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A NOTE ON ABELIAN GROUP EXTENSIONS

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R. J. NUNKE

In Exercise 21 page 248 of his book *Abelian Groups* L. Fuchs asks for a proof of the following

THEOREM. *If A is a torsion-free and C a torsion group, then $\text{Ext}(A, C)$ is either 0 or contains an element of infinite order.*

Unfortunately the hint given with the exercise leads only to the conclusion that every countable subgroup of A is free. Professor Fuchs has informed me that he meant to assume A countable. The purpose of this note is to prove this theorem.

LEMMA. *If C_1, C_2, \dots is a sequence of abelian groups, $\prod C_i$ their direct product and $\sum C_i$ their direct sum, then $\text{Ext}(A, \prod C_i / \sum C_i) = 0$ for all torsion-free groups A .*

Proof. A special case of this lemma with all the $C_i = Z$ the group of integers is a consequence of Theorem 1 of [1]. The proof of the special case given in [4] makes no use of the fact that $C_i = Z$. This proof will be sketched here. It is enough to prove the case in which A is the rational numbers. Since $\text{Ext}(A, \prod C_i / \sum C_i)$ is a homomorphic image of $\text{Ext}(A, \prod C_i)$ we must show that each extension $0 \rightarrow \prod C_i \rightarrow E \rightarrow A \rightarrow 0$ splits over $\prod C_i / \sum C_i$, i.e., that there is a map $f: E \rightarrow \prod C_i / \sum C_i$ whose restriction to $\prod C_i$ is the canonical projection. With A the rationals we choose elements e^1, e^2, \dots in E such that e^n maps onto $1/n!$ modulo $\prod C_i$. Then E is generated by $\prod C_i$ and the e 's with relations

$$e^n = (n + 1)e^{n+1} + c^n \qquad n = 1, 2, \dots$$

where $c^n \in \prod C_i$. We choose $b^n \in \sum C_i$ such that the first n coordinates of $c^n + b^n$ are 0 and put

$$x^n = \sum_{k \geq n} (k!/n!)(c^k + b^k).$$

Then

$$x^n = (n + 1)x^{n+1} + c^n + b^n$$

and we can define f to be the projection on $\prod C_i$ and by $f(e^n) = x^n + \sum C_i$.

PROPOSITION. *If C is the direct sum of infinitely many copies of*

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D and if A is torsion-free with $\text{Ext}(A, D) \neq 0$, then $\text{Ext}(A, C)$ has an element of infinite order.

Proof. Since D is a direct summand of C we have $\text{Ext}(A, C) \neq 0$. The sequence

$$\text{Ext}(A, \Sigma C) \rightarrow \text{Ext}(A, \Pi C) \rightarrow \text{Ext}(A, \Pi C / \Sigma C) \rightarrow 0$$

is exact where ΣC is the direct sum and ΠC the direct product of countably many copies of C . By the lemma $\text{Ext}(A, \Pi C / \Sigma C) = 0$ so that the left-most map in the sequence is an epimorphism. Since A is torsion-free $\text{Ext}(A, C)$ is divisible and hence has elements of arbitrarily large finite order if it has nonzero elements of finite order at all. Hence $\text{Ext}(A, \Pi C) \cong \Pi \text{Ext}(A, C)$ has an element of infinite order. It follows that $\text{Ext}(A, \Sigma C)$ also has an element of infinite order. Since C is the direct sum of infinitely many copies of D we have $\Sigma C \cong C$ so that $\text{Ext}(A, \Sigma C) \cong \text{Ext}(A, C)$ proving the proposition.

Now to prove the theorem we suppose that A is torsion-free, C is torsion and that $\text{Ext}(A, C)$ is a nonzero torsion group. Then $\text{Ext}(A, C)$ has a nonzero p -primary component for some prime p . Since $C = C' \oplus E$ where C' is the p -primary component of C and E is the sum of the other primary components we have

$$\text{Ext}(A, C) = \text{Ext}(A, C') \oplus \text{Ext}(A, E).$$

Multiplication by p is an automorphism of E , hence also an automorphism of $\text{Ext}(A, E)$. It follows that $\text{Ext}(A, C')$ is a nonzero torsion group. Hence in proving the theorem we may assume that C is p -primary.

In [3] it was shown that, for A torsion-free and C p -primary,

$$\text{Ext}(A, C) \cong \text{Ext}(A, M)$$

where M is a direct sum of copies of $\Sigma Z / p^n Z$, the number of copies being equal to the final rank of C . If C has bounded order, then $\text{Ext}(A, C) = 0$ for all torsion-free groups A . Otherwise the final rank of C is infinite. This last case is the one to be considered. Then M is the direct sum of countably many copies of itself and the proposition shows that $\text{Ext}(A, M)$ is either 0 or has an element of infinite order.

The referee has pointed out that a stronger form of the lemma in this paper has been proved by A. Hulanicki (Bull. Acad. Pol. Sci. Ser. Sci. Math. Astr. Phys., 10 (1962), 77-80.) He showed that each element of infinite height in $\Pi C_i / \Sigma C_i$ is in the maximal divisible subgroup, hence this group is algebraically compact.

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