

# Pacific Journal of Mathematics

**CENTERS OF PURITY IN ABELIAN GROUPS**

RICHARD SCOTT PIERCE

# CENTERS OF PURITY IN ABELIAN GROUPS

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This note is a supplement to the paper [5]<sup>1</sup> of J. D. Reid "On subgroups of an abelian group maximal disjoint from a given subgroup." Our main result is based on the observation that in the case of primary groups, a bit of extra information can be gleaned from Reid's Theorem 2.1. We are led to the following characterization of the "centers of purity" in a  $p$ -group.

**THEOREM 1.** *Let  $G$  be a  $p$ -group. For each integer  $k \geq 0$ , define  $P_k = G[p] \cap p^k G$ . Let  $P_\infty = G[p] \cap G^1$ , and  $P_{\infty+1} = P_{\infty+2} = 0$ . Let  $H$  be a subgroup of  $G$ . Then  $H$  is a center of purity in  $G$  (that is, every subgroup of  $G$  which is maximal with respect to disjointness from  $H$  is pure) if and only if there exists  $k$  with  $0 \leq k \leq \infty$  such that*

$$P_k \supseteq H[p] \supseteq P_{k+2}.$$

It is easy to see that if  $G$  is a torsion group and  $H$  is a subgroup of  $G$ , then  $H$  is a center of purity in  $G$  if and only if every  $p$ -component  $H_p$  of  $H$  is a center of purity in the corresponding  $p$ -component  $G_p$  of  $G$ . Thus, Theorem 1 can be used to determine the centers of purity in torsion groups. The following result shows that the centers of purity in arbitrary groups can also be characterized.

**THEOREM 2.** *A subgroup  $H$  of an abelian group  $G$  is a center of purity in  $G$  if and only if the following two conditions are satisfied:*

- (i) *the torsion subgroup  $H_t$  of  $H$  is a center of purity in the torsion subgroup  $G_t$  of  $G$ ;*
- (ii) *either  $G/H$  is a torsion group, or else, for all primes  $p$ ,*

$$H[p] \subseteq \bigcap_{n=0}^{\infty} p^n G.$$

The problem of characterizing centers of purity in  $p$ -groups was first posed by J. M. Irwin in [2]. Irwin showed that any subgroup of a  $p$ -group  $G$  which is maximal disjoint from  $G^1$  is pure in  $G$ . In [3], Irwin and Walker extended this result to arbitrary abelian groups. They also showed that if  $G$  is a torsion group and  $H$  is a subgroup

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<sup>1</sup> In this issue.

of  $G^1$ , then  $H$  is a center of purity in  $G$ . Charles pointed out that the proof given in [1] of Erdélyi's theorem (see p. 81) shows that the subgroups  $pG, p^2G, p^3G, \dots$  of a  $p$ -group  $G$  are centers of purity. Khabbaz (in [4]) showed how the proof of Erdélyi's theorem could be modified to obtain a short proof of Irwin and Walker's result. Finally, Reid established the general sufficient condition 2.1 of [5] for a subgroup  $H$  of an arbitrary group  $G$  to be a center of purity. It was Reid who introduced the term "center of purity." In the lemma below, we show that Reid's condition is necessary as well as sufficient for  $H$  to be a center of purity in a  $p$ -group  $G$ . This lemma is then used to prove Theorem 1, from which Theorem 2 follows easily. The author is indebted to Professor Reid for sending him a pre-print of the paper [5]. It was the reading of this paper which inspired the present work.

The notation and terminology of [5] will be used in this paper. In addition, we let  $O(x)$  denote the order of the element  $x$ .

LEMMA. *Let  $G$  be a  $p$ -group, and suppose that  $H$  is a subgroup of  $G$ . Then there is a subgroup  $M$  of  $G$  such that  $M$  is maximal with respect to disjointness from  $H$ , and  $M$  is not pure in  $G$ , if and only if the following condition is satisfied.*

- (\*) *There exists  $h \in H$  and  $m \in G$  such that*
  - (i)  $O(m) > O(h) = p$ ;
  - (ii)  $h_p(m) = h_p(h) < h_p(m + h)$ ;
  - (iii)  $\{m\} \cap H = 0$ .

*Proof.* Suppose that  $M$  is a subgroup of  $G$  which is maximal disjoint from  $H$  and not pure in  $G$ . Using the fact that two subgroups of a  $p$ -group are disjoint if and only if their  $p$ -layers are disjoint, it is easy to see that  $M$  is maximal disjoint from  $H[p]$ . Therefore (\*) is satisfied by Theorem 2.1 of [5]. (It is clear from the proof of 2.1 that  $pm \neq 0$ , so that  $O(m) > O(h)$ .)

Assume conversely that the condition (\*) is satisfied. Let  $r$  be a natural number such that  $h_p(m) < r \leq h_p(m + h)$ . Define  $P_r = p^r G \cap G[p]$ . Let  $O(m) = p^j$ , where, by (i),  $j > 1$ . Then by (i),  $n = p^{j-1}m = p^{j-1}(m + h)$  has height  $\geq r + 1$ . Thus,  $n \in P_r$ . However, by (iii),  $n \notin H[p]$ . Consequently, there is a vector space decomposition

$$P_r = S \oplus (P_r \cap H[p]), \quad n \in S.$$

By (i) and (ii),  $h \in H[p]$  and  $h \notin P_r \cap H[p]$ . Therefore, there is a decomposition

$$H[p] = T \oplus (P_r \cap H[p]), \quad h \in T.$$

Clearly,

$$P_r + H[p] = S \oplus T \oplus (P_r \cap H[p]) .$$

Finally, choose a decomposition

$$G[p] = R \oplus (P_r + H[p]) .$$

Define

$$M_0 = R \oplus S .$$

Then we have

$$G[p] = M_0 \oplus H[p] , \quad \text{with } n \in M_0 .$$

Let  $\pi$  be the projection mapping determined by this decomposition:

$$\pi: G[p] \rightarrow H[p] .$$

Note that by the construction,  $\pi(P_r) = P_r \cap H[p]$ . Define

$$K = \{M_0, m\} .$$

It is easy to see that since  $p^{j-1}m = n \in M_0$ , the  $p$ -layer of  $K$  is  $M_0$ . Thus,  $K[p] \cap H[p] = M_0 \cap H[p] = 0$ , and therefore  $K \cap H = 0$ . Let  $M$  be maximal containing  $K$  and disjoint from  $H$ . The proof of the lemma is completed by showing that  $h_p^M(pm) \leq r$ . Indeed, this will imply that  $M$  is not pure, because

$$h_p(pm) = h_p(p(m+h)) \geq h_p(m+h) + 1 \geq r + 1 .$$

Suppose that  $h_p^M(pm) \geq r + 1$ . Then  $z \in M$  exists satisfying

$$p^{r+1}z = pm .$$

Consequently,

$$u = p^r z - m \in M \cap G[p] = M \cap (M_0 + H[p]) = M_0 + (M \cap H[p]) = M_0 .$$

Since  $h_p(m+h) \geq r$ , we can write

$$m+h = p^r y$$

for some  $y \in G$ . Thus,

$$p^r(y-z) = h - u \in G[p] \cap p^r G = P_r ,$$

and therefore since  $u \in M_0$ ,

$$h = \pi(h-u) \in \pi(P_r) = P_r \cap H[p] \subseteq P_r .$$

However,  $h_p(h) < r$  by the choice of  $r$ . This contradiction shows that  $h_p^M(pm) > r$  is impossible, so that the proof of the lemma is complete.

We can now prove Theorem 1. Suppose that  $P_k \supseteq H[p] \supseteq P_{k+2}$ . If  $k = \infty$ , there cannot be any  $h \in H[p]$  satisfying condition (ii) of the lemma. Suppose therefore that  $k$  is finite. Assume that  $h \in H$  and  $m \in G$  exist satisfying conditions (i), (ii) and (iii) of (\*) in the lemma. Let  $h_p(h) = j$ . Then  $k \leq j < h_p(m+h) \leq \infty$ . Let  $O(m) = p^f$ , where  $f \geq 2$  by (i). Write  $x = m+h$ . Then  $h_p(x) \geq k+1$ . Consequently,  $h_p(p^{f-1}x) \geq k+2$ . Therefore,  $p^{f-1}m \in P_{k+2} \subseteq H[p]$ . This is contrary to (iii). It follows that  $H$  is a center of purity in  $G$ . Conversely, suppose that  $P_k \supseteq H[p] \supseteq P_{k+2}$  is not satisfied for any  $k$ . Then in particular,  $H[p] \not\subseteq P_\infty$ . Since  $P_\infty = \bigcap_{k < \omega} P_k$ , it follows that  $H[p] \not\subseteq P_j$  for some finite  $j$ . Let  $k \geq 0$  be the largest natural number such that  $H[p] \subseteq P_k$ . The maximality of  $k$  and the fact that  $P_k \supseteq H[p] \supseteq P_{k+2}$  is false implies that

$$H[p] \not\subseteq P_{k+1}, \quad H[p] \subseteq P_k, \quad \text{and} \quad P_{k+2} \not\subseteq H[p].$$

Therefore, there is an element  $h \in H[p]$  such that  $h_p(h) = k$ , and there exists  $u \in P_{k+2}$  such that  $u \notin H[p]$ . Let  $u = pv$ , where  $v \in G$  and  $h_p(v) \geq k+1$ . Define  $m = v-h$ . Then  $O(m) = p^2 > p = O(h)$ ,  $h_p(m) = k$ ,  $h_p(m+h) = h_p(v) \geq k+1$ , and  $\{m\} \cap H = 0$ , since  $pm = pv = u \notin H[p]$ . It follows from the lemma that  $H$  is not a center of purity in  $G$ . The proof of Theorem 1 is therefore complete.

Theorem 2 is obtained with the help of Theorem 1, by refining the proof of Lemmas 3.5 and 3.7 in [5]. Suppose that  $G/H$  is a torsion group, and  $H_i$  is a center of purity in  $G_i$ . If  $M$  is maximal disjoint from  $H$ , then  $M \subseteq G_i$ , and a short calculation shows that  $M$  is maximal disjoint from  $H_i$  in  $G_i$ . Therefore  $M$  is pure in  $G_i$ , and hence also in  $G$ . Next, suppose that  $H[p] \subseteq \bigcap_{n=0}^{\infty} p^n G$  for all primes  $p$ . If  $H$  is not a center of purity in  $G$ , then by Theorem 2.1 in [5], there exists  $h \in H_p$  such that  $h_p(h) < \infty$ . Let  $O(h) = p^r$ . Using the same argument that was given in the last paragraph of the proof of 2.1 in [5], we can show that  $h_p(p^{r-1}h) < \infty$ . This contradiction proves that  $H$  must be a center of purity. Suppose conversely that  $H$  is a center of purity in  $G$ . It is a routine exercise to show that  $H_i$  is a center of purity in  $G_i$ . Assume that  $G/H$  is not a torsion group and for some prime  $p$ ,  $H[p] \not\subseteq \bigcap_{n=0}^{\infty} p^n G$ . Let  $k$  be the largest integer such that  $p^k G \supseteq H[p]$ . Then by Theorem 1

$$p^k G \cap G[p] \supseteq H[p] \supseteq p^{k+2} G \cap G[p], \quad p^{k+1} G \cap G[p] \not\supseteq H[p].$$

Let  $t \in H[p]$  satisfy  $h_p(t) = k$ . Since  $G/H$  is not a torsion group, an element  $x \in G$  exists satisfying  $O(x) = \infty$  and  $\{x\} \cap H = 0$ . Consequently  $\{p^{k+2}x + t\} \cap H = 0$ . Let  $M$  be maximal disjoint from  $H$ , with  $p^{k+2}x + t \in M$ . Then  $p^{k+3}x = p(p^{k+2}x + t) \in M$ . Since  $H$  is a center of purity,  $M$  is pure. Consequently,  $m \in M$  exists satisfying  $p^{k+3}m =$

$p^{k+3}x$ . Thus,  $p^{k+2}(x - m) \in p^{k+2}G \cap G[p] \cong H[p]$ . Therefore,

$$p^{k+2}x + t - p^{k+2}m = p^{k+2}(x - m) + t \in H \cap M = 0,$$

so that  $h_p(t) \cong k + 2$ . However,  $h_p(t) = k$  by choice. The contradiction shows that the condition (ii) must hold.

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