

Pacific Journal of Mathematics

ON THE NÖRLUND SUMMABILITY OF FOURIER SERIES

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1. Let $f(x)$ be a function integrable $-L$ over the interval $(-\pi, \pi)$ and periodic with period 2π , outside this interval.

Let

$$(1.1) \quad \phi(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2s(x)\},$$

and

$$(1.2) \quad \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier Series of the function $\phi(t)$.

Nörlund Summability of Fourier Series (1.2) has been considered by Woronoi [6] and later on by Nörlund [4]. These results have been extended by Hille and Tamarkin [2], [3], and later on by Astrachan [1]. Recently, extending a result due to Hille and Tamarkin [3], Varshney [5] has proved the following:

THEOREM. V. *If the sequence $\{p_n\}$ satisfies the following conditions:*

$$(1.3) \quad \frac{n |pn|}{\log n} < c |P_n|,$$

$$(1.4) \quad \sum_{k=0}^n \frac{k |p_k - p_{k-1}|}{\log(k+1)} < c |P_n|$$

and

$$(1.5) \quad \sum_{k=0}^n \frac{P_k}{k \log(k+1)} < c |P_n|$$

and also if

$$(1.6) \quad \bar{\Phi}_1(t) = \int_0^t |\phi(u)| du = o\left(t \log \frac{1}{t}\right)$$

then the Fourier Series (1.2) associated with the function $\phi(t)$ is summable by Nörlund means i.e. summable (N, p_n) to the sum zero at the point $t = x$.

The object here is to prove the following:

THEOREM. *If the sequence $\{p_n\}$ satisfies the following conditions*

Received January 3, 1961.

$$(1.7) \quad \frac{n |p_n|}{(\log n)^r} < c |P_n|$$

and

$$(1.8) \quad \sum_{k=0}^n \frac{k |p_k - p_{k-1}|}{\{\log(k+1)\}^r} < c |P_n|$$

and also if

$$(1.9) \quad \bar{\Phi}_1(t) = \int_0^t |\phi(u)| du = o\left\{t / \left(\log \frac{1}{t}\right)^r\right\}$$

and

$$(1.10) \quad \frac{1}{P(n)} \int_{\pi/n}^{\delta} \frac{|\phi(t + \pi/n) - \phi(t)|}{t} P\left(\frac{1}{t}\right) dt = o(1)$$

then the Fourier Series (1.2) associated with the function $\phi(t)$ is summable by Nörlund means i.e. summable (N, p_n) to the sum zero at the point $t = x$ for all $0 \leq r \leq 1$.

2. The following notations will be used in the sequel.

We write $S_n(x)$ as the n th partial sum of the series (1.2) and the Nörlund transform of the partial sum of the series (1.2) we denote by $\sigma_n(x)$.

Also we write ; where $P_n \equiv P(n)$,

$$(2.1) \quad N_n(t) = \frac{1}{\pi P_n} \sum_{k=0}^n p_{n-k} \frac{\text{Sin}(k + 1/2)t}{t} .$$

We recall that the conditions of regularity of the method of summation are

$$(2.2) \quad \sum_{k=0}^n r_k = \sum_{k=0}^n |p_k| < c |P_n|$$

and

$$(2.3) \quad \{p_n/P_n\} \rightarrow 0 \quad \text{as } n \rightarrow \infty .$$

3. If we write

$$(3.1) \quad S_n(x) = \frac{1}{\pi} \int_0^\pi \phi(t) \frac{\text{Sin}(n + 1/2)t}{t} dt$$

then we have

$$\begin{aligned}
 \sigma_n(x) &= \frac{1}{\pi P(n)} \int_0^\pi \phi(t) \left(\sum_{k=0}^n p_{n-k} \frac{\text{Sin}(k + 1/2)t}{t} \right) dt \\
 &= \int_0^\pi \phi(t) N_n(t) dt \\
 &= \left(\int_0^{\pi/n} + \int_{\pi/n}^\delta + \int_\delta^\pi \right) \phi(t) N_n(t) dt \\
 (3.2) \quad &= I_1 + I_2 + I_3, \quad \text{say}
 \end{aligned}$$

where δ is fixed.

Hence

$$\begin{aligned}
 I_2 &= \frac{1}{\pi P(n)} \int_{\pi/n}^\delta \frac{\phi(t)}{t} \sum_{k=0}^n (p_k \text{Sin}(n - k + \frac{1}{2})t) dt \\
 &= \frac{1}{\pi P(n)} \int_{\pi/n}^\delta \frac{\phi(t)}{t} \sum_{k=0}^n p_k \{ \text{Sin}(n + \frac{1}{2})t \cdot \cos kt - \cos(n + \frac{1}{2})t \cdot \text{Sin} kt \} dt \\
 (3.3) \quad &= I_{2,1} - I_{2,2} \quad \text{say.}
 \end{aligned}$$

Now, if we write

$$\begin{aligned}
 I_{2,1} &= \frac{1}{\pi P(n)} \int_{\pi/n}^\delta \frac{\phi(t)}{t} \text{Sin}(n + \frac{1}{2})t \cdot \left\{ \sum_{kt \leq 1} + \sum_{kt > 1} \right\} p_k \cos kt \cdot dt \\
 &= \frac{1}{\pi P(n)} \int_{\pi/n}^\delta \frac{\phi(t)}{t} \text{Sin}(n + \frac{1}{2})t \sum_{kt \leq 1} p_k \cos kt dt \\
 (3.4) \quad &+ \frac{1}{\pi P(n)} \int_{\pi/n}^\delta \frac{\phi(t)}{t} \cdot \text{Sin}(n + \frac{1}{2})t \sum_{kt > 1} p_k \cos kt \cdot dt \\
 &= \frac{1}{\pi P(n)} \int_{\pi/n}^\delta \frac{\phi(t)}{t} \text{Sin}(n + \frac{1}{2})t \Sigma_1 dt \\
 &+ \frac{1}{\pi P(n)} \int_{\pi/n}^\delta \frac{\phi(t)}{t} \text{Sin}(n + \frac{1}{2})t \Sigma_2 dt \\
 (3.5) \quad &= I_{2,1,1} + I_{2,1,2}, \quad \text{say.}
 \end{aligned}$$

4. We shall require the following lemma.

LEMMA. *If we write*

$$(4.1) \quad |p_n| = r_n, \quad R_n = r_0 + r_1 + r_2 + \dots + r_n$$

and

$$(4.2) \quad r(u) = r_{[u]}, \quad R(u) = R_{[u]}$$

where $[u]$ denotes the integer (largest) $\leq u$, and

$$(4.3) \quad V_0 \equiv 0, \quad V_n \equiv \sum_{k=0}^n |p_k - p_{k-1}|, \quad V_u \equiv V_{[u]}$$

then we have, from (3.4) ,

$$(4.4) \quad \Sigma_1 > P_t \cos 1 > \frac{1}{2} P\left(\frac{1}{t}\right)$$

and

$$(4.5) \quad |\Sigma_2| = \frac{A}{t} \left\{ r\left(\frac{1}{t}\right) + r(n) + V(n) - V\left(\frac{1}{t} - 1\right) \right\}.$$

This is known Hille and Tamarkin [3].

5. Now we shall prove the theorem.

Proof. Since

$$\begin{aligned} I_1 &= \frac{1}{\pi} \int_0^{\pi/n} \phi(t) N_n(t) dt \\ &= \frac{1}{\pi} \int_0^{\pi/n} |\phi(t)| O(n) dt \\ &= O(n) [\bar{\Phi}_1(t)]_0^{\pi/n} \quad \text{by (1.9)} \\ &= O(n) \left[0 \left\{ t / \left(\log \frac{1}{t} \right)^r \right\} \right]_0^{\pi/n} \\ (5.1) \quad &= O(1), \quad \text{as } n \rightarrow \infty. \end{aligned}$$

From (3.5) and Lemma, above, we have

$$\begin{aligned} I_{2,1,1} &= \frac{1}{\pi P(n)} \int_{\pi/n}^{\delta} \frac{\phi(t)}{t} \text{Sin} \left(n + \frac{1}{2} \right) t P\left(\frac{1}{t}\right) dt \\ &= \frac{1}{\pi P(n)} \int_{\pi/n}^{\delta} \frac{\phi(t)}{t} \text{Sin } nt P\left(\frac{1}{t}\right) dt + o(1), \end{aligned}$$

by the regularity of the method of summation.

$$\begin{aligned} &= -\frac{1}{\pi P(n)} \int_0^{\delta - \pi/n} \frac{\phi(t + \pi/n)}{(t + \pi/n)} P\left(\frac{1}{t + \pi/n}\right) \text{Sin } nt dt + o(1) \\ &= \frac{1}{2\pi P(n)} \int_{\pi/n}^{\delta} \frac{\phi(t)}{t} \text{Sin } nt P\left(\frac{1}{t}\right) dt - \frac{1}{2\pi P(n)} \int_0^{\delta - \pi/n} \frac{\phi(t + \pi/n)}{t + \pi/n} \text{Sin } nt \\ &\quad \cdot P\left(\frac{1}{t + \pi/n}\right) dt + o(1) \\ &= \frac{1}{2\pi P(n)} \int_{\pi/n}^{\delta - \pi/n} \frac{\phi(t)}{t} \text{Sin } nt P\left(\frac{1}{t}\right) dt + \frac{1}{2\pi P(n)} \int_{\delta - \pi/n}^{\delta} \frac{\phi(t)}{t} \text{Sin } nt \\ &\quad \cdot P\left(\frac{1}{t}\right) dt - \frac{1}{2\pi P(n)} \int_0^{\pi/n} \frac{\phi(t + \pi/n)}{(t + \pi/n)} \text{Sin } nt P\left(\frac{1}{t + \pi/n}\right) dt \\ &\quad - \frac{1}{2\pi P(n)} \int_{\pi/n}^{\delta - \pi/n} \frac{\phi(t + \pi/n)}{(t + \pi/n)} \text{Sin } nt P\left(\frac{1}{t + \pi/n}\right) dt + o(1) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2\pi P(n)} \int_{\pi/n}^{\delta-\pi/n} \left\{ \frac{\phi(t)}{t} P\left(\frac{1}{t}\right) - \frac{\phi(t+\pi/n)}{t+\pi/n} P\left(\frac{1}{t+\pi/n}\right) \right\} \text{Sin } nt \, dt \\
 &\quad - \frac{1}{2\pi P(n)} \int_0^{\pi/n} \frac{\phi(t+\pi/n)}{t+\pi/n} \text{Sin } nt P\left(\frac{1}{t+\pi/n}\right) dt \\
 &\quad + \frac{1}{2\pi P(n)} \int_{\delta-\pi/n}^{\delta} \frac{\phi(t)}{t} P\left(\frac{1}{t}\right) \text{Sin } nt \, dt + o(1) \\
 &= \frac{1}{2\pi P(n)} \int_{\pi/n}^{\delta-\pi/n} \left[\left\{ \frac{\phi(t)}{t} P\left(\frac{1}{t}\right) - \frac{\phi(t+\pi/n)}{t} P\left(\frac{1}{t}\right) \right\} \right. \\
 &\quad + \left. \left\{ \frac{\phi(t+\pi/n)}{t} P\left(\frac{1}{t}\right) - \frac{\phi(t+\pi/n)}{t} P\left(\frac{1}{t+\pi/n}\right) \right\} \right. \\
 &\quad + \left. \left\{ \frac{\phi(t+\pi/n)}{t} P\left(\frac{1}{t+\pi/n}\right) - \frac{\phi(t+\pi/n)}{t+\pi/n} P\left(\frac{1}{t+\pi/n}\right) \right\} \right] \text{Sin } nt \, dt \\
 &\quad - \frac{1}{2\pi P(n)} \int_0^{\pi/n} \frac{\phi(t+\pi/n)}{t+\pi/n} \text{Sin } nt P\left(\frac{1}{t+\pi/n}\right) dt \\
 &\quad + \frac{1}{2\pi P(n)} \int_{\delta-\pi/n}^{\delta} \frac{\phi(t)}{t} \text{Sin } nt P\left(\frac{1}{t}\right) dt + o(1) \\
 &= \frac{1}{2\pi P(n)} \left(\int_{\pi/n}^{\delta-\pi/n} \left\{ \frac{[\phi(t) - \phi(t+\pi/n)]}{t} P\left(\frac{1}{t}\right) dt \right. \right. \\
 &\quad + \left. \left[P\left(\frac{1}{t}\right) - P\left(\frac{1}{t+\pi/n}\right) \right] \frac{\phi(t+\pi/n)}{t} \right. \\
 &\quad + \left. \left. P\left(\frac{1}{t+\pi/n}\right) \phi(t+\pi/n) \frac{\pi/n}{t(t+\pi/n)} \right\} \text{Sin } nt \, dt \right. \\
 &\quad - \int_0^{\pi/n} \frac{\phi(t+\pi/n)}{t+\pi/n} \text{Sin } nt P\left(\frac{1}{t+\pi/n}\right) dt \\
 &\quad + \int_{\delta-\pi/n}^{\delta} \frac{\phi(t)}{t} \text{Sin } nt P\left(\frac{1}{t}\right) dt \Big) \\
 &\quad + o(1)
 \end{aligned}$$

(5.2) = $(P_1 + P_2 + P_3) + P_4 + P_5$, say.

By virtue of (1.10) we have

(5.3) $P_1 = o(1)$, as $n \rightarrow \infty$.

Also

$$P_2 = \frac{1}{2\pi P(n)} \int_{\pi/n}^{\delta-\pi/n} \frac{\phi(t+\pi/n)}{t} \left\{ P\left(\frac{1}{t}\right) - P\left(\frac{1}{t+\pi/n}\right) \right\} \text{Sin } nt \, dt .$$

Since, for all $0 < 1/\alpha < 1/\beta$, we have

$$P\left(\frac{1}{\beta}\right) - P\left(\frac{1}{\alpha}\right) = \int_{1/\alpha}^{1/\beta} p(\beta) ds + O\left\{ p\left(\frac{1}{\alpha}\right) + p\left(\frac{1}{\beta}\right) \right\} .$$

Hence

$$\begin{aligned}
 P_2 &= O\left(\frac{1}{P(n)}\right)\left\{\int_{\pi/n}^{\delta} |\phi(t + \pi/n)| \frac{dt}{t} \int_{1/(t+\pi/n)}^{1/t} r(s) ds\right\} \\
 &\quad + O\left(\frac{1}{P(n)}\right) \int_{\pi/n}^{\delta} |\phi(t + \pi/n)| r\left(\frac{1}{t}\right) \frac{dt}{t} \\
 &\quad + O\left(\frac{1}{P(n)}\right) \int_{\pi/n}^{\delta} |\phi(t + \pi/n)| r\left(\frac{1}{t + \pi/n}\right) \frac{dt}{t} \\
 (5.4) \quad &= P_{2,1} + P_{2,2} + P_{2,3} \quad \text{say.}
 \end{aligned}$$

Now

$$\begin{aligned}
 P_{2,1} &= O\left(\frac{1}{P(n)}\right)\left\{\bar{\Phi}_1(t + \pi/n) \frac{1}{t} \int_{1/(t+\pi/n)}^{1/t} r(s) ds\right\}_{\pi/n}^{\delta} \\
 &\quad + O\left(\frac{1}{P(n)}\right) \int_{\pi/n}^{\delta} \bar{\Phi}_1(t + \pi/n) \frac{dt}{t^2} \int_{1/(t+\pi/n)}^{1/t} r(s) ds \\
 &\quad + O\left(\frac{1}{P(n)}\right) \int_{\pi/n}^{\delta} \bar{\Phi}_1(t + \pi/n) \frac{1}{t} r\left(\frac{1}{t}\right) \frac{dt}{t^2} \\
 (5.5) \quad &= P_{2,1,1} + P_{2,1,2} + P_{2,1,3}, \quad \text{say.}
 \end{aligned}$$

We have

$$\begin{aligned}
 P_{2,1,1} &= O\left(\frac{1}{P(n)}\right)\left\{o\left(\frac{1}{(\log 1/t)^r}\right) \int_{1/(t+\pi/n)}^{1/t} r(s) ds\right\}_{\pi/n}^{\delta} \quad \text{by (1.9)} \\
 &= o\left(\frac{1}{P(n)}\right)\left\{\frac{1}{(\log n)^r} \int_{n/2\pi}^{n/\pi} r(s) ds\right\} \\
 &\quad + o\left(\frac{1}{P(n)}\right)\left\{\frac{1}{(\log 1/\delta)^r} \int_{1/(\delta+\pi/n)}^{1/\delta} r(s) ds\right\} \\
 (5.6) \quad &= o(1), \quad \text{as } n \rightarrow \infty, \quad \text{by (1.8).}
 \end{aligned}$$

And

$$\begin{aligned}
 P_{2,1,2} &= O\left(\frac{1}{P(n)}\right) \int_{\pi/n}^{\delta} o\left(\frac{t}{(\log 1/t)^r}\right) \frac{dt}{t^2} \int_{1/(t+\pi/n)}^{1/t} r(s) ds \quad (1.9) \\
 &= o\left(\frac{1}{P(n)}\right) \int_{\pi/n}^{\delta} \frac{dt}{t(\log 1/t)^r} \int_{1/(t+\pi/n)}^{1/t} r(s) ds \\
 &= o\left(\frac{1}{P(n)}\right) \int_{1/(\delta+\pi/n)}^{n/\pi} r(s) ds \int_{(1/s)-\pi/n}^{1/s} \frac{dt}{t(\log 1/t)^r} + o(1),
 \end{aligned}$$

by change of order of integration.

$$\begin{aligned}
 &= o\left(\frac{1}{P(n)}\right) \int_{1/(\delta+\pi/n)}^{n/\pi} \frac{r(s) ds}{(\log 1/s)^r} \int_{(1/s)-\pi/n}^{1/s} \frac{dt}{t} + o(1) \\
 &= o\left(\frac{1}{P(n)}\right) \int_{1/(\delta+\pi/n)}^{n/\pi} \frac{r(s) ds}{(\log 1/s)^r} (s\pi/n) + o(1)
 \end{aligned}$$

$$\begin{aligned}
 &= o\left(\frac{1}{P(n)}\right) \sum_{k=0}^n \frac{r(k)}{(\log(k+1))^r} + o(1) \\
 (5.7) \quad &= o(1), \text{ as } n \rightarrow \infty, \text{ by (1.8).}
 \end{aligned}$$

Finally, considering $P_{2.1,3}$, we have

$$\begin{aligned}
 P_{2.1,3} &= O\left(\frac{1}{P(n)}\right) \int_{\pi/n}^{\delta} o\left(\frac{t}{(\log 1/t)^r}\right) \frac{dt}{t} r\left(\frac{1}{t}\right) \frac{1}{t^2}, \text{ by (1.9)} \\
 &= o\left(\frac{1}{P(n)}\right) \int_{\pi/n}^{\delta} \frac{r(1/t)}{(\log 1/t)^r t^2} dt \\
 &= o\left(\frac{1}{P(n)}\right) \int_{n/\pi}^{1/\delta} \frac{r(s)}{(\log s)^r} ds \\
 (5.8) \quad &= o\left(\frac{1}{P(n)}\right) \sum_{k=0}^n \frac{r(k)}{\{\log(k+1)\}^r} \\
 &= o(1) \text{ as } n \rightarrow \infty, \text{ by (1.8).}
 \end{aligned}$$

Thus from (5.5), (5.6), (5.7) and (5.8) we see that

$$(5.9) \quad P_{2,1} = o(1) \text{ as } n \rightarrow \infty.$$

Estimating $P_{2,2}$ we find that

$$\begin{aligned}
 P_{2,2} &= O\left(\frac{1}{P(n)}\right) \int_{\pi/n}^{\delta} |\phi(t + \pi/n)| r\left(\frac{1}{t}\right) dt \\
 &= O\left(\frac{1}{P(n)}\right) \left\{ \left[\bar{\phi}_1(t) r\left(\frac{1}{t}\right) \frac{1}{t} \right]_{\pi/n}^{\delta} + \int_{\pi/n}^{\delta} \bar{\phi}_1(t) r\left(\frac{1}{t}\right) \frac{dt}{t^2} \right. \\
 &\quad \left. - \int_{\pi/n}^{\delta} \bar{\phi}_1(t) \frac{1}{t} dr\left(\frac{1}{t}\right) \right\} + o(1).
 \end{aligned}$$

Here, the integrated part is $o(1)$, by virtue of (1.7) and the fact that $P(n) \rightarrow \infty$ as $n \rightarrow \infty$. The second part is

$$\begin{aligned}
 &o\left(\frac{1}{P(n)}\right) \int_{\pi/n}^{\delta} o\left(\frac{1}{(\log 1/t)^r}\right) r\left(\frac{1}{t}\right) \frac{dt}{t}, \text{ by (1.9)} \\
 &= o\left(\frac{1}{P(n)}\right) \int_{n/\pi}^{1/\delta} \frac{r(s)}{s(\log s)^r} ds \\
 &= o(1), \text{ by (1.8).}
 \end{aligned}$$

The third term is

$$\begin{aligned}
 &O\left(\frac{1}{P(n)}\right) \int_{\pi/n}^{\delta} o\left(\frac{t}{(\log 1/t)^r}\right) \frac{1}{t} dr\left(\frac{1}{t}\right) \\
 &= o\left(\frac{1}{P(n)}\right) \sum_{k=0}^n \frac{|r_k - r_{k-1}|}{(\log(k+1))^r} \\
 &= o(1) \text{ as } n \rightarrow \infty, \text{ by (1.8).}
 \end{aligned}$$

Thus we see that

$$(5.10) \quad P_{2,2} = o(1) \quad \text{as } n \rightarrow \infty.$$

Similarly, we can show that

$$(5.11) \quad P_{2,3} = o(1) \quad \text{as } n \rightarrow \infty.$$

Hence by (5.4), (5.9), (5.10) and (5.11) we get

$$(5.12) \quad P_2 = o(1) \quad \text{as } n \rightarrow \infty.$$

Evaluating P_3 we have

$$\begin{aligned} P_3 &= O\left(\frac{\pi}{nP(n)}\right) \int_{\pi/n}^{\delta} |\phi(t + \pi/n)| P\left(\frac{1}{t + \pi/n}\right) \frac{dt}{t(t + \pi/n)} \\ &= O\left(\frac{1}{n}\right) \int_{\pi/n}^{\delta} |\phi(t + \pi/n)| \frac{dt}{t^2} \\ &= O\left(\frac{1}{n}\right) \left[\left\{ \bar{\Phi}_1(t + \pi/n) \frac{1}{t^2} \right\}_{\pi/n}^{\delta} + 2 \int_{\pi/n}^{\delta} \bar{\Phi}_1(t + \pi/n) \frac{dt}{t^3} \right] \\ &= o\left(\frac{1}{n}\right) + o\left(\frac{1}{(\log n)^r}\right) \quad \text{by (1.9)} \\ (5.13) \quad &= o(1), \quad \text{as } n \rightarrow \infty. \end{aligned}$$

And

$$\begin{aligned} P_4 &= \frac{1}{2\pi P(n)} \int_0^{\pi/n} \phi(t + \pi/n) \text{Sin } nt P\left(\frac{1}{t + \pi/n}\right) \frac{dt}{t + \pi/n} \\ &= -\frac{1}{2\pi P(n)} \int_{\pi/n}^{2\pi/n} \phi(t) \text{Sin } nt P\left(\frac{1}{t}\right) \frac{dt}{t} \\ &= O\left(\frac{1}{P(n)}\right) \int_{\pi/n}^{2\pi/n} |\phi(t)| O(nt) P\left(\frac{1}{t}\right) \frac{dt}{t} \\ &= O(n) \int_{\pi/n}^{2\pi/n} |\phi(t)| dt \\ &= o\left(\frac{1}{(\log n)^r}\right), \quad \text{by (1.9)} \\ (5.14) \quad &= o(1) \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Also

$$\begin{aligned} P_5 &= O\left(\frac{1}{P(n)}\right) \int_{\delta - \pi/n}^{\delta} |\phi(t)| P\left(\frac{1}{t}\right) \frac{dt}{t} \\ (5.15) \quad &= o(1), \end{aligned}$$

by the regularity of the method of summation and since the interval $(\delta - \pi/n, \delta)$ tends to zero as $n \rightarrow \infty$.

Consequently from (5.2), (5.3), (5.12), (5.13), (5.14) and (5.15) we have

$$(5.16) \quad I_{2,1,1} = o(1) , \quad \text{as } n \rightarrow \infty .$$

Now

$$\begin{aligned} I_{2,1,2} &= \frac{1}{\pi P(n)} \int_{\pi/n}^{\delta} \frac{\phi(t)}{t} \text{Sin} (n + \frac{1}{2})t \Sigma_2 dt \\ &= O\left(\frac{1}{P(n)}\right) \int_{\pi/n}^{\delta} \frac{|\phi(t)|}{t} \left\{ \frac{r(1/t)}{t} + \frac{r(n)}{t} \right. \\ &\quad \left. + \frac{1}{t} \left[V(n) - V\left(\frac{1}{t} - 1\right) \right] \right\} dt , \quad \text{by (4.5)} \\ (5.17) \quad &= Q_1 + Q_2 + Q_3 , \quad \text{say.} \end{aligned}$$

We have

$$\begin{aligned} Q_1 &= O\left(\frac{1}{P(n)}\right) \int_{\pi/n}^{\delta} \frac{|\phi(t)|}{t^2} r\left(\frac{1}{t}\right) dt \\ &= O\left(\frac{1}{P(n)}\right) \left\{ \left[o\left(\frac{t}{\{\log 1/t\}^r}\right) r\left(\frac{1}{t}\right) \frac{1}{t^2} \right]_{\pi/n}^{\delta} + \int_{\pi/n}^{\delta} o\left(\frac{t}{\{\log 1/t\}^r}\right) r\left(\frac{1}{t}\right) \frac{dt}{t^3} \right. \\ &\quad \left. + \int_{\pi/n}^{\delta} o\left(\frac{t}{\{\log 1/t\}^r}\right) \frac{1}{t^2} dr\left(\frac{1}{t}\right) \right\} \quad \text{by (1.9).} \end{aligned}$$

Here the integrated part is $o(1)$, by (1.7) and the fact that $P(n) \rightarrow \infty$ as $n \rightarrow \infty$. Also the second term is

$$\begin{aligned} &o\left(\frac{1}{P(n)}\right) \int_{\pi/n}^{\delta} r\left(\frac{1}{t}\right) \frac{dt}{t^2} \frac{1}{(\log 1/t)^r} \\ &= o\left(\frac{1}{P(n)}\right) \int_{n/\pi}^{1/\delta} \frac{r(s)}{(\log s)^r} ds \\ &= o\left(\frac{1}{P(n)}\right) \sum_{k=0}^n \frac{k |p_k - p_{k-1}|}{(\log(k+1))^r} \\ &= o(1) \quad \text{by (1.8).} \end{aligned}$$

The third part is

$$\begin{aligned} &o\left(\frac{1}{P(n)}\right) \int_{n/\pi}^{1/\delta} \frac{s dr(s)}{(\log s)^r} \\ &= o\left(\frac{1}{P(n)}\right) \sum_{k=0}^n \frac{k |p_k - p_{k-1}|}{(\log(k+1))^r} \\ &= o(1) \quad \text{as } n \rightarrow \infty , \quad \text{by (1.8).} \end{aligned}$$

Thus we see that

$$(5.18) \quad Q_1 = o(1) \quad \text{as } n \rightarrow \infty .$$

Now

$$\begin{aligned}
 Q_2 &= O\left(\frac{r(n)}{P(n)}\right) \int_{\pi/n}^{\delta} \frac{|\phi(t)|}{t^2} dt \\
 &= O\left(\frac{r(n)}{P(n)}\right) \left\{ \left(\frac{\bar{\Phi}_1(t)}{t^2}\right)_{\pi/n}^{\delta} + 2 \int_{\pi/n}^{\delta} \bar{\Phi}_1(t) \frac{dt}{t^3} \right\} \\
 &= o\left(\frac{r(n)}{P(n)}\right) + o\left(\frac{nr(n)}{P(n)(\log n)^r}\right) \\
 &\quad + o\left(\frac{r(n)}{P(n)}\right) \int_{\pi/n}^{\delta} \frac{dt}{t^2(\log 1/t)^r} \\
 &= o\left\{\frac{(\log n)^r}{n}\right\} + o(1) + o\left\{\frac{r(n)n}{P(n)(\log n)^r}\right\} \\
 &= o(1) + o\left(\frac{1}{n}\right) \\
 (5.19) \quad &= o(1) \text{ as } n \rightarrow \infty.
 \end{aligned}$$

Lastly

$$\begin{aligned}
 Q_3 &= O\left(\frac{1}{P(n)}\right) \int_{\pi/n}^{\delta} \frac{|\phi(t)|}{t^2} \left[V(n) - V\left(\frac{1}{t} - 1\right) \right] dt \\
 &= O\left(\frac{1}{P(n)}\right) \left\{ \left[\frac{\bar{\Phi}_1(t)}{t^2} \left(V(n) - V\left(\frac{1}{t} - 1\right) \right) \right]_{\pi/n}^{\delta} \right. \\
 &\quad + 2 \int_{\pi/n}^{\delta} \frac{\bar{\Phi}_1(t)}{t^3} \left[V(n) - V\left(\frac{1}{t} - 1\right) \right] dt \\
 &\quad \left. - \int_{\pi/n}^{\delta} \frac{\bar{\Phi}_1(t)}{t^2} dV\left(\frac{1}{t} - 1\right) \right\}.
 \end{aligned}$$

The integrated part is $o(1)$, by (1.7) and the fact that

$$W_n \equiv \sum_{k=0}^n \frac{k |p_k - p_{k-1}|}{(\log(k+1))^r}; \quad W_0 \equiv 0.$$

Then, by (1.8) we have

$$\begin{aligned}
 V_n &= \sum_{k=0}^n |p_k - p_{k-1}| \\
 &= \sum_{k=0}^n \frac{\{\log(k+1)\}^r}{k} (W_k - W_{k-1}) \\
 &= \sum_{k=0}^{n-1} W_k \left\{ \Delta \left[\frac{\{\log(k+1)\}^r}{k} \right] \right\} + \frac{W_n \{\log(n+1)\}}{n} \\
 &= o\{R(n)\}.
 \end{aligned}$$

Now the second term is

$$\begin{aligned} & o\left[\frac{1}{P(n)} \int_{\pi/n}^{\delta} \left\{V(n) - V\left(\frac{1}{t} - 1\right)\right\} \frac{dt}{t^2(\log 1/t)^r}\right] \\ &= o\left[\frac{1}{P(n)} \int_{1/\delta}^{n/\pi} \frac{ds}{(\log s)^r} [V(n) - V(s - 1)]\right] \\ &= o\left\{\frac{1}{P(n)} \int_0^n \frac{s}{\{\log(s + 1)\}^r} dV(s)\right\} \\ &\quad + o\left[\frac{1}{R(n)} \{V(n) - V(s - 1)\} \frac{s}{\{\log(s + 1)\}^r}\right]_0^n \\ &= o\left\{\frac{1}{P(n)} \sum_{k=0}^n \frac{k|p_k - p_{k-1}|}{\{\log(k + 1)\}^r}\right\} + o\left\{\frac{1}{P(n)} \frac{n|p_n - p_{n-1}|}{\{\log(n + 1)\}^r}\right\} \end{aligned}$$

which is $o(1)$, by virtue of (1.7), (1.8) and the fact that $V_n = o\{P(n)\}$.

The third term is

$$\begin{aligned} & o\left\{\frac{1}{P(n)} \int_{\pi/n}^{\delta} \frac{1}{t(\log 1/t)^r} \left|dV\left(\frac{1}{t} - 1\right)\right|\right\}, \quad \text{by (1.9)} \\ &= o\left\{\frac{1}{P(n)}\right\} \int_0^n \frac{s|dV(s - 1)|}{\{\log(s + 1)\}^r} \end{aligned}$$

which is $o(1)$, as in the case of second term.

Thus we have

$$(5.20) \quad Q_3 = o(1) \quad \text{as } n \rightarrow \infty .$$

From (5.17), (5.18), (5.19) and (5.20), we have

$$(5.21) \quad I_{2,1,2} = o(1) \quad \text{as } n \rightarrow \infty .$$

From (3.4) (3.5), (3.16) and (5.21), we see that

$$(5.22) \quad I_{2,1} = o(1) \quad \text{as } n \rightarrow \infty .$$

Similarly, we can show that

$$(5.23) \quad I_{2,2} = o(1) \quad \text{as } n \rightarrow \infty .$$

From (3.3), (5.22) and (5.23) we get

$$(5.24) \quad I_2 = o(1) \quad \text{as } n \rightarrow \infty .$$

Lastly by Riemann Lebesgue Theorem and the regularity of the method of summation, we have, as $n \rightarrow \infty$

$$(5.25) \quad I_3 = o(1) .$$

Collection of (3.2), (5.24) and (5.25) as $n \rightarrow \infty$, completes the proof of the theorem.

I am much indebted to Professor M. L. Misra for his valuable guidance during the preparation of this paper.

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Printed in Japan by International Academic Printing Co., Ltd., Tokyo Japan

Pacific Journal of Mathematics

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