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**ON CERTAIN PROJECTIONS IN SPACES OF CONTINUOUS  
FUNCTIONS**

S. P. LLOYD

# ON CERTAIN PROJECTIONS IN SPACES OF CONTINUOUS FUNCTIONS

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**1. Introduction.** Let  $X$  be a compact Hausdorff space and let  $C(X)$  be the Banach space of continuous real or complex valued functions on  $X$ , with supremum norm. We are concerned with the set  $\mathcal{P}$  of positive bounded constant decreasing projections in  $C(X)$ . That is,  $\mathcal{P}$  is the set of bounded linear operators  $T: C(X) \rightarrow C(X)$  which have the properties  $T^2 = T$ ,  $Tf \geq 0$  if  $f \geq 0$ ,  $T1 \leq 1$ . A great deal is known about the structure of such  $T$  when the range of  $T$  is a closed self-adjoint subalgebra of  $C(X)$  containing constants [1] [4] [5]. In the present paper we develop a corresponding representation theory for members of  $\mathcal{P}$ . An application to Markov processes is given.

**2. Representation theory.** Let  $\mathcal{X}$  denote the  $\sigma$ -field of Borel subsets of  $X$ . We represent the conjugate space of bounded linear functionals on  $C(X)$  as the space of regular real or complex Borel measures in  $X$ , with variation norm. In all that follows, the topology in  $C^*(X)$  will be the  $C(X)$  (weak\*) topology.

**THEOREM 1.** *The members of  $\mathcal{P}$  correspond 1–1 to certain  $C^*(X)$  valued functions on  $X$ , as follows. Suppose  $t: X \rightarrow C^*(X)$  corresponds to  $T \in \mathcal{P}$ . Then  $t$  and  $T$  are related by (i), and  $t$  has properties (ii)–(iv):*

$$(i) \quad Tf(x) = \int f(x')t_x(dx'), \quad x \in X, f \in C(X)$$

(ii)  $t: X \rightarrow C^*(X)$  is continuous (with the  $C(X)$  topology in  $C^*(X)$ ).

$$(iii) \quad t_x \geq 0, \quad t_x(X) \leq 1, \quad x \in X$$

$$(iv) \quad t_x = \int t_x t_x(dx'), \quad x \in X.$$

*Proof.* Suppose  $T \in \mathcal{P}$  is given. Standard representation theory for bounded linear transformations into  $C(X)$  gives (i) and (ii) immediately [2, p. 490]. Property (iii) is a consequence of  $T \geq 0$ ,  $T1 \leq 1$ . It is to be noted that the conditions  $T \geq 0$ ,  $T1 \leq 1$ ,  $T \neq 0$  which characterize the nonzero members of  $\mathcal{P}$  are equivalent to the conditions  $T \geq 0$ ,  $\|T\| = 1$ . The function  $t$  is simply the restriction of the adjoint  $T^*: C^*(X) \rightarrow C^*(X)$  to domain  $X$ , regarding  $X$  as the set of unit point measures in  $C^*(X)$ . The adjoint itself has the representation

$$(2) \quad T^*\lambda = \int t_x \lambda(dx), \lambda \in C^*(X),$$

where the integration is in the weak\* sense [3]. (That is, for given  $\lambda \in C^*(X)$  the value of the integral in (2) is the element of  $C^*(X)$  whose values for  $f \in C(X)$  are

$$\int \lambda(dx) \int f(x') t_x(dx'), f \in C(X).$$

Condition (iv) is a consequence of  $T^2 = T$ ; the integration again is in the weak\* sense. Conversely, any  $t$  with properties (ii)–(iv) determines a  $T \in \mathcal{S}$  according to (i), and the theorem is proved.

Let  $\varphi$  be the equivalence in  $X$  defined by  $x_1 \varphi x_2$  if and only if  $t_{x_1} = t_{x_2}$ . On the quotient space  $Y = X/\varphi$  define  $\bar{t}: Y \rightarrow C^*(X)$  by  $\bar{t}_y = t_x$  if  $y = \pi x, x \in X$  where  $\pi: X \rightarrow Y$  is the quotient mapping. General considerations show that  $\bar{t}$  is a homeomorphism of compact Hausdorff  $Y$  and the set  $K = \{\bar{t}_y: y \in Y\}$  of various distinct values of  $t$ . The quotient mapping is closed, so that the decomposition  $\{\pi^{-1}y: y \in Y\}$  of  $X$  into closed equivalence classes is upper semicontinuous.

Denote by  $K_1$  the closed convex hull of  $K \cup \{0\}$ , where 0 is the zero measure. Since  $K \cup \{0\}$  is compact,  $K_1$  is compact, and is hence the closed convex hull of its extreme points. Denote by  $Y_0$  the set of all  $y \in Y$  such that  $\bar{t}_y \neq 0$  is an extreme point of  $K_1$ ; all extreme points of  $K_1$  are to be found in  $\{\bar{t}_y: y \in Y_0\} \cup \{0\}$  [2, p. 440].

**THEOREM 2.** *For each  $y \in Y_0$  the measure  $\bar{t}_y$  lives on  $\pi^{-1}y$ ; that is,  $\bar{t}_y(E) = \bar{t}_y(E \cap \pi^{-1}y), E \in X, y \in Y_0$ . Moreover,  $\bar{t}_y(X) = 1, y \in Y_0$ .*

*Proof.* Property (1.iv) is

$$(3) \quad \bar{t}_y = \int \bar{t}_{\pi x} \bar{t}_y(dx), y \in Y,$$

in terms of  $\bar{t}$ . Fix  $y \in Y_0$ , and suppose there exists a closed set  $F$  disjoint from  $\pi^{-1}y$  such that  $\bar{t}_y(F) > 0$ . Since  $\bar{t}$  is one-to-one and continuous,  $\bar{t}\pi F = \{\bar{t}_{\pi x}: x \in F\}$  is a compact set which does not contain  $\bar{t}_y$ . The closed convex hull of  $\bar{t}\pi F$  does not contain  $\bar{t}_y$ , either (otherwise  $\bar{t}_y \in \bar{t}\pi F, \bar{t}_y$  being extreme [2, p. 440]). Thus there exists  $f \in C(X)$  which separates  $\bar{t}_y$  and  $\bar{t}\pi F$  strictly. Expressing (3) as

$$\bar{t}_y = \bar{t}_y(F) \int_F \bar{t}_{\pi x} \frac{\bar{t}_y(dx)}{\bar{t}_y(F)} + \int_{X-F} \bar{t}_{\pi x} \bar{t}_y(dx) + [1 - \bar{t}_y(X)]0,$$

we see that  $\bar{t}_y$  is expressed as a proper convex combination of elements of  $K_1$  distinct from  $\bar{t}_y$ . This contradicts the assumption that  $\bar{t}_y$  is an extreme point of  $K_1$ . The regularity of each  $\bar{t}_y$  shows that  $\bar{t}_y$  lives

on  $\pi^{-1}y$  when  $y \in Y_0$ . The same sort of argument shows that if  $\bar{t}_y(X) \neq 0$ , then  $\bar{t}_y$  is not an extreme point of  $K_1$  unless  $\bar{t}_y(X) = 1$ .

**THEOREM 3.**  $Y_0$  is closed.

*Proof.* Define  $u: Y \rightarrow C^*(Y)$  by  $u_y(E) = \bar{t}_y(\pi^{-1}E)$ ,  $E \in Y$ ,  $y \in Y$ . The continuity of  $\bar{t}$  implies that  $u$  is continuous with the  $C(Y)$  topology in  $C^*(Y)$ . From Theorem 2,  $u_y$  is for each  $y \in Y_0$  the unit point measure at  $y$ . Thus for each  $f \in C(Y)$  we have

$$(4) \quad f(y) = \int f(y')u_y(dy'), \quad y \in Y_0.$$

Since for each  $f \in C(Y)$  the members of (4) are continuous in  $y$ , the equality (4) persists for  $y \in \bar{Y}_0$ . This implies that  $u_y$  is the unit point measure at  $y$  for each  $y \in \bar{Y}_0$ . It follows that  $\bar{t}_y$  lives on  $\pi^{-1}y$  for each  $y \in \bar{Y}_0$ . It should be clear that each such  $\bar{t}_y$ ,  $y \in \bar{Y}_0$ , is necessarily an extreme point of  $K_1$ , and the theorem follows.

**THEOREM 4.** For each  $y \in Y$  the measure  $\bar{t}_y$  lives on  $\pi^{-1}Y_0$ ; that is,  $\bar{t}_y(E) = \bar{t}_y(E \cap \pi^{-1}Y_0)$ ,  $E \in X$ ,  $y \in Y$ .

*Proof.* Since  $\{\bar{t}_y, y \in Y\}$  is in the closed convex hull of compact  $\{\bar{t}_y, y \in Y_0\} \cup \{0\}$ , for each  $y \in Y$  there exists a Borel measure  $\nu_y \geq 0$  on compact  $Y_0$  such that

$$(5) \quad \bar{t}_y = \int_{X_0} \bar{t}_y \nu_y(dy')$$

in the weak\* sense. Let  $F$  be an arbitrary closed subset of  $X - \pi^{-1}Y_0$ , and let  $f \in C(X)$  satisfy  $f(F) = 1$ ,  $f(\pi^{-1}Y_0) = 0$ ,  $0 \leq f \leq 1$ . From (5) and Theorem 2 one has  $\int f(x)\bar{t}_y(dx) = 0$ ,  $y \in Y$ , and hence  $\bar{t}_y(F) = 0$ ,  $y \in Y$ . Since each  $\bar{t}_y$  is regular, the theorem follows.

**3. Invariant measures and functions.** We now characterize the ranges of  $T^*$  and  $T$ . From (2), any invariant measure  $T^*\lambda$  is contained in the weak\* closed subspace spanned by  $\{t_x, x \in X\}$ . From (1.iv), each  $t_x$  is invariant,  $x \in X$ . Thus range ( $T^*$ ) is the weak\* closed subspace spanned by  $\{t_x, x \in X\}$ . The extreme points  $\{\bar{t}_y, y \in Y_0\}$  constitute a minimal spanning set, clearly.

From (1.i), any invariant function  $Tf$  is constant on equivalence classes, and so determines an element of  $C(Y)$ . Restriction of domain to  $Y_0$  gives an element of  $C(Y_0)$ . Conversely, let  $f_0$  be an arbitrary element of  $C(Y_0)$ . Define function  $f$  by

$$(6) \quad f(x) = \int_{\pi^{-1}Y_0} f_0(\pi x')t_x(dx'), \quad x \in X.$$

It follows from Theorem 3 and the Tietze extension theorem that  $f \in C(X)$  and hence that  $Tf = f$ . From Theorem 2, the contraction procedure described above applied to  $f$  gives  $f_0$  back again. It should then be clear that (6) establishes an isometric order isomorphism of  $C(Y_0)$  and range  $(T)$ . The isomorphism is algebraic if and only if  $Y_0 = Y$  [4].

**4. Application to Markov chains.** Let  $(X_1, \mathcal{F})$  be a measurable space, and let  $p(x, E)$ ,  $x \in X_1$ ,  $E \in \mathcal{F}$ , be a transition subprobability. That is,  $p(x, \cdot)$  is a measure on  $\mathcal{F}$  for each  $x \in X_1$  and  $0 \leq p(\cdot, E) \leq 1$  is a measurable function for each  $E \in \mathcal{F}$ . Denote by  $B(X_1, \mathcal{F})$  the Banach space of all bounded real or complex measurable functions on  $X_1$ , with supremum norm. Then  $P: B(X_1, \mathcal{F}) \rightarrow B(X_1, \mathcal{F})$  defined by

$$Pf(x) = \int f(x')p(x, dx'), \quad x \in X_1, f \in B(X_1, \mathcal{F}),$$

has the properties  $P \geq 0$ ,  $\|P\| \leq 1$ . Suppose there is an operator  $T$  (necessarily unique) in the closed convex hull of  $\{P^n, n = 1, 2, \dots\}$  in the weak operator topology with the properties  $TP = PT = T$ . Then  $T$  has the properties  $T \geq 0$ ,  $\|T\| \leq 1$ , and is the projection onto the subspace of invariant functions of  $P$ .

We assume without essential loss of generality that  $B(X_1, \mathcal{F})$  separates the points of  $X_1$ . Then there is a totally disconnected compact Hausdorff space  $X$  containing  $X_1$  as a dense subset such that each element of  $B(X_1, \mathcal{F})$  extends uniquely to an element of  $C(X)$  [2, p. 276]. Operator  $P$  becomes an operator  $P: C(X) \rightarrow C(X)$  with the properties  $P \geq 0$ ,  $\|P\| \leq 1$ . Such an operator necessarily has the form

$$Pf(x) = \int f(x')p_x(dx'), \quad x \in X, f \in C(X),$$

where  $p: X \rightarrow C^*(X)$  is continuous with the  $C(X)$  topology in  $C^*(X)$  and has the properties  $p_x \geq 0$ ,  $p_x(X) \leq 1$ ,  $x \in X$ . Clearly,  $p$  is the extension of the given transition subprobability to all of  $X$ .

The operator  $T$  becomes a projection in  $C(X)$  to which our results apply. Each set  $\pi^{-1}y$ ,  $y \in Y_0$  is an ergodic set and  $X - \pi^{-1}Y_0$  is the dissipative set, according to

**THEOREM 5.** *If  $y \in Y_0$  then for almost all  $x \in \pi^{-1}y$  with respect to  $\bar{t}_y$  the measure  $p_x$  lives on  $\pi^{-1}y$ .*

*Proof.* From  $TP = T$  we obtain

$$\int \bar{t}_y(dx) \int p_x(dx') f(dx') = \int \bar{t}_y(x') f(x'), \quad y \in Y, f \in C(X).$$

Fix  $y \in Y_0$  and let  $F$  be any closed set disjoint from  $\pi^{-1}y$ . Let  $f \in C(X)$  be such that  $f(F) = 1$ ,  $f(\pi^{-1}y) = 0$ ,  $0 \leq f \leq 1$ . The right-hand side above vanishes, from Theorem 2, which requires  $p_x(F) = 0$  for almost all  $x$  with respect to  $\bar{t}_y$ . Since  $p_x$  is regular, the theorem follows.

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