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THE INVERSE OF THE ERROR FUNCTION

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THE INVERSE OF THE ERROR FUNCTION

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1. Introduction. In a recent paper [3] J. R. Philip has discussed some properties of the function inverfc θ defined by means of

$$\theta = \text{erfc (inverfc } \theta) . \tag{1.1}$$

Since

$$\tfrac{1}{2}\pi^{1/2}(1 - \text{erfc } x) = x - \frac{x^3}{3} + \frac{x^5}{2!5} - \frac{x^7}{3!7} + \frac{x^9}{4!9} \cdots \tag{1.2}$$

it follows that

$$\text{inverfc } \theta = u + \frac{1}{3}u^3 + \frac{7}{30}u^5 + \frac{127}{630}u^7 + \frac{4369}{22680}u^9 + \cdots , \tag{1.3}$$

where

$$u = \tfrac{1}{2}\pi^{1/2}(1 - \theta) .$$

The coefficients in (1.3) are rational numbers. It is therefore of some interest to look for arithmetic properties of these numbers.

It will be convenient to change the notation slightly. Put

$$f(x) = \int_0^{\infty} e^{-t^2/2} dt , \tag{1.4}$$

so that

$$f(x) = \left(\frac{\pi}{2}\right)^{1/2}(1 - \text{erfc } 2^{1/2}x)$$

and let $g(x)$ denote the inverse function:

$$f(g(u)) = g(f(u)) = u , \tag{1.5}$$

where

$$g(u) = \sum_{n=0}^{\infty} A_{2n+1}\frac{u^{2n+1}}{(2n+1)!} \qquad (A_1 = 1) . \tag{1.6}$$

It follows from (1.4) and (1.5) that

$$g'(u) = \exp(\tfrac{1}{2}g^2(u)) . \tag{1.7}$$

Differentiating again, we get

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$$g''(u) = g(u)(g'(u))^2 . \tag{1.8}$$

It follows from (1.6) and (1.8) that

$$A_{2n+3} = \sum_{r+s\leq n} \frac{(2n+1)!}{(2r)!\,(2s)!\,(2n-2r-2s+1)!} A_{2r+1}A_{2s+1}A_{2n-2r-2s+1} . \tag{1.9}$$

Since $A_1 = 1$ it is evident from (1.9) that all the coefficients A_{2n+1} are positive integers. It is easily verified that the first few values of A_{2n+1} are

$$A_1 = A_3 = 1,\ A_5 = 7,\ A_7 = 127,\ A_9 = 4369 = 17.257 .$$

We shall show that

$$A_{2n+p} \equiv -\, 2.4.6 \cdots (p-1) A_{2n+1} \pmod{p} , \tag{1.10}$$

where p is an arbitrary prime and that

$$A_{2n+5} \equiv -\, A_{2n+1} \pmod{8} \tag{1.11}$$

and indeed

$$A_{2n+9} \equiv A_{2n+1} \pmod{16} . \tag{1.12}$$

We also find certain congruences (mod p) for a sequence of integers e_{2n} related to the A_{2n+1} (see Theorems 2 and 3 below).

Finally we put

$$\frac{u}{g(u)} = \sum_0^\infty \beta_{2n} \frac{u^{2n}}{(2n)!}$$

and obtain a theorem of the Staudt-Clausen type for the β_{2n}, namely

$$\beta_{2n} = G_{2n} - \frac{b}{3} - \sum_{p-1/2n} \frac{1}{p} A_p^{2n/(p-1)} ,$$

where G_{2n} is an integer, $b = 2$ or 1 according as $n \equiv 1$ or $\not\equiv 1 \pmod{3}$ and the summation is over all primes $p > 3$ such that $p - 1/2n$. Moreover

$$A_p \equiv -\, 2.4.6 \cdots (p-1) \pmod{p} .$$

2. A series of the form [2]

$$H(x) = \sum_{n=0}^\infty a_n \frac{x^n}{n!} , \tag{2.1}$$

where the a_n are rational integers, is called a Hurwitz series, or briefly an H-series. It is easily verified that sum, difference and product of two H-series is again an H-series. Also the derivative and the definite integral of the H-series define by (2.1):

$$H'(x) = \sum_{n=0}^{\infty} a_{n+1} \frac{x^n}{n!}, \quad \int_0^x H(t)dt = \sum_{n=1}^{\infty} a_{n-1} \frac{x^n}{n!}$$

are H-series. If $H_1(x)$ denotes an H-series without constant term then $H_1^k(x)/k!$ is an H-series for $k = 1, 2, 3, \cdots$; it follows that $H(H_1(x))$ is an H-series, where $H(x)$ is an arbitrary series of the form (2.1).

By the statement

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \equiv \sum_{n=0}^{\infty} b_n \frac{x^n}{n!} \pmod{m},$$

where the a_n, b_n are integers, is meant the system of congruences

$$a_n \equiv b_n \pmod{m} \qquad (n = 0, 1, 2, \cdots).$$

Thus the above statement about $H_1^k(x)/k!$ can be written in the form

$$H_1^k(x) \equiv 0 \pmod{k!}. \tag{2.2}$$

Returning to (1.4) it is evident that

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2^n(2n+1)n!} = \sum_{n=0}^{\infty} c_{2n+1} \frac{x^{2n+1}}{(2n+1)!}, \tag{2.3}$$

where

$$c_{2n+1} = (-1)^n \frac{(2n)!}{2^n n!} = (-1)^n\, 1.\, 3.\, 5 \cdots (2n-1), \tag{2.4}$$

so that $f(x)$ is an H-series without constant term.

If p is an odd prime, it follows from (2.4) that

$$c_{2n+1} \equiv 0, \pmod{p} \qquad (2n+1 > p). \tag{2.5}$$

Thus (1.5) implies

$$\sum_{n=0}^{1/2(p-1)} c_{2n+1} \frac{g^{2n+1}(u)}{(2n+1)!} \equiv u \pmod{p}. \tag{2.6}$$

We now compute the coefficient of $u^p/p!$ in the left member of (2.6). Clearly the terms with $1 \leqq n < (p-1)/2$ contribute nothing. Hence (2.6) yields

$$A_p + c_p \equiv 0 \pmod{p}.$$

Using (2.4) this becomes

$$A_p \equiv -(-1)^m\, 1.\, 3.\, 5 \cdots (p-2) \pmod{p}, \tag{2.7}$$

or if we prefer

$$(2.8)\qquad A_p \equiv -2.\ 4.\ 6 \cdots 2m \equiv -\left(\frac{2}{p}\right)m! \pmod p ,$$

where $p = 2m + 1$ and $(2/p)$ is the Legendre symbol. For example we have

$$A_5 \equiv -1.\ 3 \equiv 2 \pmod 5 ,$$
$$A_7 \equiv 1.\ 3.\ 5 \equiv 1 \pmod 7 ,$$
$$A_{11} \equiv 1.\ 3.\ 5.\ 7.\ 9 \equiv -1 \pmod{11} .$$

We consider next the residue (mod p) of A_{p+2n}. If $2n < p$ we have

$$\frac{(p+2n)!}{(2r)!\,(2s)!\,(p+2n-2r-2s)!} \equiv \frac{(2n)!}{(2r)!\,(2s)!\,(2n-2r-2s)!} \pmod p$$

by a familiar property of multinomial coefficients. Thus (1.9) implies (for $2n < p$)

$$(2.9)\qquad A_{p+2n+2} \equiv \sum_{r+s\leqq n} \frac{(2n)!}{(2r)!\,(2s)!\,(2n-2r-2s)!} \cdot A_{2r+1}A_{2s+1}A_{p+2n-2r-2s} \pmod p .$$

Since $A_p \not\equiv 0 \pmod p$ we may put

$$(2.10)\qquad A_{p+2n} \equiv A_p e_{2n} \pmod p \quad (2n \leqq p+1) .$$

Then (2.9) becomes

$$(2.11)\qquad e_{2n+2} \equiv \sum_{r+s\leqq n} \frac{(2n)!}{(2r)!\,(2s)!\,(2n-2r-2s)!} \cdot A_{2r+1}A_{2s+1}e_{2n-2r-2s} \pmod p$$

provided $2n < p$.

We now define a set of positive integers e_{2n} by means of $e_0 = 1$,

$$(2.12)\qquad e_{2n+2} = \sum_{r+s\leqq n} \frac{(2n)!}{(2r)!\,(2s)!\,(2n-2r-2s)!} A_{2r+1}A_{2s+1}e_{2n-2r-2s} \quad (n = 0, 1, 2, \cdots) .$$

If we put

$$\phi(x) = \sum_{n=0}^{\infty} e_{2n}\frac{x^{2n}}{(2n)!} ,$$

then (2.12) is equivalent to

$$(2.13)\qquad \phi''(x) = \phi(x)(g'(x))^2 .$$

Comparing (2.13) with (1.8) we get

$$\frac{\phi''(x)}{\phi(x)} = \frac{g''(x)}{g(x)} . \tag{2.14}$$

It follows that

$$\phi(x)g'(x) - g(x)\phi'(x) = 1 .$$

A little manipulation yields

$$\phi(x) = - g(x) \int \frac{dx}{g^2(x)} = - g(x) \int \frac{g'(x) \exp(-\frac{1}{2}g^2(x))dx}{g^2(x)}$$

and we get

$$\phi(x) = 1 - \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n!} \frac{g^{2n}(x)}{2n-1} . \tag{2.15}$$

Since

$$\frac{(2n)!}{2^n(2n-1)n!} = 1.\,3.\,5 \cdots (2n-3) ,$$

it follows from (2.2) and (2.15) that

$$\phi(x) \equiv 1 - \sum_{n=1}^{m+1} \frac{(-1)^n}{2^n n!} \frac{g^{2n}(x)}{2n-1} \pmod{p} , \tag{2.16}$$

where $p = 2m + 1$.

We notice also that (1.7) gives

$$g'(u) \equiv \sum_{n=0}^{\infty} \frac{g^{2n}(x)}{2^n n!} \pmod{p} , \tag{2.17}$$

while (1.8) yields

$$g''(u) \equiv \sum_{n=0}^{m-1} \frac{g^{2n+1}(x)}{n!} \pmod{p} . \tag{2.18}$$

3. We may rewrite (1.8) as

$$g''(u) = g(u) \exp g^2(u) . \tag{3.1}$$

Differentiating again and using (1.7) we get

$$g'''(u) = (1 + 2g^2(u)) \exp \left(\frac{3}{2}g^2(u)\right) . \tag{3.2}$$

Since

$$\exp\left(\frac{3}{2}g^2(u)\right) \equiv 1 \pmod 3 ,$$

it is clear that (3.2) implies

$$g'''(u) \equiv 1 + 2g^2(u) \pmod 3 .$$

On the other hand (1.7) gives

$$g'(u) \equiv 1 + \tfrac{1}{2}g^2(u) \equiv 1 + 2g^2(u) \pmod 3 .$$

We have therefore

$$(3.3) \qquad g'''(u) \equiv g'(u) \pmod 3 .$$

Comparison with (1.6) yields

$$(3.4) \qquad A_{2n+1} \equiv 1 \pmod 3 \qquad (n = 0, 1, 2, \cdots) .$$

If we differentiate (3.2) two more times we get

$$(3.5) \qquad \begin{cases} D^4g(u) = (7g(u) + 6g^3) \exp(2g^2(u)) , \\ D^5g(u) = (7 + 46g^2(u) + 24g^4(u)) \exp\left(\frac{5}{2}g^2(u)\right) , \end{cases}$$

where $D = d/du$. From the last equation it follows easily that

$$D^5g(u) \equiv 2 + g^2(u) + 4g^4(u) \pmod 5 .$$

Since by (1.7)

$$Dg(u) \equiv 1 + \frac{1}{2}g^2(u) + \frac{1}{8}g^4(u) \equiv 1 + 3g^2(u) + 2g^4(u) \pmod 5 ,$$

it follows that

$$(3.5) \qquad (D^5 - 2D)g(u) \equiv 0 \pmod 5 .$$

This is equivalent to

$$(3.6) \qquad A_{2n+5} \equiv 2A_{2n+1} \pmod 5 \qquad (n = 0, 1, 2, \cdots) .$$

Since $A_1 = A_3 = 1$, (2.6) implies

$$(3.7) \qquad A_{4n+1} \equiv A_{4n+3} \equiv 2^n \pmod 5 \qquad (n = 0, 1, 2, \cdots) .$$

It is clear from (3.1), (3.2) and (3.5) that

$$(3.8) \qquad D^n g(u) = \psi_{n-1}(g(u)) \exp\left(\frac{n}{2}g^2(u)\right) ,$$

where $\psi_n(z)$ is a polynomial of degree n in z with positive integral coefficients. Differentiating (3.8) we find that $\psi_n(z)$ satisfies the recurrence

(3.9) $$\psi_n(z) = \psi'_{n-1}(z) + nz\psi_{n-1}(z) .$$

We shall require the residue (mod p) of $\psi_{p-1}(z)$. It is not evident how to obtain this residue using (3.8) and (3.9). We shall therefore use a different method.

The writer has proved [1, §6] that if

$$g(x) = \sum_1^\infty a_n \frac{x^n}{n!} \quad (a_1 = 1)$$

is an H-series without constant term, if

$$\lambda(x) = \sum_1^\infty b_n \frac{x^n}{n!} \quad (b_1 = 1)$$

is the inverse of $g(x)$ and in addition

(3.10) $$b_n \equiv 0 \pmod{p} \qquad (n > p) ,$$

where p is an arbitrary prime, then

(3.11) $$a_{n+p} \equiv a_p a_{n+1} \pmod{p} \qquad (n \geqq 0) .$$

Clearly (3.10) is satisfied in the present case and therefore (3.11) implies

(3.12) $$A_{2n+p} \equiv A_p A_{2n+1} \pmod{p} .$$

Making use of (2.8) we may now state

THEOREM 1. *The coefficients of* $g(u)$ *defined by* (1.6) *satisfy*

(3.13) $$A_{2n+p} \equiv -2.4.6 \cdots (p-1)A_{2n+1} \pmod{p} \; (n = 0, 1, 2, \cdots) ,$$

where p *is an arbitrary odd prime.*

It is easily verified that (3.4) and (3.6) are in agreement with (3.13).

Since (3.12) is equivalent to

$$(D^p - A_p D)g(u) \equiv 0 \pmod{p} ,$$

comparison with (3.8) yields

$$\psi_{p-1}(g(u)) \equiv A_p \exp\left(\tfrac{1}{2}g^2(u)\right) \equiv A_p \sum_{n=0}^{m} \frac{g^{2n}(u)}{2^{n}n!} \pmod{p} ,$$

where $p = 2m + 1$.

If we put

$$(g(u))^k = \sum_{n=k}^{\infty} A_n^{(k)} \frac{u^n}{n!} \qquad (k = 1, 2, 3, \cdots) ,$$

we can show [1, Theorem 10] that $A_n^{(k)}$ satisfies

$$A_{n+p}^{(k)} \equiv A_p A_{n+1}^{(k)} \pmod p \quad (n \geqq 0) \tag{3.14}$$

for all $k \geqq 1$.

We shall apply this result to the series $\phi(u)$ defined by (2.15). Since (3.14) is equivalent to

$$(D^p - A_p D) g^k(u) \equiv 0 \pmod p ,$$

it is clear that (2.16) implies

$$\begin{aligned}(D^p - A_p D)\phi(u) &\equiv \frac{(-1)^m}{2^{m+1}(m-1)!} \frac{g^{p+1}(u)}{p} \\ &\equiv A_p (D^p - A_p D) \frac{g^{p+1}(u)}{p} \pmod p ,\end{aligned} \tag{3.15}$$

where $p = 2m + 1$.

Now by [1, (6.12)] we have

$$g(u) \equiv \sum_{n=0}^{m} A_{2n+1} \frac{g_1^{2n+1}(u)}{(2n+1)!} \pmod p ,$$

where

$$g_1(u) = u + A_p \frac{g^p(u)}{p!} ; \tag{3.16}$$

moreover

$$\frac{g_1^p(u)}{p!} \equiv \sum_{n=0}^{\infty} A_p^n \frac{x^{n(p-1)+1}}{(n(p-1)+1)!} \pmod p . \tag{3.17}$$

It follows from (3.16) and (3.17) that

$$(D^p - A_p D) \frac{g^p(u)}{p!} \equiv 1 \pmod p .$$

Thus (3.15) becomes

$$(D^p - A_p D)\phi(u) \equiv -A_p g(u) \pmod p ,$$

which is equivalent to

$$e_{2n+p+1} \equiv A_p (e_{2n+2} - A_{2n+1}) \pmod p \qquad (n = 0, 1, 2, \cdots) . \tag{3.18}$$

We may state

THEOREM 2. *The coefficients* e_{2n} *defined by* (2.12) *satisfy* (3.18).

In view of (2.10) we may rewrite (3.18) as

$$A_{2n+p+2} \equiv A_p A_{2n+1} + e_{2n+p+1} \quad (2n < p) . \tag{3.19}$$

Since

$$A_p A_{2n+1} \equiv A_{2n+p} ,$$

(3.19) is equivalent to

$$A_{2n+p+2} \equiv A_{2n+p} + e_{2n+p+1} \pmod p \quad (2n < p) . \tag{3.20}$$

We notice also that repeated application of (3.18) yields

$$e_{2n+k}(p-1) \equiv A_p^k e_{2n} - kA_{2n+k}(p-1) - 1 \pmod p ; \tag{3.21}$$

in particular we have for $k = p$

$$e_{2n+p(p-1)} \equiv A_p e_{2n} \pmod p . \tag{3.22}$$

It is also easy to extend (3.20) to

$$A_{2n+k(p-1)+1} \equiv kA_{2n+k(p-1)-1} + e_{2n+k(p-1)} \pmod p \quad (0 < 2n \leqq p+1 ;\ k = 1, 2, 3, \cdots) . \tag{3.23}$$

Indeed it follows from (3.23) and (3.18) that

$$\begin{aligned} e_{2n+(k+1)(p-1)} &\equiv A_p(e_{2n+k(p-1)} - A_{2n+k(p-1)-1}) \\ &\equiv A_p e_{2n+k(p-1)} - A_{2n+(k+1)(p-1)-1} \\ &\equiv A_p(A_{2n+k(p-1)+1} - kA_{2n+k(p-1)-1}) - A_{2n+(k+1)(p-1)-1} \\ &\equiv A_{2n+(k+1)(p-1)+1} - (k-1)A_{2n+(k+1)(p-1)-1} . \end{aligned}$$

Note that (3.23) does not hold for $k = 0$.

We may state the following theorem which supplements Theorem 2.

THEOREM 3. *The coefficients* e_{2n} *defined by* (2.12) *satisfy* (3.21), (3.22) *and* (3.23).

4. We now derive congruences for $A_{2n+1} \pmod 8$. From the first of (3.5) we have

$$\begin{aligned} D^4 g(u) &\equiv (-g(u) + 6g^3(u)) \exp(2g^2(u)) \\ &\equiv (-g(u) + 6g^3(u))(1 + 2g^2(u)) \\ &\equiv -g(u) + 4g^3(u) + 4g^5(u) \pmod 8 , \end{aligned}$$

so that

$$D^4 g(u) \equiv -g(u) \pmod 8 . \tag{4.1}$$

This is equivalent to

$$A_{2n+5} \equiv -A_{2n+1} \pmod 8 \qquad (n = 0, 1, 2, \cdots) , \tag{4.2}$$

which implies

$$A_{4n+1} \equiv A_{4n+3} \equiv (-1)^n \pmod 8 \qquad (n = 0, 1, 2, \cdots) . \tag{4.3}$$

This result can however be improved without much difficulty. Working modulo 16 we find that the $\psi_n(z)$ defined by (3.8) and (3.9) satisfy

$$\begin{aligned} &\psi_3(z) \equiv 7z + 6z^3 , && \psi_4(z) \equiv 7 - 2z^2 , \\ &\psi_5(z) \equiv -z + 6z^3 , && \psi_6(z) \equiv -1 + 12z^2 , \\ &\psi_7(z) \equiv z + 4z^3 ; \end{aligned}$$

note that the $\psi_n(z)$ are here treated as finite H-series. Then by (3-8)

$$\begin{aligned} D^8 g(u) &\equiv (g(u) + 4g^3(u)) \exp(4g^2(u)) \\ &\equiv (g(u) + 4g^3(u))(1 + 4g^2(u)) , \end{aligned}$$

so that

$$D^8 g(u) \equiv g(u) \pmod{16} . \tag{4.4}$$

This is equivalent to

$$A_{2n+9} \equiv A_{2n+1} \pmod{16} . \tag{4.5}$$

Since $A_1 = A_3 = 1$, $A_5 = 7$, $A_7 \equiv 7 \pmod{16}$, (4.5) implies

$$\begin{cases} A_{8n+1} \equiv A_{8n+3} \equiv 1 \pmod{16} , \\ A_{8n+5} \equiv A_{8n+7} \equiv 7 \pmod{16} . \end{cases} \tag{4.6}$$

We may state

THEOREM 4. *The coefficients* A_{2n+1} *satisfy* (4.2), (4.3), (4.5), (4.6).

5. We now put

$$\frac{u}{g(u)} = \sum_{n=0}^{\infty} \beta_{2n} \frac{u^{2n}}{(2n)!} , \tag{5.1}$$

so that

$$\sum_{r=0}^{n} \binom{2n+1}{2r} A_{2n-2r+1} \beta_{2r} = 0 \quad (n > 0) . \tag{5.2}$$

It follows from (5.2) that the β_{2n} are rational numbers with odd denominators.

From (5.1) and (2.3) we have

$$\frac{u}{g(u)} = \sum_{n=0}^{\infty} \frac{c_{2n+1}}{2n+1} \frac{g^{2n}(u)}{(2n)!} . \tag{5.3}$$

By (2.4)

$$c'_{2n+1} = \frac{c_{2n+1}}{2n+1} = (-1)^n \frac{1.3.5 \cdots (2n-1)}{2n+1} . \tag{5.4}$$

Let p be an odd prime. Then for $2n+1 > p$, c'_{2n+1} is integral (mod p) except possibly when $p/2n+1$. Let

$$2n+1 = kp^r , \quad p \nmid k , \quad r \geqq 1 .$$

If $k > 1$ it is obvious from (5.4) that c'_{2n+1} is integral (mod p). If $k = 1$, the numerator of c'_{2n+1} is divisible by at least p^w, where $w = (p^{r-1} - 1)/2$. But since

$$\tfrac{1}{2}(p^{r-1} - 1) \geqq r$$

except when $p = 3$, $r = 2$, it follows that

$$p \frac{u}{g(u)} \equiv c_p \frac{g^{p-1}(u)}{(p-1)!} \pmod{p} \quad (p > 3) , \tag{5.5}$$

$$3 \frac{u}{g(u)} \equiv -\frac{g^2(u)}{2!} - \frac{g^8(u)}{8!} \pmod{3} . \tag{5.6}$$

In the next place we have [1, (6.2)]

$$\frac{g^{p-1}(u)}{(p-1)!} \equiv \sum_{n=1}^{\infty} A_p^{n-1} \frac{u^{n(p-1)}}{(n(p-1))!} \pmod{p} \tag{5.7}$$

for all p. As for $g^8(u)/8!$, we have by (3.16)

$$\frac{g^3(u)}{3!} g_1(u) - u \equiv \sum_{1}^{\infty} \frac{u^{2n+1}}{(2n+1)!} ,$$

$$g_1'(u) \equiv 1 + \tfrac{1}{2} g^2(u) g'(u) \equiv 1 + \tfrac{1}{2} g^2(u) \equiv g'(u) \pmod{3} .$$

It follows that

$$\frac{g^4(u)}{4!} \equiv \sum_{2}^{\infty} (n-2) \frac{u^{2n}}{(2n)!} \pmod{3}$$

and a little manipulation leads to

$$\frac{g^8(u)}{8!} \equiv \sum_{1}^{\infty} \frac{u^{6n+2}}{(6n+2)!} \pmod{3} . \tag{5.8}$$

If we recall that

$$c_p \equiv -A_p \pmod{p}$$

and make use of (5.1), (5.3), (5.5), (5.6), (5.7) and (5.8) we get the following analog of the Staudt-Clausen theorem:

THEOREM 5. *The coefficients* β_{2n} *defined by* (5.1) *satisfy*

$$\beta_{2n} = G_{2n} - \frac{b}{3} - \sum_{\substack{p-1/2n \\ p>3}} \frac{A_p^{2n/(p-1)}}{p}, \tag{5.9}$$

where G_{2n} *is an integer,*

$$b = \begin{cases} 2 & n \equiv 1 \pmod 3 \\ 1 & n \not\equiv 1 \pmod 3 \end{cases}$$

and the summation is over all primes $p > 3$ *such that* $p - 1 \mid 2n$.

6. The following values of A_n were computed by R. Carlitz in the Duke University Computing Laboratory.

$$\begin{aligned}
A_5 &= 7, \quad A_7 = 127, \\
A_9 &= 17.257, \\
A_{11} &= 7.34807, \\
A_{13} &= 20036983, \\
A_{15} &= 17.134138639, \\
A_{17} &= 7.49020204823, \\
A_{19} &= 127.163.467.6823703, \\
A_{21} &= 23.109.6291767620181, \\
A_{23} &= 7.655889589032992201^*, \\
A_{25} &= 17.9402069019103587 3697^*,
\end{aligned}$$

The numbers marked with an asterisk have not been factored completely but at any rate have no prime divisors $< 10^4$.

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