ON THE GENERATION OF DISCONTINUOUS GROUPS

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In a paper in this Journal (v. 11, p. 675) M. I. Knopp remarked that $G(j)$, the principal congruence subgroup of level $j \geq 2$ of the modular group, can be generated exclusively by parabolic transformations if and only if it is of genus zero. The following natural generalization is easily proved:

Let $\Gamma$ be a horocyclic group of genus $g$. Then $\Gamma$ possesses a system of generators consisting entirely of parabolic and elliptic elements if and only if $g = 0$.

Knopp's result is a special case, since $G(j)$ has no elliptic substitutions.

For the proof we appeal to the classical result that $\Gamma$ has a canonical fundamental region whose sides are conjugated by elliptic and parabolic substitutions and $2g$ hyperbolic substitutions $A_1, B_1, \ldots, A_g, B_g$ (cf. [1], p. 182 ff). These substitutions generate $\Gamma$. If $g = 0$, the hyperbolic ones are absent and the conclusion follows.

Conversely, let $\Gamma$ be generated by elliptic and parabolic transformations $T_1, \ldots, T_s$. Let the domain of existence of $\Gamma$ be, for example, the upper half-plane $H$. Denote by $H^+$ the union of $H$ and the parabolic cusps of $\Gamma$. If $g > 0$ there exists an abelian integral of the first kind, that is, a function $F$ regular in $H^+$ such that

\[(*) \quad F(Lz) = F(z) + C(L)\]

for all $L \in \Gamma$. Each $T_i$ has a fixed point lying in $H^+$. Letting $z$ tend to this fixed point in $(*)$, we see that $C(T_i) = 0, \ i = 1, \ldots, s$. Since

\[C(L_1 L_s) = C(L_1) + C(L_s),\]

and the $T_i$ generate $\Gamma$, we have

\[C(L) = 0\]

for all $L \in \Gamma$. The abelian integral $F$ has zero periods and is therefore an automorphic function on $\Gamma$. Since it is regular in the closed fundamental region, it is a constant. Differentiating, we conclude that there are no abelian differentials of the first kind except 0.

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1 A discontinuous group $\Gamma$ is called horocyclic (Grenzkreisgruppe) if there is a fixed disk (or half-plane) preserved by each element of $\Gamma$ and every boundary point of the disk is a limit point of $\Gamma$. 
whence $\Gamma$ is of genus 0. This completes the proof.

That a group of genus 0 cannot always be generated entirely by parabolic elements is shown by the following example, supplied by Morris Newman. Let $H$ be the group generated by $G = G(3)$ and $T$, where $T\tau = -1/\tau$. Since $T$ is of period 2 and commutes with $G$, we have

$$H = G + TG.$$ 

Now $G$ is of genus 0, as is known. Let $f(\tau)$ be a univalent function on $G$ with a simple pole at $\tau_0 \neq i$. Then $f(\tau) + f(-1/\tau)$ is univalent on $H$, which is therefore of genus 0. A parabolic element $P$ of $H$ cannot lie in $TG$, for $P$ has trace $\pm 2$ whereas $TG \equiv T (\text{mod } 3)$ has trace divisible by 3. Hence $P$ is in $G$, and therefore every product of parabolic elements of $H$ is also in $G$. It follows that $H$ cannot be generated by parabolic elements alone.

Instead of $G(3)$ we could also have used $G(4)$ or $G(5)$.

**Reference**

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