Pacific Journal of Mathematics

# A NOTE ON ABELIAN GROUP EXTENSIONS

RONALD JOHN NUNKE

Vol. 12, No. 4 April 1962

# A NOTE ON ABELIAN GROUP EXTENSIONS

R. J. NUNKE

In Exercise 21 page 248 of his book Abelian Groups L. Fuchs asks for a proof of the following

THEOREM. *If $A$ is a torsion-free and $C$ a torsion group, then* $\operatorname{Ext}(A, C)$ *is either* 0 *or contains an element of infinite order.*

Unfortunately the hint given with the exercise leads only to the conclusion that every countable subgroup of $A$ is free. Professor Fuchs has informed me that he meant to assume $A$ countable. The purpose of this note is to prove this theorem.

LEMMA. *If $C_1, C_2, \cdots$ is a sequence of abelian groups, $\Pi C_i$ their direct product and $\Sigma C_i$ their direct sum, then* $\operatorname{Ext}(A, \Pi C_i/\Sigma C_i) = 0$ *for all torsion-free groups $A$.*

*Proof.* A special case of this lemma with all the $C_i = Z$ the group of integers is a consequence of Theorem 1 of [1]. The proof of the special case given in [4] makes no use of the fact that $C_i = Z$. This proof will be sketched here. It is enough to prove the case in which $A$ is the rational numbers. Since $\operatorname{Ext}(A, \Pi C_i/\Sigma C_i)$ is a homomorphic image of $\operatorname{Ext}(A, \Pi C_i)$ we must show that each extension $0 \to \Pi C_i \to E \to A \to 0$ splits over $\Pi C_i/\Sigma C_i$, i.e., that there is a map $f: E \to \Pi C_i/\Sigma C_i$ whose restriction to $\Pi C_i$ is the canonical projection. With $A$ the rationals we choose elements $e^1, e^2, \cdots$ in $E$ such that $e^n$ maps onto $1/n!$ modulo $\Pi C_i$. Then $E$ is generated by $\Pi C_i$ and the $e$'s with relations

$$e^n = (n+1)e^{n+1} + c^n \qquad n = 1, 2, \cdots$$

where $c^n \in \Pi C_i$. We choose $b^n \in \Sigma C_i$ such that the first $n$ coordinates of $c^n + b^n$ are 0 and put

$$x^n = \textstyle\sum_{k \geqq n}(k!/n!)(c^k + b^k) .$$

Then

$$x^n = (n+1)x^{n+1} + c^n + b^n$$

and we can define $f$ to be the projection on $\Pi C_i$ and by $f(e^n) = x^n + \Sigma C_i$.

PROPOSITION. If $C$ is the direct sum of infinitely many copies of

Received April 25, 1962. This work was supported by the National Science Foundation research grant NSF-G 11098.

$D$ and if $A$ is torsion-free with $\operatorname{Ext}(A, D) \neq 0$, then $\operatorname{Ext}(A, C)$ has an element of infinite order.

*Proof.* Since $D$ is a direct summand of $C$ we have $\operatorname{Ext}(A, C) \neq 0$. The sequence

$$\operatorname{Ext}(A, \Sigma C) \to \operatorname{Ext}(A, \Pi C) \to \operatorname{Ext}(A, \Pi C/\Sigma C) \to 0$$

is exact where $\Sigma C$ is the direct sum and $\Pi C$ the direct product of countably many copies of $C$. By the lemma $\operatorname{Ext}(A, \Pi C/\Sigma C) = 0$ so that the left-most map in the sequence is an epimorphism. Since $A$ is torsion-free $\operatorname{Ext}(A, C)$ is divisible and hence has elements of arbitrarily large finite order if it has nonzero elements of finite order at all. Hence $\operatorname{Ext}(A, \Pi C) \cong \Pi \operatorname{Ext}(A, C)$ has an element of infinite order. It follows that $\operatorname{Ext}(A, \Sigma C)$ also has an element of infinite order. Since $C$ is the direct sum of infinitely many copies of $D$ we have $\Sigma C \cong C$ so that $\operatorname{Ext}(A, \Sigma C) \cong \operatorname{Ext}(A, C)$ proving the proposition.

Now to prove the theorem we suppose that $A$ is torsion-free, $C$ is torsion and that $\operatorname{Ext}(A, C)$ is a nonzero torsion group. Then $\operatorname{Ext}(A, C)$ has a nonzero $p$-primary component for some prime $p$. Since $C = C' \oplus E$ where $C'$ is the $p$-primary component of $C$ and $E$ is the sum of the other primary components we have

$$\operatorname{Ext}(A, C) = \operatorname{Ext}(A, C') \oplus \operatorname{Ext}(A, E) .$$

Multiplication by $p$ is an automorphism of $E$, hence also an automorphism of $\operatorname{Ext}(A, E)$. It follows that $\operatorname{Ext}(A, C')$ is a nonzero torsion group. Hence in proving the theorem we may assume that $C$ is $p$-primary.

In [3] it was shown that, for $A$ torsion-free and $C$ $p$-primary,

$$\operatorname{Ext}(A, C) \cong \operatorname{Ext}(A, M)$$

where $M$ is a direct sum of copies of $\Sigma Z/p^n Z$, the number of copies being equal to the final rank of $C$. If $C$ has bounded order, then $\operatorname{Ext}(A, C) = 0$ for all torsion-free groups $A$. Otherwise the final rank of $C$ is infinite. This last case is the one to be considered. Then $M$ is the direct sum of countably many copies of itself and the proposition shows that $\operatorname{Ext}(A, M)$ is either 0 or has an element of infinite order.

The referee has pointed out that a stronger form of the lemma in this paper has been proved by A. Hulanicki (Bull. Acad. Pol. Sci. Ser. Sci. Math. Astr. Phys., 10 (1962), 77–80.) He showed that each element of infinite height in $\Pi C_i/\Sigma C_i$ is in the maximal divisible subgroup, hence this group is algebraically compact.

## References

1. S. Balcerzyk, *On factor groups of some subgroups of a complete direct sum of infinite cyclic groups,* Bull. Acad. Polon. Sci. Ser. Sci. Math. Astr. Phys., **7** (1959), 141–142.
2. L. Fuchs, *Abelian Groups,* Budapest, Publishing house of the Hungarian Academy of Sciences, 1958.
3. R. J. Nunke, *On the extensions of a torsion module,* Pacific J. Math., **10** (1960), 597–606.
4. ———, *Slender groups,* Acta Sci. Math. Szeged, **23** (1962), 67–73.

UNIVERSITY OF WASHINGTON

# PACIFIC JOURNAL OF MATHEMATICS





Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

# Pacific Journal of Mathematics

Vol. 12, No. 4 April, 1962