If $aP$, $bP$ and $cP$ are not distinct, say $bP = cP$, then $\{k \in K | kP = P\}$ is of order 3 so that $P$ and $aP = bP = cP$ are the only two components of $F(H, X) - B$. Now $F(H, Z) - B = F(H, Z) - (F(a) \cup F(b) \cup F(c))$ and $F(a), F(b), F(c)$ are manifold over $Z$ of dimension one less than the dimension of $F(H)$. Hence $F(H, X) \cap B = F(a) \cap F(b) \cap F(c) = F(G, X)$.

This is impossible, because the intersection of $F(H, X)$ and a 2-dimensional orbit is contained in $B$ but not contained in $F(G, X)$. From this result it follows that $P, aP, bP, cP$ are distinct components of $F(H, X) - B$.

Hence $P, aP, bP, cP, bcP, cbP$ are all the distinct components of $F(H, X) - B$.

Now it is clear that for every $x^* \in U^*$, $\pi'^{-1}x^*$ contains exactly one point. Hence $\pi'$ is a homeomorphism.

(3.10) Let $P$ be a component of $F(H, X) - B$. Then the map of $G/H \times P$ onto $U$ defined by $(gH, x) \to gx$ is a homeomorphism onto. Hence $U$ is homeomorphic to the topological product of a principal orbit and $U^*$.

This is an immediate consequence of (3.5) and (3.9).

(3.11) The closure of $F(a) - F(G, X)$ is equal to $F(a)$. Hence $\dim_{x^*}F(G, X) \leq \dim_x F(G, X) \leq n - 5$.

Suppose that the closure of $F(a) - F(G, X)$ is not equal to $F(a)$. Then there is a point $z$ of $F(G, X)$ and a neighborhood $A$ of $z$ such that $A \cap F(a) = A \cap F(G, X)$. Since $A \cap F(G, X) \subset F(b)$ and since, by (3.8), both $A \cap F(G, X)$ and $F(b)$ are cohomology $(n - 4)$-manifolds over $Z$, $A \cap F(G, X)$ is open in $F(b)$ so that we may assume that $A \cap F(G, X) = A \cap F(b)$. Similarly, we may assume that $A \cap F(G, X) = A \cap F(e)$. Hence $A \cap F(G, X) = A \cap F(H, X) \cap B$. By (3.1) and (3.8), we may
also assume that $K A = A$ and $A \cap (F(H, X) - F(a))$ contains exactly two components $Q$ and $Q'$. Now both $Q$ and $Q'$ are contained in $F(H, X) - B$ and $aQ = bQ = Q'$. Therefore $abQ = Q$ so that $ab$ maps the component of $F(H, X) - B$ containing $Q$ into itself, contrary to (3.9).

Since, by (3.8), $F(a)$ is a cohomology $(n - 4)$-manifold over $\mathbb{Z}$ and since $F(G, X)$ is nowhere dense in $F(a)$, it follows that $\dim_x F(G, X) \leq \dim_x F(G, X) \leq n - 5$.

(3.12) If $n = 4$, then $F(G, X)$ is null.

This is a direct consequence of (3.11).

(3.13) Let $T$ be a circle group in $G$, let $N$ be the normalizer of $T$ and let $A$ be an orbit. If $A$ is a projective plane, then $A/T$ is an arc and $N/T$ acts trivially on $A/T$ so that $F(N/T, A/T) = A/T = A/N$. If $A$ is 3-dimensional, then $A/T$ is a 2-sphere and $A/N$ is a closed 2-cell so that $F(N/T, A/T)$ is a circle.

If $A$ is a projective plane, it is clear that $A/T$ is an arc and $N/T$ acts trivially on $A/T$. Therefore $A/N = A/T = F(N/T, A/T)$.

Now let $A$ be 3-dimensional. By (3.6), we may let $A = G/H = \{gH|g \in G\}$. Therefore $A/T$ is the double coset space $(G/H)/T$ and $(G/T)/H$ are homeomorphic. Since $G/T$ is a 2-sphere and since every element of $H$ preserves the orientation of $G/T$, it follows that $(G/T)/H$ is a 2-sphere. Hence $A/T$ is a 2-sphere.

As seen in [3], the double coset space $(G/N)/H$ is a closed 2-cell. Since $A/N$ may be regarded as the double coset space $(G/H)/N$ which is homeomorphic to $(G/N)/H$, we infer that $A/N$ is a closed 2-cell.

From these results, it follows that $f(N/T, A/T)$ is a circle.

(3.14) $X^*$ is cohomological trivial over $\mathbb{Z}$.

Let $N$ be the normalizer of a circle group $T$ in $G$. Then $N/T$ is a cyclic group of order 2 which acts on $X/T$ with $(X/T)/(N/T) = X^*$. Since, by (2.6), $H^*(F(T, X); Z) = H^*(S^{n-1}; Z)$, it follows that $H(X/T; Z) = H^*(S^{n-1}; Z)$ [1; p. 65].

By (3.13), $F(N/T, B/T) = B/T$ and for every singular orbit $A$, $A/T$ is either a single point or an arc. It follows from the Vietoris map theorem that $H^*(B/T; Z) = H^*(B^*; Z) = H^*(S^{n-1}; Z)$ (see (3.3)). By (3.10) and (3.13), $F(N/T, U/T)$ is homeomorphic to the topological product of a circle and $U^*$ so that $H^*\{F(N/T, U/T); Z\} \neq 0$. Therefore $H^*(F(N/T, X/T); Z) = H^*(S^{n-2}; Z)$. Hence $H^*(X/N; Z) = 0$. By (3.13), for every orbit $A$, $A/N$ is either a single point or an arc or a closed 2-cell. It follows from the Vietoris map theorem that $H^*(X^*; Z) = H^*(X/N; Z) = 0$. 


A THEOREM ON THE ACTION OF $SO(3)$

This follows from (3.3), (3.14) and the cohomology sequence of $(X^*, B^*)$.

$$H^k(U^*; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{for } k = n - 3, n ; \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } k = n - 2, n - 1 ; \\ 0 & \text{otherwise}. \end{cases}$$

Since for a principal orbit $A$, we have

$$H^k(A; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{for } k = 0, 3 ; \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } k = 1, 2 ; \\ 0 & \text{otherwise}, \end{cases}$$

our assertion follows from (3.10) and (3.15).

As a consequence of (3.16) and the cohomology sequence of $(X, B)$, we have

$$H^k(B; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{for } k = 0, n - 4 ; \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } k = n - 3, n - 2 ; \\ 0 & \text{otherwise}. \end{cases}$$

(3.18) Let $T$ be a circle group in $G$ and let $n \geq 5$. Then

$$H^k(F(T, X) - F(G, X); \mathbb{Z}_2) = \begin{cases} \tilde{H}^{k-1}(F(G, X); \mathbb{Z}_2) & (\text{the reduced group}) \\ H^{k-1}(F(G, X); \mathbb{Z}_2) \oplus \mathbb{Z}_2 & \text{for } k = n - 4 ; \\ H^{k-1}(F(G, X); \mathbb{Z}_2) & \text{otherwise}. \end{cases}$$

This follows from (2.6) and the cohomology sequence of $(F(T, X), F(G, X))$.

(3.19) Let $n > 5$. Then

$$H^k(B - F(G, X); \mathbb{Z}_2) = \begin{cases} H^k(B; \mathbb{Z}_2) & \text{for } k > n - 4 ; \\ H^k(B; \mathbb{Z}_2) \oplus H^{k-1}(F(G, X); \mathbb{Z}_2) & \text{for } k = n - 4 ; \\ H^{k-1}(F(G, X); \mathbb{Z}_2) & \text{for } k = 2, \ldots, n - 5 ; \\ \tilde{H}^{k-1}(F(G, X); \mathbb{Z}_2) & \text{for } k = 1. \end{cases}$$

This follows from the cohomology sequence of $(B, F(G, X))$.

(3.20) $B - F(G, X)$ is homeomorphic to the topological product of a projective plane and $F(T, X) - F(G, X)$. Hence
\[ H^k(B - F(G, X); Z_2) = H^k(F(T, X) - F(G, X); Z_2) \oplus H^{k-i}(F(T, X) - F(G, X); Z_2) \]

The first part follows from the that \( F(T, X) - F(G, X) \) is a cross-section of the transformation group \((G, B - F(G, X))\) on which the isotropy group is constant. The second part follows from the first part and the fact that if \( A \) is a projective plane, then

\[ H^k(A; Z_2) = \begin{cases} Z_2 & \text{for } k = 0, 1, 2; \\ 0 & \text{otherwise.} \end{cases} \]

(3.21) \( \dim_{Z_2} F(G, X) = n - 5. \) If \( n = 4, \) then \( B \) contains exactly two projective planes. If \( n = 5, \) then \( F(G, X) \) contains exactly two points. If \( n > 5, \) then \( H^{n-i}(F(G, X); Z_2) = Z_2, \) so that \( F(G, X) \) is not null.

Setting \( k = n - 2 \) in (3.20), we have, by (2.6) and (3.17),

\[ Z_2 \oplus Z_2 = H^{n-i}(F(T, X) - F(G, X); Z_2). \]

If \( n = 4, \) then, by (3.12), \( H^i(F(T, X); Z_2) = Z_2 \) so that \( F(T, X) \) contains exactly two points. Hence \( B \) contains exactly two projective planes.

If \( n = 5, \) then \( H^i(F(T, X) - F(G, X); Z_2) = H^i(F(T, X); Z_2) \oplus H^i(F(G, X); Z_2) \) so that \( H^i(F(G, X); Z_2) = Z_2. \) Hence \( F(G, X) \) contains exactly two points.

If \( n > 5, \) it follows from (3.18) that \( H^{n-i}(F(G, X); Z_2) = Z_2. \) Hence \( F(G, X) \) is not null.

(3.22) \( H^*(F(G, X); Z_2) = H^*(S^{n-3}; Z_2). \)

For \( n = 4 \) and \( 5, \) the result has been shown in (3.12) and (3.21). For \( n > 5, \) our assertion follows from (3.18), (3.19), (3.20) and (3.21).

(3.23) \( F(G, X) \) is a compact cohomology \((n - 5)\)-manifold over \( Z_2. \)

To prove (3.23), we have only to localize the preceding computations. Details are omitted.

**Remark.** There is no difficulty to use \( Z \) in place of \( Z_2 \) in these computations. However, the computations over \( Z \) will not strengthen our final results (3.22) and (3.23).

4. Case that the 2-dimensional orbits are all 2-spheres.

Let \( X \) be a compact cohomology \( n \)-manifold over \( Z \) with \( H^*(X; Z) = H^*(S^n; Z) \) and let \( G = SO(3) \) act nontrivially on \( X \) with \( \dim_x B = n - 2. \)
Throughout this section, we assume that for some \( x \in U \), \( G_x \) is of odd order.

(4.1) Let \( H \) be a dihedral subgroup of \( G \) of order 4. Then \( F(H, X) \) is a compact cohomology \((n - 6)\)-manifold over \( \mathbb{Z} \) with \( H^*(F(H, X); \mathbb{Z}_2) = H^*(S^{n-6}; \mathbb{Z}_2) \). Hence \( n \geq 5 \).

Let \( g \in G \) be of order 2 and let \( T \) be the circle group in \( G \) containing \( g \). Since for some \( x \in U \), \( G_x \) is of odd order, \( F(g, X) \subset B \) so that \( F(g, X) = F(T, X) \) is a compact cohomology \((n - 4)\)-manifold over \( \mathbb{Z} \) with \( H^*(F(g, X); \mathbb{Z}_2) = H^*(S^{n-4}; \mathbb{Z}_2) \). By Borel's theorem [1; p. 175], \( F(H, X) \) is a compact cohomology \((n - 6)\)-manifold over \( \mathbb{Z} \) with \( H^*(F(H, X); \mathbb{Z}_2) = H^*(S^{n-6}; \mathbb{Z}_2) \). From this result it follows that \( n - 6 \geq -1 \). Hence \( n \geq 5 \).

(4.2) The 2-dimensional orbit are all 2-spheres.

Suppose that this assertion is false. Then there is, by (2.3), a projective plane \( Gz \). Denote by \( T \) the identity component of \( Gz \) and by \( H \) a dihedral subgroup of \( Gz \) of order 4. Let \( S \) be a connected slice at \( z \). Then \( S \) is a cohomology \((n - 2)\)-manifold over \( Z \) and \( Gz \) acts on \( S \). Moreover, \( F(T, S) = F(T, X) \cap S \) is open in \( F(T, X) \) so that it is a cohomology \((n - 4)\)-manifold over \( Z \). Hence we may let \( S \) be so chosen that \( F(T, S) \) is connected and that both \( S \) and \( F(T, S) \) are orientable.

Since \( T \) is a circle group and since \( \dim S - \dim F(T, S) = 2 \), it follows that \( S/T \) is a connected cohomology \((n - 3)\)-manifold over \( Z \) with boundary \( F(T, S) \) [1; p. 196]. Hence we have a connected cohomology \((n - 3)\)-manifold \( Y \) over \( Z \) obtained by doubling \( S/T \) on \( F(T, S) \) [1; p. 196]. Since \( S \) is orientable, so is \( S/T - F(T, S) \). It follows from the connectedness of \( F(T, S) \) that \( Y \) is orientable.

It is clear that \( K = Gz/T \) is a cyclic group of order 2 which acts on \( S/T \) with \( KF(T, S) = F(T, S) \). Since \( F(K, F(T, S)) = F(H, S) \) is a cohomology \((n - 6)\)-manifold over \( Zz \), we infer from the dimensional parity that \( K \) preserves the orientation of \( F(T, S) \) [1; p. 79].

The action of \( K \) on \( S/T \) defines a natural action of \( K \) on \( Y \) which also preserves the orientation of \( Y \). Hence \( \dim F(K, Y) > n - 6 \) so that for some \( y^* = Ty \in S/T - F(T, S) \), \( Ky^* = y^* \). But this implies that \( Gz y = Ty \) so that \( y \) is a point of \( D \), contrary to (2.9). Hence (4.2) is proved.

(4.3) \( F(G, X) \) is a compact cohomology \((n - 6)\)-manifold over \( Zz \) with \( H^*(F(G, X); \mathbb{Z}_2) = H^*(S^{n-6}; \mathbb{Z}_2) \).

By (4.2), \( F(G, X) = F(H, X) \). Hence our assertion follows from (4.1).

(4.4) Whenever \( x \in U \), \( G_x \) is the identity group.
If $X$ is strongly paracompact, the result can be found in [5]. But an unpublished result of Yang shows that it is true in general.

(4.5) $B^*$ is a compact cohomology $(n - 4)$-manifold over $Z$ with $H^*(B^*; Z) = H^*(S^{n-4}; Z)$.

Proof. Let $T$ be a circle group in $G$ and $N$ its normalizer. Then $F(T, X)$ is a compact cohomology $(n - 4)$-manifold over $Z$ with $H^*(F(T, X); Z) = H^*(S^{n-4}; Z)$ and $N/T$ is a cyclic group of order 2 acting on $F(T, X)$ with $F(T, X)/(N/T) = B^*$. Therefore $H^*(B^*; Z)$ is finitely generated [1; p. 44]. If $H$ is a dihedral subgroup of $N$ of order 4, it is easily seen that $F(N/T, F(T, X)) = F(H, X)$ so that $F(N/T, F(T, X))$ is a compact cohomology $(n - 6)$-manifold over $Z_2$ with $H^*(F(N/T, F(T, X)); Z_2) = H^*((S^{n-6}; Z_2)$. Hence, by the dimensional parity theorem, $N/T$ preserves the orientation of $F(T, X)$.

By [1; pp. 63–64],

$$H^*(B^*; Z_2) = H^*(F(T, X)/(N/T); Z_2) = H^*(S^{n-4}; Z_2).$$

We now use the following diagram from [1; p. 45]

$$
\begin{array}{ccc}
\cdots & \longrightarrow & H^i(B^*; Z) \\
\downarrow \pi^* & & \downarrow \mu \\
& H^i(F(T, X); Z) & \longrightarrow \cdots
\end{array}
$$

in which the horizontal sequence is exact and the triangle is commutative.

For $k \neq 0, n - 4$, we have $H^k(B^*; Z_2) = 0$ and $H^*(F(T, X); Z) = 0$; hence $H^k(B^*; Z) = 0$. For $k = 0$, we have $H^0(B^*; Z) = Z$, because $B^*$ is clearly connected. For $k = n - 4$, $H^{n-4}(B^*; Z)$ is a finitely generated group with $H^{n-4}(B^*; Z) \otimes Z_2 = H^{n-4}(B^*; Z_2) = Z_2$. It follows from the universal coefficient theorem that there is a finite subgroup $K$ of $H^{n-4}(B^*; Z)$ of odd order such that $H^{n-4}(B^*; Z)/K$ is $Z$ or $Z_2$. Since $K = 2K = \mu \pi^*K = 0$, $H^{n-4}(B^*; Z) = Z$ or $Z_2$. But $H^{n-4}(B^*; Z) \neq Z_2$, because $N/T$ preserves the orientation of $F(T, X)$. Hence $H^{n-4}(B^*; Z) = Z$.

By localizing this result, we can show that $B^*$ is a cohomology $(n - 4)$-manifold over $Z$ near every point of $F(G, X)$. (This result is also shown in [2].) Since the projection of $F(T, X) - F(G, X)$ onto $B^* - F(G, X)$ is a local homeomorphism, $B^*$ is a cohomology $(n - 4)$-manifold over $Z$ near every point of $B^* - F(G, X)$. Hence $B^*$ is a compact cohomology $(n - 4)$-manifold over $Z$.

(4.6) Let $T$ be a circle group in $G$ and let $N$ be the normalizer of $T$. Then $H^*(B^*/N; Z) = H^*(S^{n-4}; Z)$.

Let $A$ be a singular orbit. If $A$ is a single point, so is $A/N$. If $A$
is a 2-sphere, we may let \( A = G/T \). Therefore \( A/N = (G/T)/N \) is homeomorphic to \((G/N)/T\) which is known to be a closed 2-cell [3]. Hence \( A/N \) is a closed 2-cell.

Since, by (2.1) and (4.2), every singular orbit is either a single point or a 2-sphere, it follows from Vietoris map theorem that \( H^*(B/N; Z) = H^*(B^*; Z) \). Hence our assertion follows from (4.5).

\[
H^*(X/N; Z) = \begin{cases} 
Z & \text{for } k = 0 \\
Z_2 & \text{for } k = n - 1 \\
0 & \text{otherwise}.
\end{cases}
\] (4.7)

Since \( H^*(F(T, X); Z) = H^*(S^{n-4}; Z) \), it follows that \( H^*(X/T; Z) = H^*(S^{n-4}; Z) \). Now \( N/T \) is a cyclic group of order 2 acting on \( X/T \) with \( (X/T)/(N/T) = X/N \).

Let \( A \) be an orbit. If \( A \) is 3-dimensional, then, by (4.4), \( A/T \) is a 2-sphere and \( N/T \) acts freely on \( A/T \). If \( A \) is a 2-sphere, then \( A/T \) is an arc and \( F(N/T, A/T) \) is a single point. If \( A \) is a point, then \( F(N/T, A/T) = A/T = A \). Hence \( F(N/T, X/T) \) is homeomorphic to \( B^* \) so that, by (4.5), \( H^*(F(N/T, X/T); Z_2) \).

As in the proof of (4.5), we can show that

\[
H^x(U/N; Z) = \begin{cases} 
Z & \text{for } k = n - 3 \\
Z_2 & \text{for } k = n - 1 \\
0 & \text{otherwise}.
\end{cases}
\] (4.8)

(4.9) \textbf{There is an exact sequence}

\[ \cdots \rightarrow H^{x-3}(U^*; Z_2) \rightarrow H^{x-2}(U^*; Z) \rightarrow H^{x}(U/N; Z) \rightarrow H^{x+1}(U^*; Z_2) \rightarrow \cdots \]

By (4.4), \( G \) acts freely on \( U \). Hence we have the desired exact sequence as seen in [3].

\[
H^x(U^*; Z) = \begin{cases} 
Z & \text{for } k = n - 3 \\
0 & \text{otherwise}.
\end{cases}
\] (4.10)

Since \( \dim U^* = n - 3 \), we have

\( H^x(U^*; Z) = 0 \) for \( k > n - 3 \).

It follows from (4.9) and (4.8) that \( H^{x-3}(U^*; Z_2) = H^{x-3}(U/N; Z) = Z_2 \).

From (4.9), it is easily seen that \( H^{x-3}(U^*; Z) = Z \oplus I \), where \( I = \text{im}(H^{x-3}(U^*; Z_2) \rightarrow H^{x-3}(U^*; Z)) \) so that every element of \( I \) different from 0 is of order 2. By the universal coefficient theorem,

\[
Z_2 = H^{x-3}(U^*; Z_2) = H^{x-3}(U^*; Z) \otimes Z_2 \oplus \text{Tor}(H^{x-3}(U^*; Z), Z_2)
\]

\[ = Z_2 \oplus I. \]

Hence \( I = 0 \), proving that
If $k < n - 3$, then by (4.8) and (4.9), $H^k(U^*; Z) = H^{k-3}(U^*; Z)$. Hence for $k < n - 3$,

$$H^k(U^*; Z) = 0.$$  

(4.11) $X^*$ is cohomologically trivial over $Z$.

This is an easy consequence of (4.5), (4.10) and the cohomology sequence of $(X^*, B^*)$.

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<table>
<thead>
<tr>
<th>Author(s)</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tsuyoshi Andô</td>
<td><em>On fundamental properties of a Banach space with a cone</em></td>
<td>1163</td>
</tr>
<tr>
<td>Sterling K. Berberian</td>
<td><em>A note on hyponormal operators</em></td>
<td>1171</td>
</tr>
<tr>
<td>Errett Albert Bishop</td>
<td><em>Analytic functions with values in a Frechet space</em></td>
<td>1177</td>
</tr>
<tr>
<td>(Sherman) Elwood Bohn</td>
<td><em>Equicontinuity of solutions of a quasi-linear equation</em></td>
<td>1193</td>
</tr>
<tr>
<td>Andrew Michael Bruckner and E. Ostrow</td>
<td><em>Some function classes related to the class of convex functions</em></td>
<td>1203</td>
</tr>
<tr>
<td>J. H. Curtiss</td>
<td><em>Limits and bounds for divided differences on a Jordan curve in the complex domain</em></td>
<td>1217</td>
</tr>
<tr>
<td>P. H. Doyle, III and John Gilbert Hocking</td>
<td><em>Dimensional invertibility</em></td>
<td>1235</td>
</tr>
<tr>
<td>David G. Feingold and Richard Steven Varga</td>
<td><em>Block diagonally dominant matrices and generalizations of the Gerschgorin circle theorem</em></td>
<td>1241</td>
</tr>
<tr>
<td>Leonard Dubois Fountain and Lloyd Kenneth Jackson</td>
<td>*A generalized solution of the boundary value problem for ( y'' = f(x, y, y') ) *</td>
<td>1251</td>
</tr>
<tr>
<td>Robert William Gilmer, Jr.</td>
<td><em>Rings in which semi-primary ideals are primary</em></td>
<td>1273</td>
</tr>
<tr>
<td>Ruth Goodman</td>
<td><em>( K )-polar polynomials</em></td>
<td>1277</td>
</tr>
<tr>
<td>Israel Halperin and Maria Wonenburger</td>
<td><em>On the additivity of lattice completeness</em></td>
<td>1289</td>
</tr>
<tr>
<td>Robert Winship Heath</td>
<td><em>Arc-wise connectedness in semi-metric spaces</em></td>
<td>1301</td>
</tr>
<tr>
<td>Isidore Heller and Alan Jerome Hoffman</td>
<td><em>On unimodular matrices</em></td>
<td>1321</td>
</tr>
<tr>
<td>Robert G. Heyneman</td>
<td><em>Duality in general ergodic theory</em></td>
<td>1329</td>
</tr>
<tr>
<td>Charles Ray Hobby</td>
<td><em>Abelian subgroups of ( p )-groups</em></td>
<td>1343</td>
</tr>
<tr>
<td>Kenneth Myron Hoffman and Hugo Rossi</td>
<td><em>The minimum boundary for an analytic polyhedron</em></td>
<td>1347</td>
</tr>
<tr>
<td>Adam Koranyi</td>
<td><em>The Bergman kernel function for tubes over convex cones</em></td>
<td>1355</td>
</tr>
<tr>
<td>Pesi Rustom Masani and Jack Max Robertson</td>
<td><em>The time-domain analysis of a continuous parameter weakly stationary stochastic process</em></td>
<td>1361</td>
</tr>
<tr>
<td>William Schumacher Massey</td>
<td><em>Non-existence of almost-complex structures on quaternionic projective spaces</em></td>
<td>1379</td>
</tr>
<tr>
<td>Deane Montgomery and Chung-Tao Yang</td>
<td><em>A theorem on the action of ( \text{SO}(3) )</em></td>
<td>1385</td>
</tr>
<tr>
<td>Ronald John Nunke</td>
<td><em>A note on Abelian group extensions</em></td>
<td>1401</td>
</tr>
<tr>
<td>Carl Mark Pearcy</td>
<td><em>A complete set of unitary invariants for operators generating finite ( W^</em> )-algebras of type 1*</td>
<td>1405</td>
</tr>
<tr>
<td>Edward C. Posner</td>
<td><em>Integral closure of rings of solutions of linear differential equations</em></td>
<td>1417</td>
</tr>
<tr>
<td>Duane Sather</td>
<td><em>Asymptotics. III. Stationary phase for two parameters with an application to Bessel functions</em></td>
<td>1423</td>
</tr>
<tr>
<td>J. Śladkowska</td>
<td><em>Bounds of analytic functions of two complex variables in domains with the Bergman-Shilov boundary</em></td>
<td>1435</td>
</tr>
<tr>
<td>Joseph Gail Stampfl!</td>
<td><em>Hyponormal operators</em></td>
<td>1453</td>
</tr>
<tr>
<td>George Gustave Weill</td>
<td><em>Some extremal properties of linear combinations of kernels on Riemann surfaces</em></td>
<td>1459</td>
</tr>
<tr>
<td>Edward Takashi Kobayashi</td>
<td>Errata: “A remark on the Nijenhuis tensor”</td>
<td>1467</td>
</tr>
</tbody>
</table>