

# Pacific Journal of Mathematics

**$E^3$  MODULO A 3-CELL**

DONALD VERN MEYER

# $E^3$ MODULO A 3-CELL

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If  $A$  is a compact continuum in  $E^n$ , then  $E^n/A$  is the decomposition of  $E^n$  whose only nondegenerate element is  $A$ . If  $C$  is an  $n$ -cell in  $E^n$ , let  $N(C)$  be the set of points on  $BdC$  at which  $BdC$  is not locally polyhedral.

In [1], Andrews and Curtis proved that if  $A$  is an arc in  $E^n$ , then  $E^n/A \times E^1$  is homeomorphic to  $E^{n+1}$ . In Theorem 2 of this paper it is proved that if  $C$  is a 3-cell in  $E^3$  such that there exists an arc  $A$  on  $BdC$  containing  $N(C)$ , then  $E^3/A$  is homeomorphic to  $E^3/C$ . It follows that  $E^3/C \times E^1$  is homeomorphic to  $E^4$ .

$J$  denotes the set of all positive integers and  $d$  is the usual metric for  $E^3$ . An  $n$ -manifold is a separable metric space  $K$  such that each point of  $K$  has a neighborhood which is homeomorphic to  $E^n$ . An  $n$ -manifold-with-boundary is a separable metric space  $M$  such that each point of  $M$  lies in an open set  $V$  such that the closure of  $V$  is an  $n$ -cell (the homeomorphic image of  $\{(x_1, x_2, \dots, x_n): x_1^2 + x_2^2 + \dots + x_n^2 \leq 1\}$ ). If  $M$  is an  $n$ -manifold-with-boundary, then the boundary of  $M$  is the set of points of  $M$  which do not have neighborhoods homeomorphic to  $E^n$ . The boundary of  $M$  is denoted by  $BdM$ .

The term "interior" is used in two different ways. The interior of an  $n$ -manifold-with-boundary  $M$  is  $M - BdM$ . If  $T$  is a compact connected 2-manifold in  $E^3$  such that  $E^3 - T$  is the union of two disjoint open sets each having  $T$  as its boundary, then the interior of  $T$  is the bounded component of  $E^3 - T$ . In either case the interior of a set  $L$  is denoted by  $(\text{int } L)$ . The exterior of  $T$  is the unbounded component of  $E^3 - T$  and is denoted by  $(\text{ext } T)$ . If  $X$  is a set in  $E^3$  and  $e$  is a positive number, let  $Cl(X)$  be the closure of  $X$  and  $V(X, e)$  be  $\{y: y \in E^3 \text{ and } d(X, y) < e\}$ .

**THEOREM 1.** *Let  $C$  and  $A$  be compact sets in  $E^3$  such that there exist sequences  $U$  and  $V$  of open sets in  $E^3$  and a sequence  $h$  of homeomorphisms of  $E^3$  onto itself such that*

- (1)  $Cl(U_{i+1}) \subset U_i, \cap \{U_j: j \in J\} = C, U_1$  is bounded,
- (2)  $Cl(V_{i+1}) \subset V_i, \cap \{V_j: j \in J\} = A, V_1$  is bounded, and
- (3)  $h_i[U_i - Cl(U_{i+1})] = V_i - Cl(V_{i+1})$ , and  $h_i = h_{i-1}$  on  $E^3 - U_i$ .

*Then  $E^3/C$  is homeomorphic to  $E^3/A$ .*

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*Proof.* If  $x \in (E^3 - C)$ , let  $g(\{x\})$  be  $\{\lim h_i(x)\}$ , and let  $g(C)$  be  $A$ . Then  $g$  is a homeomorphism of  $E^3/C$  onto  $E^3/A$ .

**THEOREM 2.** *Let  $C$  be a 3-cell in  $E^3$  such that there exists an arc  $A$  on  $BdC$  such that  $N(C) \subset A$ . Then  $E^3/C$  is homeomorphic to  $E^3/A$ .*

*Proof.* Let  $C$  and  $A$  satisfy the hypothesis of Theorem 2.

**LEMMA 1.** *If  $\epsilon$  is a positive number, there exist a 3-manifold-with-boundary  $S$  and a homeomorphism  $h_\epsilon$  of  $E^3$  onto itself such that (1)  $C \subset (\text{int } S)$ , (2) if  $x \in [E^3 - V(C, \epsilon)] \cup A$ ,  $h_\epsilon(x) = x$ , and (3)  $h_\epsilon[Cl(\text{int } S)] \subset V(A, \epsilon)$ .*

*Proof of Lemma 1.* Let  $P$  be the solid parallelepiped with the set of vertices

$$\{((-1)^n, (-1)^m, 0) : m, n \in J\} \cup \{((-1)^n, (-1)^m, -1) : m, n \in J\}.$$

There exists a homeomorphism  $g$  of  $C$  onto  $P$  such that  $g[A] = \{(x, 0, 0) : -1 \leq x \leq 1\}$ . There exists a number  $b$ ,  $0 < b < 1$ , such that  $\{(x, y, z) : y^2 + z^2 \leq b^2 \text{ and } (x, y, z) \in P\} \subset g[V(A, \epsilon)]$ . Let  $E$  be  $\{(x, y, z) : y^2 + z^2 = b^2 \text{ and } (x, y, z) \in P\}$ .

Let  $D$  be  $g^{-1}[E]$ ,  $D_1$  be the component of  $BdC - D$  containing  $A$ , and  $D_2$  be  $BdC - Cl(D_1)$ . Notice that each of  $D \cup D_1$  and  $D \cup D_2$  is a 2-sphere which bounds a 3-cell, and  $Cl(\text{int}(D \cup D_1)) \subset V(A, \epsilon)$ .

Now  $BdD$  is a simple closed curve which lies on a tame disk, and therefore  $BdD$  is a tame simple closed curve. It follows from Theorem 7 of [2] that, without loss of generality, it can be assumed that  $D$  is locally polyhedral at each point of  $(\text{int } D)$ . But then  $D$  is tame ([3]). Thus it can be assumed that  $D$  is a tame disk.

Since  $D$  and  $Cl(D_2)$  are tame disks which intersect in the boundary of each,  $D \cup D_2$  is a tame 2-sphere ([3]). Thus there exists a homeomorphism  $f$  of  $E^3$  onto itself such that  $f[Cl(\text{int}(D \cup D_2))] = P$ ,  $f[D] = \{(x, y, 0) : (x, y, 0) \in P\}$ , and  $f[Cl(\text{int}(D \cup D_1)) - D] \subset \{(x, y, z) : z > 0\}$ . Let  $U$  be  $f[V(C, \epsilon)]$  and  $W$  be  $f[V(A, \epsilon)]$ . Since

$$Cl(\text{int}(D \cup D_1)) \subset V(A, \epsilon), f[Cl(\text{int}(D \cup D_1))] \subset W.$$

There exists a positive number  $c$  such that  $Cl(V(P, c)) \subset U$ . Let  $T_0$  be  $Cl(V(P, c))$ . If  $x \in (f[C] - T_0)$ , let  $T_x$  be a polyhedral 3-cell such that  $x \in (\text{int } T_x)$  and  $T_x \subset (W \cap \{(x, y, z) : z > 0\})$ . Then there exists a finite subcollection  $\{T_1, T_2, \dots, T_n\}$  of  $\{T_x : x \in (f[C] - T_0)\}$

such that  $\{T_0, T_1, T_2, \dots, T_n\}$  covers  $f[C]$ . Assuming that  $BdT_0, BdT_1, \dots$ , and  $BdT_n$  are in relative general position, let  $H$  be  $\cup \{T_i: i = 0, 1, 2, \dots, n\}$ .  $H$  is a polyhedral 3-manifold-with-boundary and  $f[C] \subset (\text{int } H) \subset H \subset U$ . Furthermore, since  $(H - \{(x, y, z): z < 0\}) \subset W$  and  $H \cap \{(x, y, z): z \leq 0\}$  is  $Cl(V(P, c)) \cap \{(x, y, z): z \leq 0\}$ , there exists a homeomorphism  $k$  of  $E^3$  onto itself such that if  $x \in (E^3 - U) \cup \{(x, y, z): z \geq 0\}$ ,  $k(x) = x$ , and  $k[H] \subset W$ .

Let  $h_e$  be  $f^{-1}kf$  and  $S$  be  $f^{-1}[H]$ . Then  $h_e$  and  $S$  satisfy the conclusion of Lemma 1.

LEMMA 2. *There exist a sequence  $S_1, S_2, \dots$  of 3-manifolds-with-boundary and a sequence  $h$  of homeomorphisms of  $E^3$  onto itself such that*

- (1)  $S_1 \subset V(C, 1)$ ,
- (2)  $S_{i+1} \subset (\text{int } S_i)$ ,
- (3)  $\cap \{(\text{int } S_j): j \in J\} = C$ ,
- (4)  $\cap \{(\text{int } h_j[S_j]): j \in J\} = A$ , and
- (5) if  $x \in ((\text{int } S_k) - S_{k+1})$ ,  $h_{k+1}(x) = h_k(x)$ .

*Proof of Lemma 2.* Lemma 2 follows immediately by repeated application of Lemma 1.

For each positive integer  $i$ , let  $U_i$  be  $(\text{int } S_i)$  and  $V_i$  be  $h_i[(\text{int } S_i)]$ . Then the sequences  $U, V$ , and  $h$  satisfy the hypothesis of Theorem 1. Thus  $E^3/C$  is homeomorphic to  $E^3/A$ .

COROLLARY 1. *If  $C$  satisfies the hypothesis of Theorem 2, then  $E^3/C \times E^1$  is homeomorphic to  $E^4$ .*

COROLLARY 2. *Let  $C$  be a 3-cell in  $E^3$  such that  $N(C)$  is a 0-dimensional set. Then  $E^3/C \times E^1$  is homeomorphic to  $E^4$ .*

*Proof.*  $N(C)$  is a compact 0-dimensional set on  $BdC$ . Thus there exists an arc  $A$  on  $BdC$  such that  $N(C) \subset A$ . Then the result follows from Corollary 1.

THEOREM 3. *Let  $C$  be a 3-cell in  $E^3$  such that there exists a disk  $D$  on  $BdC$  containing  $N(C)$ . Then  $E^3/C$  is homeomorphic to  $E^3/D$ .*

*Proof.* The proof of Theorem 3 is analogous to the proof of Theorem 2.

## REFERENCES

1. J. J. Andrews and M. L. Curtis, *n-space modulo an arc*, Ann. of Math., **75** (1962), 1-7.
2. R. H. Bing, *Approximating surfaces by polyhedral ones*, Ann. of Math., **65** (1957), 456-483.
3. E. E. Moise, *Affine structures in 3-manifolds VIII. Invariance of the knot-types; local tame imbedding*, Ann. of Math., **59** (1954), 159-170.

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