

Pacific Journal of Mathematics

HYPONORMAL OPERATORS

JOSEPH GAIL STAMPFLI

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We say a bounded linear transformation T on a Hilbert space H is hyponormal if $\|Tx\| \geq \|T^*x\|$ for all $x \in H$ or equivalently if $T^*T - TT^* \geq 0$. The notion of hyponormality was introduced in [6], through under another name. In [8], Putnam studied properties of the operator $J_\theta = e^{i\theta}T + e^{-i\theta}T^*$, where T is hyponormal. Lemmas 1 through 6 which appear below occur as exercises in [1], and will be quoted without proof. Henceforth the term operator will mean bounded linear transformation.

LEMMA 1. *Let T be a hyponormal operator on the Hilbert space H , then $\|(T - zI)x\| \geq \|(T^* - \bar{z}I)x\|$ for $x \in H$, i.e. $T - zI$ is hyponormal.*

LEMMA 2. *Let T be hyponormal on H ; then $Tx = zx$ implies $T^*x = \bar{z}x$.*

LEMMA 3. *Let T be hyponormal on H with $Tx_1 = z_1x_1$, $Tx_2 = z_2x_2$ and $z_1 \neq z_2$. Then $(x_1, x_2) = 0$.*

LEMMA 4. *If T is hyponormal on H and $M \subset H$ is invariant under T ; then $T|_M$ is hyponormal.*

LEMMA 5. *Let T be hyponormal on H , with $M \subset H$ invariant under T and let $T|_M$ be normal. Then M reduces T .*

LEMMA 6. *Let T be hyponormal on H and Let $M = \{x \in H : Tx = zx\}$. Then M reduces T and $T|_M$ is normal.*

THEOREM 1. *Let T be hyponormal on H , then $\|T\| = R_{sp}(T)$ (the spectral radius of T).*

Proof. For $x \in H$, $\|x\| = 1$ we have

$$\|Tx\|^2 = (Tx, Tx) = (T^*Tx, x) \leq \|T^*Tx\| \leq \|T^2x\|.$$

But then $\|T\|^2 \leq \|T^2\| \leq \|T\|^2$ which implies $\|T\|^2 = \|T^2\|$.

Now

$$\begin{aligned} \|T^n x\|^2 &= (T^n x, T^n x) = (T^* T^n x, T^{n-1} x) \\ &\leq \|T^* T^n x\| \cdot \|T^{n-1} x\| \leq \|T^{n-1} x\| \cdot \|T^{n-1} x\|. \end{aligned}$$

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Thus $\|T^n\|^2 \leq \|T^{n-1}\| \cdot \|T^{n-1}\|$, and combining this with the equality above, a simple induction argument yields $\|T^n\| = \|T\|^n$ for $n = 1, 2, \dots$. Since $R_{\sigma_p}(T) = \lim_{n \rightarrow \infty} \|T^n\|^{1/n} = \lim_{n \rightarrow \infty} \|T\|$ the proof is finished.

COROLLARY. *The only quasi-nilpotent hyponormal operator is the transformation which is identically zero.*

THEOREM 2. *Let T be hyponormal on the Hilbert space H and let z_0 be an isolated point in the spectrum of T . Then $z_0 \in \sigma_p(T)$, the point spectrum of T .*

Proof. By Lemma 1 we may assume $z_0 = 0$. Choose $R > 0$ sufficiently small that 0 is the only point of $\sigma(T)$ contained in or on the circle $|z| = R$. Define

$$E = \int_{|z|=R} (T - zI)^{-1} dz .$$

Then E is a nonzero projection which commutes with T (see [9]; projection as used here does not necessarily mean self-adjoint).

Thus EH is invariant under T and $T|_{EH}$ is hyponormal. Also

$$\sigma(T|_{EH}) = \sigma(T) \cap \{|z| < R\}$$

by, [9] p. 421, so $\sigma(T|_{EH}) = \{0\}$.

From the last corollary we may conclude that $T|_{EH}$ is the zero transformation. In fact, it is now clear that $EH = \{x \in H : Tx = 0\}$ which implies EH actually reduces T .

THEOREM 3. *If T is hyponormal on H with a single limit point in its spectrum, then T is normal.*

Proof. We may assume by Lemma 1 that the limit point is 0. By Theorem 1 there exists $z_1 \in \sigma(T)$ such that $|z_1| = \|T\|$.

Let $M_1 = \{x \in H : Tx = z_1x\}$; M_1 is not empty by theorem 2 and since M_1 reduces T , we conclude from theorem 2 that $T|_{M_1^\perp}$ does not have z_1 in its spectrum. We note also by Lemma 6 that $T|_{M_1}$ is normal. We continue in this way, selecting points in $\sigma(T)$ ordered by absolute value, setting $M_i = \{x \in H : Tx = z_ix\}$.

Then $M_1 \oplus \dots \oplus M_n$ reduces T and $T|_{M_1 \oplus \dots \oplus M_n}$ is normal. We observe that $T|_{(M_1 \oplus \dots \oplus M_n)^\perp}$ is hyponormal with its spectral radius equal to its norm. Thus, since 0 is the only limit point of $\sigma(T)$, the normal operators $T|_{M_1 \oplus \dots \oplus M_n}$ must converge to T in the uniform operator topology. Hence, T is normal.

COROLLARY 1. *If T is a hyponormal, completely continuous opera-*

tor, then T is normal.

COROLLARY 2. *If T is hyponormal on H with only a finite number of limit points in its spectrum; then T is normal.*

Proof. Let z_1 be a limit point of $\sigma(T)$ and choose a smooth simple closed curve G which does not intersect $\sigma(T)$ and contains only the limit point z_1 in its interior. Now define

$$E_1 = \int_G (T - zI)^{-1} dz.$$

Then T is invariant on E_1H and

$$\sigma(T) \cap [\text{Interior } G] = \sigma(T|_{E_1H})$$

so $\sigma(T|_{E_1H})$ can have only one limit point.

We now apply theorem 3 to $T|_{E_1H}$ to conclude that it is normal. Then by Lemma 5, T is reduced by E_1H . We may thus turn our attention to T on $(E_1H)^\perp$ and continue this process until the limit points are exhausted.

Theorem 1 and the first corollary to Theorem 3 have been proved independently by both T. Ando and S. Berberian, and will soon appear.

The subsequent theorem generalizes the well-known result which equates similarity equivalence of normal operators with unitary equivalence. However, there is a strong restriction on the spectrum of the operator.

THEOREM 4. *If T is a hyponormal scalar operator and $\sigma(T)$ has zero area, then T is normal.*

Proof. Since T is scalar, (see [2]), $T = QAQ^{-1}$ where Q is positive self-adjoint and A is normal. Let $A = \int zdE(z)$. For $\varepsilon > 0$, there exists a set of half-open, half-closed disjoint squares $\{R_i\}_{i=1}^k$ with each R_i of dimension $1/n \times 1/n$ such that $k/n^2 = \text{area } (U_{i=1}^k R_i) < \varepsilon$ where

$$\sigma(T) \subset (U_{i=1}^k R_i). \text{ Now for } x_i \in E(R_i)H; z_i \text{ the center of } R_i$$

we have

$$\|(A - z_i I)x_i\| \leq \left[\int_{R_i} |z - z_i|^2 d \|E(z)x_i\|^2 \right]^{1/2} \leq \frac{1}{n}.$$

Thus

$$\begin{aligned} \|(A - z_i I)Q^2x_i\| &= \|A^* - \bar{z}_i I\|Q^2x_i\| = \|Q(T^* - \bar{z}_i I)Qx_i\| \\ &\leq \|(T^* - \bar{z}_i I)Qx_i\| \cdot \|Q\| \leq \|(T - z_i I)Qx_i\| \cdot \|Q\| \end{aligned}$$

$$\begin{aligned}
 &= \|Q(A - z_i I)x_i\| \cdot \|Q\| \leq \|Q\|^2 \cdot \frac{1}{n} \quad \text{and} \\
 &\|Q^2(A - z_i I)x_i\| \leq \|Q\|^2 \cdot \frac{1}{n}.
 \end{aligned}$$

Combining these we have

$$\|(AQ^2 - Q^2A)x_i\| \leq 2\|Q\|^2 \cdot \frac{1}{n} \text{ for } x_i \in E(R_i)H.$$

Now $E(R_i)H$ is orthogonal to $E(R_j)H$ for $i \neq j$ so for $y \in H$, $\|y\| = 1$ we have

$$y = \sum_{i=1}^k a_i x_i \text{ where } \sum_{i=1}^k |a_i|^2 = 1.$$

Thus

$$\begin{aligned}
 \|(AQ^2 - Q^2A)y\| &= \left\| \sum_{i=1}^k a_i (AQ^2 - Q^2A)x_i \right\| \\
 &\leq \sum_{i=1}^k |a_i| \|(AQ^2 - Q^2A)x_i\| \\
 &\leq \left\{ \sum_{i=1}^k |a_i|^2 \sum_{i=1}^k \left(2\|Q\|^2 \cdot \frac{1}{n} \right)^2 \right\}^{1/2} = 2\|Q\|^2 \cdot k^{1/2}/n \\
 &\leq 2\|Q\|^2 \varepsilon^{1/2}
 \end{aligned}$$

implying that $AQ^2 = Q^2A$.

Noting that Q is positive we may conclude from the spectral theorem that $AQ = QA$ and thus $T = A$ which completes the proof.

The author has been unable to decide whether the condition on the area of the spectrum in the last theorem may be omitted. He would conjecture that it cannot. There is a generalization of the theorem quoted above which states that if A and B are normal operators, Q an arbitrary operator such that $AQ = QB$; then $A^*Q = QB^*$. This statement does not hold if A is normal and B semi-normal. To see this, let H be a Hilbert space with the basis $\{\phi_i\}_{i=-\infty}^\infty$ and define $A\phi_i = \phi_{i+1}$, all i ; $B\phi_i = \phi_{i+1}$, $i \geq 0$ $B\phi_i = 0$, $i < 0$; and $Q = B$. Then it is clear that $AQ = QB$ but $A^*Q\phi_0 = \phi_0 \neq QB^*\phi_0 = 0$.

Before going on to the next theorem we must recall some results from the literature. Let B be a normal operator of finite spectral multiplicity n , and let T be an operator commuting with B . Then there is a finite measure $\nu(\cdot)$ defined on Borel sets of the complex plane and vanishing outside of $\sigma(B)$ and n Borel sets e_1, \dots, e_n with e_1 the plane and $e_i \subset e_{i+1}$, such that if we define $\nu_i(e) = \nu(e \cap e_i)$ for such Borel set e ; set $\hat{H} = \sum_{i=1}^n L_2(\nu_i)$ and define

$$\hat{B}f(s) = \hat{B}(f_1(s), \dots, f_n(s)) = (sf_1(s), \dots, sf_n(s)) = sf(s)$$

for $f(a) \in H$; then B and \hat{B} are unitarily equivalent. Also there exist measurable functions $a_{ij}(s)$, $j = 1, \dots, n$ such that if we define

$$Tf(a) = \begin{vmatrix} a_{1,1}(s) \cdots a_{1,n}(s) & | & f_1(s) \\ \vdots & & \vdots \\ a_{n,1}(s) \cdots a_{n,n}(s) & | & f_n(s) \end{vmatrix}$$

for $f(s) \in \hat{H}$; then T and T are unitarily equivalent. The foregoing may be found in [3] Chapter X theorem 5.10, in [4], [7] and in its earliest form is due to von Neumann.

Using results of Gonsior (see [5] Theorem 3 and remarks in section 6) we may define \hat{H} in such a way that

$$T = \begin{vmatrix} a_{11}(s) & a_{12}(s) \cdots a_{1n}(s) \\ 0 & a_{22}(s) \cdots a_{2n}(s) \\ \vdots & 0 \quad \vdots \\ 0 & a_{nn}(s) \end{vmatrix}$$

or roughly speaking \hat{T} has super diagonal form. In what follows we will identify \hat{T} with T and \hat{B} with B .

THEOREM 5. *Let T be a hyponormal operator with $T^n = B$ where n is a positive integer and B is a normal operator; then T is normal.*

Proof. For $x_0 \in H$, let $M = \text{clm}[B^i B^{*j} T^k x_0]$ (the closed linear manifold spanned by the iterates) for $k = 0, 1, \dots, n - 1$; $i, j = 0, 1, 2, \dots$.

Then M reduces B and B has spectral multiplicity n on M (we will assume $x_0, Tx_0, \dots, T^{n-1}x_0$ are linearly independent). Also M is invariant under T since $TB^i B^{*j} T^k x_0 = B^i B^{*j} T^{k+1} x_0$ and invariance holds under closure. The Fuglede theorem is used in obtaining the last equality.

Let us now consider $T|_M$ which we may write as

$$\begin{vmatrix} a_{11}(s) \cdots a_{1n}(s) \\ 0 \quad \cdot \\ \quad \cdot \\ \quad \quad \cdot \\ \quad \quad \quad a_{nn}(s) \end{vmatrix}$$

Then for the vector $f_1 = (x_{\sigma(B)}, 0, \dots, 0)$, where $x_{\sigma(B)}$ is the characteristic function of $\sigma(B)$,

we have

$$\| T|_M f_1 \|^2 = \int_{\sigma(B)} |a_{11}(s)|^2 dv_1(s)$$

and

$$\| (T|_M)^* f_1 \|^2 = \sum_{j=1}^n \int_{\sigma(B)} |a_{1j}(s)|^2 dv_j(s) .$$

But $\|T|_{\mathcal{M}}f_1\| \geq \|(T|_{\mathcal{M}})^*f_1\|$ since the restriction of a hyponormal operator to an invariant subspace is hyponormal. Thus $a_{1j}(s) = 0$ a.e. (v_j) for $j = 2, 3, \dots, n$. Continuing this argument, we find that $a_{ij}(s) = 0$ a.e. (v_j) for $i \neq j$. We conclude from this that $T|_{\mathcal{M}}$ is normal, or $\|T|_{\mathcal{M}}y\| = \|(T|_{\mathcal{M}})^*y\|$ for $y \in \mathcal{M}$. Therefore,

$$\|Tx_0\| = \|T|_{\mathcal{M}}x_0\| = \|(T|_{\mathcal{M}})^*x_0\| \leq \|T^*x_0\|.$$

But this with hyponormality implies that $\|Tx_0\| = \|T^*x_0\|$. Since x_0 is arbitrary T must be normal.

COROLLARY. *If T is hyponormal and commutes with a normal operator having finite spectral multiplicity; then T is normal.*

EXAMPLE. Let $\{\varphi_i\}_{i=-\infty}$ be a basis for the Hilbert space H and define $T\varphi_i = a_i\varphi_{i+1}$. Then if $|a_i| \leq |a_{i+1}|$, all i , T is hyponormal. T will be normal if and only if $|a_i| = |a_{i+1}|$ for all i . If $|a_k| = |a_{k+1}|$ for some fixed k and $|a_j| \neq |a_k|$ for some $j > k$ then T will not be subnormal. Another example of hyponormal operator which is not subnormal is given in [6].

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Vol. 12, No. 4

April, 1962

Tsuyoshi Andô, <i>On fundamental properties of a Banach space with a cone</i>	1163
Sterling K. Berberian, <i>A note on hyponormal operators</i>	1171
Errett Albert Bishop, <i>Analytic functions with values in a Frechet space</i>	1177
(Sherman) Elwood Bohn, <i>Equicontinuity of solutions of a quasi-linear equation</i>	1193
Andrew Michael Bruckner and E. Ostrow, <i>Some function classes related to the class of convex functions</i>	1203
J. H. Curtiss, <i>Limits and bounds for divided differences on a Jordan curve in the complex domain</i>	1217
P. H. Doyle, III and John Gilbert Hocking, <i>Dimensional invertibility</i>	1235
David G. Feingold and Richard Steven Varga, <i>Block diagonally dominant matrices and generalizations of the Gerschgorin circle theorem</i>	1241
Leonard Dubois Fountain and Lloyd Kenneth Jackson, <i>A generalized solution of the boundary value problem for $y'' = f(x, y, y')$</i>	1251
Robert William Gilmer, Jr., <i>Rings in which semi-primary ideals are primary</i>	1273
Ruth Goodman, <i>K-polar polynomials</i>	1277
Israel Halperin and Maria Wonenburger, <i>On the additivity of lattice completeness</i>	1289
Robert Winship Heath, <i>Arc-wise connectedness in semi-metric spaces</i>	1301
Isidore Heller and Alan Jerome Hoffman, <i>On unimodular matrices</i>	1321
Robert G. Heyneman, <i>Duality in general ergodic theory</i>	1329
Charles Ray Hobby, <i>Abelian subgroups of p-groups</i>	1343
Kenneth Myron Hoffman and Hugo Rossi, <i>The minimum boundary for an analytic polyhedron</i>	1347
Adam Koranyi, <i>The Bergman kernel function for tubes over convex cones</i>	1355
Pesi Rustom Masani and Jack Max Robertson, <i>The time-domain analysis of a continuous parameter weakly stationary stochastic process</i>	1361
William Schumacher Massey, <i>Non-existence of almost-complex structures on quaternionic projective spaces</i>	1379
Deane Montgomery and Chung-Tao Yang, <i>A theorem on the action of SO(3)</i>	1385
Ronald John Nunke, <i>A note on Abelian group extensions</i>	1401
Carl Mark Pearcy, <i>A complete set of unitary invariants for operators generating finite W^*-algebras of type I</i>	1405
Edward C. Posner, <i>Integral closure of rings of solutions of linear differential equations</i>	1417
Duane Sather, <i>Asymptotics. III. Stationary phase for two parameters with an application to Bessel functions</i>	1423
J. Ślaskowska, <i>Bounds of analytic functions of two complex variables in domains with the Bergman-Shilov boundary</i>	1435
Joseph Gail Stampfli, <i>Hyponormal operators</i>	1453
George Gustave Weill, <i>Some extremal properties of linear combinations of kernels on Riemann surfaces</i>	1459
Edward Takashi Kobayashi, <i>Errata: "A remark on the Nijenhuis tensor"</i>	1467