CENTERS OF PURITY IN ABELIAN GROUPS

RICHARD SCOTT PIERCE
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This note is a supplement to the paper [5]1 of J. D. Reid "On subgroups of an abelian group maximal disjoint from a given subgroup." Our main result is based on the observation that in the case of primary groups, a bit of extra information can be gleaned from Reid's Theorem 2.1. We are led to the following characterization of the "centers of purity" in a p-group.

THEOREM 1. Let G be a p-group. For each integer $k \geq 0$, define $P_k = G[p] \cap p^kG$. Let $P_\infty = G[p] \cap G^1$, and $P_{\infty+1} = P_\infty + 1 = 0$. Let $H$ be a subgroup of $G$. Then $H$ is a center of purity in $G$ (that is, every subgroup of $G$ which is maximal with respect to disjointness from $H$ is pure) if and only if there exists $k$ with $0 \leq k \leq \infty$ such that

$$P_k \supseteq H[p] \supseteq P_{k+2}.$$ 

It is easy to see that if $G$ is a torsion group and $H$ is a subgroup of $G$, then $H$ is a center of purity in $G$ if and only if every $p$-component $H_p$ of $H$ is a center of purity in the corresponding $p$-component $G_p$ of $G$. Thus, Theorem 1 can be used to determine the centers of purity in torsion groups. The following result shows that the centers of purity in arbitrary groups can also be characterized.

THEOREM 2. A subgroup $H$ of an abelian group $G$ is a center of purity in $G$ if and only if the following two conditions are satisfied:

(i) the torsion subgroup $H_t$ of $H$ is a center of purity in the torsion subgroup $G_t$ of $G$;

(ii) either $G/H$ is a torsion group, or else, for all primes $p$,

$$H[p] \subseteq \bigcap_{n=0}^{\infty} p^nG.$$ 

The problem of characterizing centers of purity in $p$-groups was first posed by J. M. Irwin in [2]. Irwin showed that any subgroup of a $p$-group $G$ which is maximal disjoint from $G^1$ is pure in $G$. In [3], Irwin and Walker extended this result to arbitrary abelian groups. They also showed that if $G$ is a torsion group and $H$ is a subgroup

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1 In this issue.
of $G$, then $H$ is a center of purity in $G$. Charles pointed out that the proof given in [1] of Erdélyi's theorem (see p. 81) shows that the subgroups $pG, p^2G, p^3G, \ldots$ of a $p$-group $G$ are centers of purity. Khabbaz (in [4]) showed how the proof of Erdélyi's theorem could be modified to obtain a short proof of Irwin and Walker's result. Finally, Reid established the general sufficient condition 2.1 of [5] for a subgroup $H$ of an arbitrary group $G$ to be a center of purity. It was Reid who introduced the term "center of purity." In the lemma below, we show that Reid's condition is necessary as well as sufficient for $H$ to be a center of purity in a $p$-group $G$. This lemma is then used to prove Theorem 1, from which Theorem 2 follows easily. The author is indebted to Professor Reid for sending him a pre-print of the paper [5]. It was the reading of this paper which inspired the present work.

The notation and terminology of [5] will be used in this paper. In addition, we let $O(x)$ denote the order of the element $x$.

**Lemma.** Let $G$ be a $p$-group, and suppose that $H$ is a subgroup of $G$. Then there is a subgroup $M$ of $G$ such that $M$ is maximal with respect to disjointness from $H$, and $M$ is not pure in $G$, if and only if the following condition is satisfied.

(*) There exists $h \in H$ and $m \in G$ such that

(i) $O(m) > O(h) = p$;
(ii) $h_p(m) = h_p(h) < h_p(m + h)$;
(iii) $\{m\} \cap H = 0$.

**Proof.** Suppose that $M$ is a subgroup of $G$ which is maximal disjoint from $H$ and not pure in $G$. Using the fact that two subgroups of a $p$-group are disjoint if and only if their $p$-layers are disjoint, it is easy to see that $M$ is maximal disjoint from $H[p]$. Therefore (*) is satisfied by Theorem 2.1 of [5]. (It is clear from the proof of 2.1 that $pm \neq 0$, so that $O(m) > O(h)$.)

Assume conversely that the condition (*) is satisfied. Let $r$ be a natural number such that $h_p(m) < r \leq h_p(m + h)$. Define $P_r = p^rG \cap G[p]$. Let $O(m) = p^j$, where, by (i), $j > 1$. Then by (i), $n = p^{j-1}m = p^{j-1}(m + h)$ has height $\geq r + 1$. Thus, $n \in P_r$. However, by (ii), $n \notin H[p]$. Consequently, there is a vector space decomposition

$$P_r = S \oplus (P_r \cap H[p]), \quad n \in S.$$ 

By (i) and (ii), $h \in H[p]$ and $h \notin P_r \cap H[p]$. Therefore, there is a decomposition

$$H[p] = T \oplus (P_r \cap H[p]), \quad h \in T.$$
Clearly,
\[ P_r + H[p] = S \oplus T \oplus (P_r \cap H[p]) . \]

Finally, choose a decomposition
\[ G[p] = R \oplus (P_r + H[p]) . \]

Define
\[ M_0 = R \oplus S . \]

Then we have
\[ G[p] = M_0 \oplus H[p] , \quad \text{with} \ n \in M_0 . \]

Let \( \pi \) be the projection mapping determined by this decomposition:
\[ \pi: G[p] \to H[p] . \]

Note that by the construction, \( \pi(P_r) = P_r \cap H[p] . \) Define
\[ K = \{ M_0 , m \} . \]

It is easy to see that since \( p^{i-1}m = n \in M_0 , \) the \( p \)-layer of \( K \) is \( M_0 . \)
Thus, \( K[p] \cap H[p] = M_0 \cap H[p] = 0 , \) and therefore \( K \cap H = 0 . \) Let \( M \) be maximal containing \( K \) and disjoint from \( H . \) The proof of the lemma is completed by showing that \( h_r(p^m) \leq r . \) Indeed, this will imply that \( M \) is not pure, because
\[ h_r(p^m) = h_r(p(m + h)) \geq h_r(m + h) + 1 \geq r + 1 . \]

Suppose that \( h_r(p^m) \geq r + 1 . \) Then \( z \in M \) exists satisfying
\[ p^{r+1}z = pm . \]

Consequently,
\[ u = p^rz - m \in M \cap G[p] = M \cap (M_0 + H[p]) = M_0 + (M \cap H[p]) = M_0 . \]

Since \( h_r(m + h) \geq r , \) we can write
\[ m + h = p^r y \]
for some \( y \in G . \) Thus,
\[ p^r(y - z) = h - u \in G[p] \cap p^r G = P_r , \]
and therefore since \( u \in M_0 , \)
\[ h = \pi(h - u) \in \pi(P_r) = P_r \cap H[p] \subseteq P_r . \]

However, \( h_r(h) < r \) by the choice of \( r . \) This contradiction shows that \( h_r(p^m) > r \) is impossible, so that the proof of the lemma is complete.
We can now prove Theorem 1. Suppose that $P_k \supseteq H[p] \supseteq P_{k+2}$. If $k = \infty$, there cannot be any $h \in H[p]$ satisfying condition (ii) of the lemma. Suppose therefore that $k$ is finite. Assume that $h \in H$ and $m \in G$ exist satisfying conditions (i), (ii) and (iii) of (*) in the lemma. Let $h_p(h) = j$. Then $k \leq j < h_p(m + h) \leq \infty$. Let $O(m) = p'$, where $f \geq 2$ by (i). Write $x = m + h$. Then $h_p(x) \geq k + 1$. Consequently, $h_p(p^{f-1}x) \geq k + 2$. Therefore, $p^{f-1}m \in P_{k+2} \subseteq H[p]$. This is contrary to (iii). It follows that $H$ is a center of purity in $G$. Conversely, suppose that $P_k \supseteq H[p] \supseteq P_{k+2}$ is not satisfied for any $k$. Then in particular, $H[p] \subsetneq P_\infty$. Since $P_\infty = \bigcap_{k \in \omega} P_k$, it follows that $H[p] \subsetneq P_j$ for some finite $j$. Let $k \geq 0$ be the largest natural number such that $H[p] \subseteq P_k$. The maximality of $k$ and the fact that $P_k \supseteq H[p] \supseteq P_{k+2}$ is false implies that

$$H[p] \not\subseteq P_{k+1}, \quad H[p] \subseteq P_k, \quad \text{and} \quad P_{k+2} \not\subseteq H[p].$$

Therefore, there is an element $h \in H[p]$ such that $h_p(h) = k$, and there exists $u \in P_{k+2}$ such that $u \in H[p]$. Let $u = pv$, where $v \in G$ and $h_p(v) \geq k + 1$. Define $m = v - h$. Then $O(m) = p > p = O(h)$, $h_p(m) = k$, $h_p(m + h) = h_p(v) \geq k + 1$, and $\{m\} \cap H = 0$, since $pm = pv = u \in H[p]$. It follows from the lemma that $H$ is not a center of purity in $G$. The proof of Theorem 1 is therefore complete.

Theorem 2 is obtained with the help of Theorem 1, by refining the proof of Lemmas 3.5 and 3.7 in [5]. Suppose that $G/H$ is a torsion group, and $H_t$ is a center of purity in $G_t$. If $M$ is maximal disjoint from $H$, then $M \subseteq G_t$, and a short calculation shows that $M$ is maximal disjoint from $H_t$ in $G_t$. Therefore $M$ is pure in $G_t$, and hence also in $G$. Next, suppose that $H[p] \subseteq \bigcap_{n=0}^\infty p^nG$ for all primes $p$. If $H$ is not a center of purity in $G$, then by Theorem 2.1 in [5], there exists $h \in H_t$ such that $h_p(h) < \infty$. Let $O(h) = p'$. Using the same argument that was given in the last paragraph of the proof of 2.1 in [5], we can show that $h_p(p^{f-1}h) < \infty$. This contradiction proves that $H$ must be a center of purity. Suppose conversely that $H$ is a center of purity in $G$. It is a routine exercise to show that $H_t$ is a center of purity in $G_t$. Assume that $G/H$ is not a torsion group and for some prime $p$, $H[p] \not\subseteq \bigcap_{n=0}^\infty p^nG$. Let $k$ be the largest integer such that $p^kG \supseteq H[p]$. Then by Theorem 1

$$p^kG \cap G[p] \supseteq H[p] \supseteq p^{k+2}G \cap G[p], \quad p^{k+2}G \cap G[p] \not\supseteq H[p].$$

Let $t \in H[p]$ satisfy $h_p(t) = k$. Since $G/H$ is not a torsion group, an element $x \in G$ exists satisfying $O(x) = \infty$ and $\{x\} \cap H = 0$. Consequently, $(p^{k+2}x + t) \cap H = 0$. Let $M$ be maximal disjoint from $H$, with $p^{k+2}x + t \in M$. Then $p^{k+2}x = p(p^{k+2}x + t) \in M$. Since $H$ is a center of purity, $M$ is pure. Consequently, $m \in M$ exists satisfying $p^{k+2}m =
Thus, $p^{k+3}x \in p^{k+2}G \cap G[p] \subseteq H[p]$. Therefore,

$$p^{k+2}x + t - p^{k+2}m = p^{k+2}(x - m) + t \in H \cap M = 0,$$

so that $h_p(t) \geq k + 2$. However, $h_p(t) = k$ by choice. The contradiction shows that the condition (ii) must hold.

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