INTEGRAL CLOSURE OF RINGS OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

Edward C. Posner
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Let $K$ be an ordinary differential field of characteristic zero with field of constants $C$. Let $R$ be a differential subring of $K$ containing $C$ and having quotient field $K$. A differential subring $V$ of an extension differential field $M$ of $K$ is called a fundamental differential ring (over $R$) if $V$ contains $R$ and if, for each $v$ in $V$, there exist $v_2, \ldots, v_n$ in $V$, $n$ depending on $v$, such that $v, v_2, \ldots, v_n$ form a fundamental system of solutions of a homogeneous linear differential equation with coefficients in $K$. Throughout this paper, $\{\cdots\}$ denotes differential ring adjunction, $\langle \cdots \rangle$ differential field adjunction.

**Theorem 1.** Let $K, C, R, M, V$ be as above. Then $V$ is a fundamental differential ring over $R$ if and only if $V = R\{v_{ai}, a \in A, 1 \leq i \leq n_a\}, A$ an indexing set, where for each $a$ in $A$, $v_{a1}, v_{a2}, \ldots, v_{an_a}$ form a fundamental system of solutions of a homogeneous linear differential equation over $K$.

**Proof.** If $V$ is a fundamental differential ring over $R$, we may let $A = V$; the interest attaches to the converse. It amounts to proving that every differential polynomial with coefficients in $R$ in the $v_{ai}$ is one element of a fundamental system of solutions of a homogeneous linear differential equation over $K$, all the elements of which system of solutions belong to $V$. By use of induction, we may reduce the problem to consideration of the four differential polynomials $s', s + t, st, \text{ and } rs, r \in K$. We treat the polynomials $s'$ and $s + t$; the polynomials $st$ and $rs$ are treated in a like manner.

Let $s^{(n)} + a_{n-1}s^{(n-1)} + \cdots + a_0s = 0, a_i \in K, 0 \leq i \leq n - 1$. (There is no loss of generality in supposing that the leading coefficient of this differential equation is 1.) If $a_0 = 0$, then $s'$ already satisfies a homogeneous linear differential equation (of order $n - 1$) over $K$; if $a_0 \neq 0$, we differentiate the expression

$$
\left(\frac{1}{a_0}\right)s^{(n)} + \left(\frac{a_{n-1}}{a_0}\right)s^{(n-1)} + \cdots + \left(\frac{a_1}{a_0}\right)s' + s
$$

to obtain a homogeneous linear differential equation of order $n$ in $s'$.
with coefficients in \( K \).

To prove the result for \( s + t \), let \( s, t \) be in \( V \) with \( s + t \neq 0 \); let \( s, s_1, \ldots, s_n \) be \( n \) elements of \( V \) forming a fundamental system of solutions of a homogeneous linear differential equation over \( K \), and the same for \( t, t_1, \ldots, t_m \). Let \( s_i = s, t_i = t \). Let \( u_1, u_2, \ldots, u_r \) be \( r \) elements of \( V \) with \( s + t \neq 0 \); let \( u_1, d_1, \ldots, u_r, d_r \) be \( r \) elements of \( V \) forming a fundamental system of solutions of a homogeneous linear differential equation over \( K \), and the same for \( t, t_1, \ldots, t_m \). Let \( W(z_1, z_2, \ldots, z_p) \) denote the wronskian of the \( p \) elements \( z_1, z_2, \ldots, z_p \).

Consider the linear differential operator of order \( r \), \( L(y) = W(y, u_1, \ldots, u_r) \). (Since \( u_1, \ldots, u_r \) are linearly independent over constants, their wronskian is nonzero.) \( L(u_i) = 0, 1 \leq \lambda \leq r, \text{ and } L \neq 0 \) since the coefficient of \( y^{(r)} \) is \( 1 = W(u_1, \ldots, u_r) \). We shall prove that all the coefficients of \( L \) are in \( K \); \( L(y) = 0 \) will then be the sought-after differential equation.

Let \( \sigma \) be a differential isomorphism of \( K(s_1, s_2, \ldots, s_n; t_1, t_2, \ldots, t_m) \) over \( K \); then \( \sigma(s_i) = \sum_{\mu=1}^n c_{\mu i} s_\mu, 1 \leq \mu \leq n \) and \( \sigma(t_j) = \sum_{\nu=1}^m d_{\nu j} t_\nu, 1 \leq \nu \leq m \), where the \( c_{\mu i} \) and \( d_{\nu j} \) are constants. This is true because \( s_1, \ldots, s_n \) span over constants the vector space of solutions of the homogeneous linear differential equation over \( K \) satisfied by \( s_i \); similarly for \( t_1, \ldots, t_m \). These two sets of equations taken together imply \( \sigma(u_i) = \sum_{\nu=1}^r e_{\nu i} u_{\nu}, 1 \leq \nu \leq r, e_{\nu i} \) constants, for each \( \sigma(u_i) \) is in the vector space spanned over the constants by \( s_1, \ldots, s_n; t_1, \ldots, t_m \).

This implies that \( W(y, \sigma u_1, \ldots, \sigma u_r) = (\det (e_{\lambda i})) W(y, u_1, \ldots, u_r), \) and similarly \( W(\sigma u_1, \ldots, \sigma u_r) = (\det (e_{\lambda i})) W(u_1, \ldots, u_r). \) Therefore the coefficients \( a_p, 0 \leq p \leq r, \) of \( L(y) \) are invariant under \( \sigma \), for all differential isomorphisms \( \sigma \) of \( K(s_1, \ldots, s_n; t_1, \ldots, t_m) \) over \( K \). By Theorem 2.6, pg. 16 of [1], \( a_p \) is in \( K \), as required. This proves the theorem.

The above theorem has the following immediate consequence.

**Corollary.** If \( M \) is a universal differential field extension of \( K \) ([2], Sec. 5, esp. pg. 771, Theorem), the set \( V \) of all elements of \( M \) satisfying a homogeneous linear differential equation over \( K \) forms a fundamental differential ring.

The following lemma isolates the key property of fundamental differential rings that will be used to prove integral closure. An element \( w \) in an extension differential field of \( K \) is called a wronskian over \( K \) if \( w \neq 0 \) and \( w'/w \) belongs to \( K \).

**Lemma.** Let \( V \) be a fundamental differential ring over \( R \). Then any nonzero differential ideal \( I \) of \( V \) contains a wronskian over \( K \).
Proof. Let \( u_1 \) be a nonzero element of the differential ideal \( I \) of \( V \), and let \( u_2, u_3, \ldots, u_n \) be \( n - 1 \) elements of \( V \) such that \( u_1, u_2, \ldots, u_n \) form a fundamental system of solutions of a homogeneous linear differential equation over \( K \). Then \( W(u_1, u_2, \ldots, u_n) \) is a nonzero element of \( I \): it is nonzero since \( u_1, u_2, \ldots, u_n \) are linearly independent over constants; it belongs to \( I \) because each term in the expansion of the determinant defining \( W(u_1, \ldots, u_n) \) contains a derivative of \( u_1 \) as a factor. Since \( W(u_1, \ldots, u_n) \) is a wronskian over \( K \), the proof is complete.

Definition. A differential ring is called differentiably simple if it has no differential ideals other than zero and itself.

Theorem 2. Let \( R \) be differentiably simple (in particular, \( R = K \)), and for every wronskian \( w \) over \( K \) belonging to \( V \), let there exist a nonzero \( h \) in \( R \) such that \( h/w \) is in \( V \). Then \( V \) too is differentiably simple. (When \( R = K \), the assumption is that \( V \) contains the inverse of every wronskian over \( K \) which belongs to \( V \).)

Proof. Let \( I \) be a nonzero differential ideal of \( V \). To prove that \( I = V \), let \( \psi \) be a wronskian over \( K \) in \( I \); such exist by the lemma. Now by hypothesis, there is a nonzero \( h \) in \( R \) with \( h/w \) in \( V \). Thus \( w \cdot h/w = h \) is in \( I \), so that \( I \cap R \) is not the zero ideal of \( R \). Since \( I \cap R \) is a differential ideal of \( R \) and \( R \) is differentiably simple, \( I \cap R = R \), so that \( 1 \in I \cap R \), and \( 1 \in I \). Thus \( I = V \) as required.

The next theorem is a sort of converse to the previous theorem. (Here \( V \) need not be a fundamental differential ring over \( R \); \( V \) can be any differential subring of \( M \) containing \( R \).)

Theorem 3. Let \( V \), but not necessarily \( R \), be differentiably simple, and let \( w \) be a wronskian over \( K \) belonging to \( V \). Then there is a nonzero \( h \) in \( R \) such that \( h/w \) is in \( V \). (Thus if \( R = K \), \( 1/w \) is in \( V \).)

Proof. Since \( K \) is the quotient field of \( R \), there exist \( b, c \) in \( R \), with \( c \neq 0 \), such that \( w' = (b/c)w \). Let \( I \) denote the set of elements of \( V \) of the form \( vc^{-p}w \), \( p \) a nonnegative integer, \( v \) an element of \( V \). \( I \) can readily be shown to be an ideal of \( V \); we shall prove that \( I \) is closed under differentiation. If \( vc^{-p}w \in I \), then \( (vc^{-p}w)' = v'c^{-p}w - pvc^{-p-1}c'w + vc^{-p}w' = (v'c)c^{-p-1}w - (pvc')c^{-p-1}w + (bv)c^{-p-1}w = (v'c - pvc')c^{-p-1}w + bv)c^{-p-1}w \) is an element of \( V \) and hence of \( I \). Thus \( I \) is a differential ideal of \( V \), and is nonzero since \( w \) is in \( I \). Since \( V \) is differentiably simple, \( I = V \), and \( 1 \in I \). Thus \( 1 = vc^{-p}w \) for some \( v \in V \), \( p \geq 0 \). Then, if \( c^p = h \), we have \( h/w = v \in V \), with \( h \) an element of \( R \). This proves the theorem.
The following theorem with $K = C$ generalizes a consequence of a result of Ritt ([4], Sec. 1, pg. 681) to the effect that if $C$ is the field of complex numbers, the ring $C[e^\lambda, \text{all complex } \lambda]$ is integrally closed in its quotient field. In fact, Theorem 4 also implies that $C[x, e^\lambda]$ is integrally closed in its quotient field.

**Theorem 4.** Let $K$ be a differential field of characteristic zero with field of constants $C$. Let $K$ be differential algebraic over $C$. Let $V$ be a fundamental differential ring over $K$ which contains the inverse of every wronskian over $K$ in it. Then $V$ is integrally closed in its quotient field (it is differentiably simple by Theorem 2).

**Proof.** Let $u$ be an element of the quotient field $M$ of $V$ integral over $V$: that is, there exist elements $v_i$ in $V$, $1 \leq i \leq n$, such that $u^n + \sum_{i=1}^n v_i u^{n-i} = 0$, and there exist $v_{n+1}, v_{n+2}$ in $V$ with $u = v_{n+1}/v_{n+2}$. Let $v_i$ be a solution of a homogeneous linear differential equation $\mathcal{L}_i(y) = 0$, $1 \leq i \leq n + 2$, where $\mathcal{L}_i(y) = \sum_{j=0}^n b_{ij} y^{(j)}$, $1 \leq i \leq n + 2$, $0 \leq j \leq n_i$; $b_{ij} = 1$, $1 \leq i \leq n + 2$. Furthermore let $v_{ik}, 1 \leq k \leq n_i$, be for each $i$ a fundamental system of solutions of $\mathcal{L}_i(y) = 0$, with $v_i = v_{i1}$. Let $Y$ be a differential indeterminate, and, for each $i, j$, let $P_{ij}(y) \in C[Y]$ be a differential polynomial of lowest order $r_{ij}$ say satisfied by $b_{ij}$ over $C$ and such that the degree of $P_{ij}$ in $Y^{(r_{ij})}$ is as small as possible among these differential polynomials of order $r_{ij}$. Define the separant $S_{ij}$ of $P_{ij}$ as the (partial) derivative of $P_{ij}$ with respect to $Y^{(r_{ij})}$. One verifies, using the minimal property of the $P_{ij}$, that $S_{ij}(b_{ij})$ is nonzero. Then $\tilde{b}_{ij} = S_{ij}(b_{ij})$ multiplied by a differential polynomial over $C$ in $b_{ij}$ of order at most $r_{ij}$. This implies that $C[b_{ij}] = C[b_{ij}^p, 0 \leq p \leq r_{ij}]$, all $i, j$. (This argument is well known.)

Now define $\tilde{V} = C[b_{ij}, S_{ij}(v_{ij}), v_{ik}, \text{all } 1 \leq i \leq n + 2, 0 \leq j \leq n_i, 1 \leq k \leq n_i]$; observe $\tilde{V} \subset V$. Since $\mathcal{L}_i$ has leading coefficient 1 and $\mathcal{L}_i(v_{ij}) = 0$, $1 \leq i \leq n + 2, 1 \leq k \leq n_i$, and because of the above property of each $C[b_{ij}]$, one concludes that $\tilde{V} = C[b_{ij}^p, S_{ij}^p(b_{ij}), v_{ij}^p, \text{all } 1 \leq i \leq n + 2, 0 \leq j \leq n_i, 1 \leq k \leq n_i 0 \leq p \leq r_{ij}, 0 \leq q \leq n_i - 1]$. This is what we were after: we have proved that $\tilde{V}$ is finitely generated as an ordinary ring over $C$. We can now apply Theorem 2 of [3] to conclude that the integral closure $\tilde{O}$ of $\tilde{V}$ in its quotient field $\tilde{M}$ is in fact a differential subring of $\tilde{M}$. But $u$ is in $\tilde{O}$; if we can prove that $\tilde{O}$ is contained in $\tilde{V}$, the proof will be completed.

So consider the ideal $I$ of $\tilde{V}$ consisting of all $h$ in $\tilde{V}$ such that $h\tilde{O} \subset \tilde{V}$. By [5], pg. 267, Theorem 9, $\tilde{I}$ is nonzero; a fortiori, the ideal $I$ of $V$ consisting of those $h$ in $V$ with $h\tilde{O} \subset V$ is also nonzero, since it contains $\tilde{I}$. We assert that $I$ is a differential ideal of $V$: let $\omega \in \tilde{O}$; then $h\omega \in \tilde{V}$, $(h\omega)' = h'\omega + h\omega' \in V$. Since $\tilde{O}$ is closed under differentiation by [3], pg. 1393, lemma, $\omega' \in \tilde{O}$, so that, since $h \in I$, 

$h \omega' \in V$. Thus $h' \omega$ is in $V$ if $\omega$ is in $\bar{O}$ and $h$ is in $I$. In other words, $I$ is a differential ideal of $V$. Since $V$ is differentiably simple by Theorem 2, and $I$ is nonzero, we conclude that $I = V$. Therefore $1 \in I$. This implies that $\bar{O} = 1 \cdot \bar{O}$ is contained in $V$, as promised. This completes the proof of Theorem 4.

(The above theorem could be strengthened by use of the following unproved result: a differentiably simple ring of characteristic zero is integrally closed in its quotient field. This result would generalize Theorem 1 of [3].)

Theorem 4 has the following corollary.

**Corollary.** Let $K$ be a differential field of characteristic zero with field of constants $C$. Let $K$ be differential algebraic over $C$. Let $M$ be a universal differential field extension of $K$. Let $V$ be the subset of $M$ comprising those elements of $M$ satisfying a homogeneous linear differential equation over $K$. Then $V$ is integrally closed in its quotient field.

**Proof.** That $V$ is a fundamental differential ring over $K$ follows from the corollary to Theorem 1. To prove $V$ integrally closed in its quotient field, we shall prove that $V$ contains the inverse of every wronskian over $K$ in it, and then apply Theorem 4.

Now if $w$ is a wronskian over $K$ in $V$, then $w \neq 0$ and $w' = kw$, $k \in K$. Then $(1/w)' = (-1/w') \cdot w' = (-1/w') \cdot kw = -k \cdot (1/w)$. So $1/w$ satisfies a (first order) homogeneous linear differential equation over $K$; by the definition of $V$, $(1/w)$ belongs to $V$, as required for the application of Theorem 4.

**Remark.** Let $V_1 = V$ and $V_{n+1}, n \geq 1$, be the differential subring of $M$ consisting of those elements of $M$ satisfying a homogeneous linear differential equation with coefficients in $V_n$. Then $V_{n+1}$ contains $L_n$ (thus $\bigcup_{n=1}^{\infty} V_n = V_\infty$ is a field), for if $f(\neq 0)$ is in $V_n$, then $(1/f)' = -f'/f \cdot 1/f$. Thus $1/f$ satisfies a first order homogeneous linear differential equation with coefficients in $L_n$ and so is in $V_{n+1}$. Since $V_{n+1}$ contains $V_n$, and now the inverse of every nonzero element in $V_n$, $V_{n+1}$ contains $L_n$. But each $L_n$ is differential algebraic over $C$, and $M$ is still a universal differential extension of $L_n$. The above corollary thus implies that each $V_n$ is integrally closed in its quotient field $L_n$, $n \geq 1$.

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