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**INTEGRAL CLOSURE OF RINGS OF SOLUTIONS OF LINEAR
DIFFERENTIAL EQUATIONS**

EDWARD C. POSNER

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Let K be an ordinary differential field of characteristic zero with field of constants C . Let R be a differential subring of K containing C and having quotient field K . A differential subring V of an extension differential field M of K is called a fundamental differential ring (over R) if V contains R and if, for each v in V , there exist v_2, \dots, v_n in V , n depending on v , such that v, v_2, \dots, v_n form a fundamental system of solutions of a homogeneous linear differential equation with coefficients in K . Throughout this paper, $\{\dots\}$ denotes differential ring adjunction, $\langle \dots \rangle$ differential field adjunction.

THEOREM 1. *Let K, C, R, M, V be as above. Then V is a fundamental differential ring over R if and only if $V = R\{v_{\alpha i}, \alpha \in A, 1 \leq i \leq n_{\alpha}\}$, A an indexing set, where for each α in A , $v_{\alpha 1}, v_{\alpha 2}, \dots, v_{\alpha n_{\alpha}}$ form a fundamental system of solutions of a homogeneous linear differential equation over K .*

Proof. If V is a fundamental differential ring over R , we may let $A = V$; the interest attaches to the converse. It amounts to proving that every differential polynomial with coefficients in R in the $v_{\alpha i}$ is one element of a fundamental system of solutions of a homogeneous linear differential equation over K , all the elements of which system of solutions belong to V . By use of induction, we may reduce the problem to consideration of the four differential polynomials $s', s + t, st$, and rs , $r \in K$. We treat the polynomials s' and $s + t$; the polynomials st and rs are treated in a like manner.

Let $s^{(n)} + a_{n-1}s^{(n-1)} + \dots + a_0s = 0$, $a_i \in K$, $0 \leq i \leq n - 1$. (There is no loss of generality in supposing that the leading coefficient of this differential equation is 1.) If $a_0 = 0$, then s' already satisfies a homogeneous linear differential equation (of order $n - 1$) over K ; if $a_0 \neq 0$, we differentiate the expression

$$\left(\left(\frac{1}{a_0}\right)s^{(n)} + \left(\frac{a_{n-1}}{a_0}\right)s^{(n-1)} + \dots + \left(\frac{a_1}{a_0}\right)s' + s\right)$$

to obtain a homogeneous linear differential equation of order n in s'

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with coefficients in K .

To prove the result for $s + t$, let s, t be in V with $s + t \neq 0$; let s, s_2, \dots, s_n be n elements of V forming a fundamental system of solutions of a homogeneous linear differential equation over K , and the same for t, t_2, \dots, t_m . Let $s_1 = s, t_1 = t$. Let $u_1 = s_1 + t_1$ and choose u_2, u_3, \dots, u_r from among $s_1, s_2, \dots, s_n; t_1, t_2, \dots, t_m$ such that u_1, u_2, \dots, u_r form a basis over the constants for the vector space spanned over the constants by $s_1, s_2, \dots, s_n; t_1, t_2, \dots, t_m$. Let $W(z_1, z_2, \dots, z_p)$ denote the wronskian of the p elements z_1, z_2, \dots, z_p . Consider the linear differential operator of order r , $\mathcal{L}(y) = W(y, u_1, \dots, u_r)/W(u_1, \dots, u_r)$. (Since u_1, \dots, u_r are linearly independent over constants, their wronskian is nonzero.) $\mathcal{L}(u_\lambda) = 0$, $1 \leq \lambda \leq r$, and $\mathcal{L} \neq 0$ since the coefficient of $y^{(r)}$ is $1 = W(u_1, \dots, u_r)/W(u_1, \dots, u_r)$. We shall prove that all the coefficients of \mathcal{L} are in K ; $\mathcal{L}(y) = 0$ will then be the sought-after differential equation.

Let σ be a differential isomorphism of $K\langle s_1, s_2, \dots, s_n; t_1, t_2, \dots, t_m \rangle$ over K ; then $\sigma(s_\mu) = \sum_{i=1}^n c_{\mu i} s_i$, $1 \leq \mu \leq n$ and $\sigma(t_\nu) = \sum_{j=1}^m d_{\nu j} t_j$, $1 \leq \nu \leq m$, where the $c_{\mu i}$ and $d_{\nu j}$ are constants. This is true because s_1, s_2, \dots, s_n span over constants the vector space of solutions of the homogeneous linear differential equation over K satisfied by s_i ; similarly for t_1, \dots, t_m . These two sets of equations taken together imply $\sigma(u_\lambda) = \sum_{k=1}^r e_{\lambda k} u_k$, $1 \leq \lambda \leq r$, $e_{\lambda k}$ constants, for each $\sigma(u_\lambda)$ is in the vector space spanned over the constants by $s_1, \dots, s_n; t_1, \dots, t_m$.

This implies that $W(y, \sigma u_1, \dots, \sigma u_r) = (\det(e_{\lambda k})) W(y, u_1, \dots, u_r)$, and similarly $W(\sigma u_1, \dots, \sigma u_r) = (\det(e_{\lambda k})) W(u_1, \dots, u_r)$. Therefore the coefficients a_p , $0 \leq p \leq r$, of $\mathcal{L}(y)$ are invariant under σ , for all differential isomorphisms σ of $K\langle s_1, \dots, s_n; t_1, \dots, t_m \rangle$ over K . By Theorem 2.6, pg. 16 of [1], a_p is in K , as required. This proves the theorem.

The above theorem has the following immediate consequence.

COROLLARY. *If M is a universal differential field extension of K ([2], Sec. 5, esp. pg. 771, Theorem), the set V of all elements of M satisfying a homogeneous linear differential equation over K forms a fundamental differential ring.*

The following lemma isolates the key property of fundamental differential rings that will be used to prove integral closure. An element w in an extension differential field of K is called a wronskian over K if $w \neq 0$ and w'/w belongs to K .

LEMMA. *Let V be a fundamental differential ring over R . Then any nonzero differential ideal I of V contains a wronskian over K .*

Proof. Let u_1 be a nonzero element of the differential ideal I of V , and let u_2, u_3, \dots, u_n be $n - 1$ elements of V such that u_1, u_2, \dots, u_n form a fundamental system of solutions of a homogeneous linear differential equation over K . Then $W(u_1, u_2, \dots, u_n)$ is a nonzero element of I : it is nonzero since u_1, u_2, \dots, u_n are linearly independent over constants; it belongs to I because each term in the expansion of the determinant defining $W(u_1, \dots, u_n)$ contains a derivative of u_1 as a factor. Since $W(u_1, \dots, u_n)$ is a wronskian over K , the proof is complete.

DEFINITION. A differential ring is called differentially simple if it has no differential ideals other than zero and itself.

THEOREM 2. *Let R be differentially simple (in particular, $R = K$), and for every wronskian w over K belonging to V , let there exist a nonzero h in R such that h/w is in V . Then V too is differentially simple. (When $R = K$, the assumption is that V contains the inverse of every wronskian over K which belongs to V .)*

Proof. Let I be a nonzero differential ideal of V . To prove that $I = V$, let w be a wronskian over K in I ; such exist by the lemma. Now by hypothesis, there is a nonzero h in R with h/w in V . Thus $w \cdot h/w = h$ is in I , so that $I \cap R$ is not the zero ideal of R . Since $I \cap R$ is a differential ideal of R and R is differentially simple, $I \cap R = R$, so that $1 \in I \cap R$, and $1 \in I$. Thus $I = V$ as required.

The next theorem is a sort of converse to the previous theorem. (Here V need not be a fundamental differential ring over R ; V can be any differential subring of M containing R .)

THEOREM 3. *Let V , but not necessarily R , be differentially simple, and let w be a wronskian over K belonging to V . Then there is a nonzero h in R such that h/w is in V . (Thus if $R = K$, $1/w$ is in V .)*

Proof. Since K is the quotient field of R , there exist b, c in R , with $c \neq 0$, such that $w' = (b/c)w$. Let I denote the set of elements of V of the form $vc^{-p}w$, p a nonnegative integer, v an element of V . I can readily be shown to be an ideal of V ; we shall prove that I is closed under differentiation. If $vc^{-p}w \in I$, then $(vc^{-p}w)' = v'c^{-p}w - pvc^{-p-1}c'w + vc^{-p}w' = (v'c)c^{-p-1}w - (pvc')c^{-p-1}w + (bv)c^{-p-1}w = (v'c - pvc' + bv)c^{-p-1}w$ is an element of V and hence of I . Thus I is a differential ideal of V , and is nonzero since w is in I . Since V is differentially simple, $I = V$, and $1 \in I$. Thus $1 = vc^{-p}w$ for some $v \in V$, $p \geq 0$. Then, if $c^p = h$, we have $h/w = v \in V$, with h an element of R . This proves the theorem.

The following theorem with $K = C$ generalizes a consequence of a result of Ritt ([4], Sec. 1, pg. 681) to the effect that if C is the field of complex numbers, the ring $C[e^{\lambda x}$, all complex λ] is integrally closed in its quotient field. In fact, Theorem 4 also implies that $C[x, e^{\lambda x}]$ is integrally closed in its quotient field.

THEOREM 4. *Let K be a differential field of characteristic zero with field of constants C . Let K be differential algebraic over C . Let V be a fundamental differential ring over K which contains the inverse of every wronskian over K in it. Then V is integrally closed in its quotient field (it is differentiably simple by Theorem 2).*

Proof. Let u be an element of the quotient field M of V integral over V : that is, there exist elements v_i in V , $1 \leq i \leq n$, such that $u^n + \sum_{i=1}^n v_i u^{n-i} = 0$, and there exist v_{n+1}, v_{n+2} in V with $u = v_{n+1}/v_{n+2}$. Let v_i be a solution of a homogeneous linear differential equation $\mathcal{L}_i(y) = 0$, $1 \leq i \leq n+2$, where $\mathcal{L}_i(y) = \sum_{j=0}^{n_i} b_{ij} y^{(j)}$, $1 \leq i \leq n+2$, $0 \leq j \leq n_i$; $b_{in_i} = 1$, $1 \leq i \leq n+2$. Furthermore let v_{ik} , $1 \leq k \leq n_i$, be for each i a fundamental system of solutions of $\mathcal{L}_i(y) = 0$, with $v_{i1} = v_i$. Let Y be a differential indeterminate, and, for each i, j , let $P_{ij}(Y) \in C\{Y\}$ be a differential polynomial of lowest order r_{ij} say satisfied by b_{ij} over C and such that the degree of P_{ij} in $Y^{(r_{ij})}$ is as small as possible among these differential polynomials of order r_{ij} . Define the separant S_{ij} of P_{ij} as the (partial) derivative of P_{ij} with respect to $Y^{(r_{ij})}$. One verifies, using the minimal property of the P_{ij} , that $S_{ij}(b_{ij})$ is nonzero. Then $b_{ij}^{(r_{ij}+1)}$ is $S_{ij}^{-1}(b_{ij})$ multiplied by a differential polynomial over C in b_{ij} of order at most r_{ij} . This implies that $C\{b_{ij}\} = C[b_{ij}^{(p)}, 0 \leq p \leq r_{ij}]$, all i, j . (This argument is well known.)

Now define $\bar{V} = C\{b_{ij}, S_{ij}^{-1}(v_{ij}), v_{ik}, \text{ all } 1 \leq i \leq n+2, 0 \leq j \leq n_i, 1 \leq k \leq n_i\}$; observe $\bar{V} \subset V$. Since \mathcal{L}_i has leading coefficient 1 and $\mathcal{L}_i(v_{ik}) = 0$, $1 \leq i \leq n+2$, $1 \leq k \leq n_i$, and because of the above property of each $C\{b_{ij}\}$, one concludes that $\bar{V} = C[b_{ij}^{(p)}, S_{ij}^{-1}(b_{ij}), v_{ik}^{(q)}, \text{ all } 1 \leq i \leq n+2, 0 \leq j \leq n_i, 1 \leq k \leq n_i, 0 \leq p \leq r_{ij}, 0 \leq q \leq n_i - 1]$. This is what we were after: we have proved that \bar{V} is finitely generated as an ordinary ring over C . We can now apply Theorem 2 of [3] to conclude that the integral closure \bar{O} of \bar{V} in its quotient field \bar{M} is in fact a differential subring of \bar{M} . But u is in \bar{O} ; if we can prove that \bar{O} is contained in \bar{V} , the proof will be completed.

So consider the ideal \bar{I} of \bar{V} consisting of all h in \bar{V} such that $h\bar{O} \subset \bar{V}$. By [5], pg. 267, Theorem 9, \bar{I} is nonzero; a fortiori, the ideal I of V consisting of those h in V with $h\bar{O} \subset V$ is also nonzero, since it contains \bar{I} . We assert that I is a differential ideal of V : let $\omega \in \bar{O}$; then $h\omega \in V$, $(h\omega)' = h'\omega + h\omega' \in V$. Since \bar{O} is closed under differentiation by [3], pg. 1393, lemma, $\omega' \in \bar{O}$, so that, since $h \in I$,

$h\omega' \in V$. Thus $h'\omega$ is in V if ω is in \bar{O} and h is in I . In other words, I is a differential ideal of V . Since V is differentially simple by Theorem 2, and I is nonzero, we conclude that $I = V$. Therefore $1 \in I$. This implies that $\bar{O} = 1 \cdot \bar{O}$ is contained in V , as promised. This completes the proof of Theorem 4.

(The above theorem could be strengthened by use of the following unproved result: a differentially simple ring of characteristic zero is integrally closed in its quotient field. This result would generalize Theorem 1 of [3].)

Theorem 4 has the following corollary.

COROLLARY. *Let K be a differential field of characteristic zero with field of constants C . Let K be differential algebraic over C . Let M be a universal differential field extension of K . Let V be the subset of M comprising those elements of M satisfying a homogeneous linear differential equation over K . Then V is integrally closed in its quotient field.*

Proof. That V is a fundamental differential ring over K follows from the corollary to Theorem 1. To prove V integrally closed in its quotient field, we shall prove that V contains the inverse of every wronskian over K in it, and then apply Theorem 4.

Now if w is a wronskian over K in V , then $w \neq 0$ and $w' = kw$, $k \in K$. Then $(1/w)' = (-1/w^2) \cdot w' = (-1/w^2) \cdot kw = -k \cdot (1/w)$. So $1/w$ satisfies a (first order) homogeneous linear differential equation over K ; by the definition of V , $(1/w)$ belongs to V , as required for the application of Theorem 4.

REMARK. Let $V_1 = V$ and V_{n+1} , $n \geq 1$, be the differential subring of M consisting of those elements of M satisfying a homogeneous linear differential equation with coefficients in V_n . Then V_{n+1} contains L_n (thus $\bigcup_{n=1}^{\infty} V_n = V_{\infty}$ is a field), for if $f (\neq 0)$ is in V_n , then $(1/f)' = -f'/f \cdot 1/f$. Thus $1/f$ satisfies a first order homogeneous linear differential equation with coefficients in L_n and so is in V_{n+1} . Since V_{n+1} contains V_n , and now the inverse of every nonzero element in V_n , V_{n+1} contains L_n . But each L_n is differential algebraic over C , and M is still a universal differential extension of L_n . The above corollary thus implies that each V_n is integrally closed in its quotient field L_n , $n \geq 1$.

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